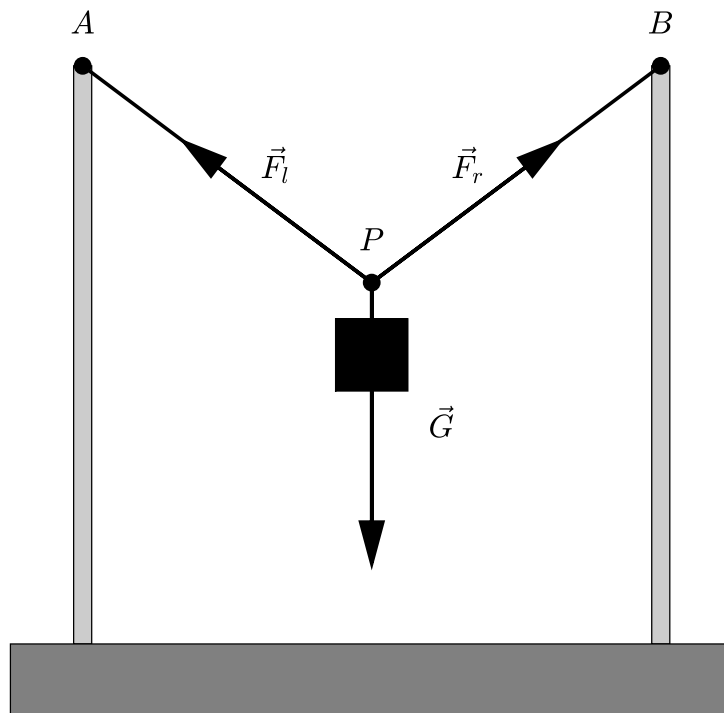


MathTraining

Problems and Solutions for
Vector Calculus



Klaus Höllig

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Aufgaben und Lösungen zur Höheren Mathematik 1 4. Auflage
by Klaus Höllig and Jörg Hörner
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Preface

The problem set of the book covers all basic topics of a course on vector calculus. It can be used to practice for exams, to facilitate the completion of homework assignments, and to review course material. Interactive variants to model problems with detailed solutions permit the student reader to test his comprehension of the relevant techniques. In addition to the collection of problems, a small mathematics lexicon contains brief descriptions of the relevant theorems, methods, and definitions.

There exists also a sportive aspect of mathematics - challenging problems requiring ideas beyond the standard techniques. The problems in the chapter *Calculus Highlights* are perhaps too difficult for undergraduates. They are included to initiate or strengthen fascination for mathematics. It is definitely not a mistake to practice substantially harder than necessary ...!

The book is partially translated from the book

[Aufgaben und Lösungen zur Höheren Mathematik 1](#)

by Jörg Hörner and the author. It supplements these textbooks by providing detailed solutions to tests for the chapters on *Vector Calculus*. Moreover, the book includes additional problems, in particular problem variants for the topics of the tests.

The author wishes the readers success in their studies **and** hopes that mathematics will become one of their favorite subjects!

Klaus Höllig

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Introduction

The book contains problems with detailed solutions, problem variants with interactive result verification, and a mathematics lexicon for the principal topics which are usually subject of a course on *Vector Calculus*:

- Vectors
- Scalar Product
- Cross Product and Scalar Triple Product
- Lines and Planes

The problem set can be used to practice for exams, to facilitate the completion of homework assignments, and to deepen the comprehension of course material. How is this accomplished most effectively? Remembering his own student days, the author makes the following recommendations to a student reader.

Consider, as an example, a problem from the chapter on *Straight Lines and Planes*:

4.2 Distance of a Point from a Straight Line

Determine the distance d of the point $Q = (6, 1, 7)$ from the straight line

$$g : (5, 6, -3)^t + t(0, 1, 2)^t$$

and the projection X of Q on g .

Resources: [Distance Point-Line](#)

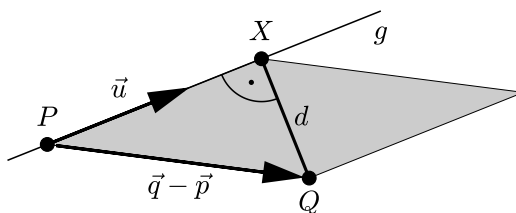
Before looking at the solution of the problem, review the relevant theory or methods (resources). Clicking on the link leads to the following brief description of the relevant formulas from the *Lexicon* in chapter 10.

Distance Point-Line

$g : \vec{p} + t\vec{u}$, X : projection of a point Q on g , i.e. $(\vec{x} - \vec{q}) \perp \vec{u}$

$$\underbrace{\text{dist}(Q, g)}_d = |\overrightarrow{QX}| = \frac{|(\vec{q} - \vec{p}) \times \vec{u}|}{|\vec{u}|}$$

$$\vec{x} = \vec{p} + \frac{(\vec{q} - \vec{p}) \cdot \vec{u}}{|\vec{u}|^2} \vec{u}$$



shaded area: norm of the cross product

Try to solve the problem with these instructions. Then compare your computations with the solution given in the book:

Solution

Distance

applying the formula $d = |(\vec{q} - \vec{p}) \times \vec{u}|/|\vec{u}|$ for the distance of a point Q from a straight line $g : \vec{p} + t\vec{u}$ with $Q = (6, 1, 7)$, $P = (5, 6 - 3)$, $\vec{u} = (0, 1, 2)^t \rightsquigarrow$

$$\vec{q} - \vec{p} = \begin{pmatrix} 1 \\ -5 \\ 10 \end{pmatrix}, \quad d = \left| \begin{pmatrix} 1 \\ -5 \\ 10 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right| / \left| \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right|$$

The cross product in the numerator equals $(-10 - 10, 0 - 2, 1 + 0)^t = (-20, -2, 1)^t$ and hence

$$d = |\overrightarrow{XQ}| = \frac{\sqrt{400 + 4 + 1}}{\sqrt{0 + 1 + 4}} = 9.$$

Projection

Since $\overrightarrow{PX} \perp \overrightarrow{XQ}$ and $\overrightarrow{PX} + \overrightarrow{XQ} = \overrightarrow{PQ}$, $|\overrightarrow{PX}|$ can be computed with the theorem of Pythagoras:

$$|\overrightarrow{PX}|^2 = |\overrightarrow{PQ}|^2 - |\overrightarrow{XQ}|^2 = |(1, -5, 10)^t|^2 - d^2 = 126 - 81 = 45$$

i.e. $|\overrightarrow{PX}| = \sqrt{45} = 3\sqrt{5}$.

\rightsquigarrow position of X on g (adding a multiple of the normalized direction vector

$\vec{u}^\circ = \vec{u}/|\vec{u}|$ to \vec{p}):

$$\vec{x} = \vec{p} + |\overrightarrow{PX}| \vec{u}^\circ = (5, 6, -3)^t + (3\sqrt{5})(0, 1, 2)^t/\sqrt{5} = (5, 9, 3)^t$$

Alternative solution

applying the formula for the component of the vector $\vec{q} - \vec{p}$ in the direction of \vec{u} \rightsquigarrow

$$\begin{aligned} \vec{x} &= \vec{p} + \frac{(\vec{q} - \vec{p}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \\ &= \begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix} + \underbrace{\frac{(1, -5, 10)^t \cdot (0, 1, 2)^t}{(0, 1, 2)^t \cdot (0, 1, 2)^t}}_{=15/5} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 3 \end{pmatrix} \end{aligned}$$

distance:

$$d = |\vec{x} - \vec{q}| = |(5, 9, 3)^t - (6, 1, 7)^t| = |(-1, 8, -4)^t| = 9$$

The solutions are written in a keyword-like style, as you would employ when you comment your solutions in an exam or for homework problems. For example, the phrase

$$“\vec{u} \perp \vec{v} \implies \dots”$$

stands for

“The orthogonality of the vectors \vec{u} and \vec{v} implies that \dots ”.

Other examples of typical phrases are “simplifying $\rightsquigarrow \dots$ ” or “alternative argument: \dots ” There is just as much detail included as is necessary for the mathematical arguments.

To gain more practice with the solution technique, it is highly recommended to solve some (preferably all \dots) of the problem variants, following the principal model problem for each topic. For *Distance of a Point from a Straight Line* the variants are:

Problem Variants

■ $Q = (-3, 1, -4), \quad g : (0, 1, 2)^t + t(2, 1, 2)^t$

$d = ?, x_3 = -?$:

check

■ $Q = (7, 6, -1), \quad g : (0, 1, 0)^t + t(3, 0, -4)^t$

$d = ?.??, x_1 = ?$:

check

■ $Q = (0, 0, 1), \quad g : (1, 0, 0)^t + t(-1, 1, 1)^t$

$d = ?.??, x_1 = ?.??$:

check

You can check your solution by typing your answer in the field adjacent to the check - box, replacing every question mark by a character (digit or letter). Convert your result to a decimal, truncated to the number of digits indicated. For example,

$$2/3 \rightarrow 0.6666 \dots \xrightarrow{?.??} 0.66, \quad \text{answer : } \span style="border: 1px solid red; padding: 2px;">066.$$

Note that the period is omitted; only the characters corresponding to the question marks are typed.

The solutions for the three problem variants are

- (1) $d = 3, X = (-4, -1, -2)$
- (2) $d = 5\sqrt{2} \approx 7.0711, X = (3, 1, -4)$
- (3) $d = \sqrt{2/3} \approx 0.8165, X = (1/3, 2/3, 2/3)$

Hence, the correct input is

- (1) $d = ?, x_3 = -? \rightarrow \span style="border: 1px solid black; padding: 2px;">32$
- (2) $d = ?.??, x_1 = ? \rightarrow \span style="border: 1px solid black; padding: 2px;">7073$
- (3) $d = ?.??, x_1 = ?.?? \rightarrow \span style="border: 1px solid black; padding: 2px;">081033$

As mentioned in the beginning, the problem set can also assist you in completing homework assignments. Just look for a similar problem and study its solution. Similarly, for methods and examples presented in class, practice with the relevant problems.

The above remarks pertain to the first five chapters which exclusively discuss the solution of standard problems. Usually, such problems constitute the major portion of an exam or homework assignment. Hence, to review the basic techniques involved is of primary importance. Applying these techniques to more advanced problems is a natural next step. The chapter *Calculus Highlights* contains examples of rather challenging applications. You do not have to be disappointed if you cannot solve any of these problems; they are definitely very difficult. It is legitimate to immediately look at the solutions and learn how the methods from the previous chapters are applied in an advanced setting. Also, as mentioned in the Preface, it is not a mistake to practice substantially harder than necessary ...!

You have solved some of the problems in chapter 9 without resorting to the solutions. Then ...

... you can take pride in your mastery of the principal techniques for solving problems in Vector Calculus!

With the previous explanations aimed at student readers, instructors could (obviously) also benefit from the interactive problem collection. The solutions of the model problems can be used as examples in class and some of the variants assigned as homework problems. Students will welcome the possibility of checking results before submitting or presenting their solutions in the exercise sections.

Disclaimer: Although the solutions and answers to the variants have been thoroughly checked, mistakes can always occur¹. Please, write to the author (Klaus.Hoellig@gmail.com) if you find any errors.

¹A statement by a teaching assistant to encourage students, which the author will always remember: "This year, the final exam is not too difficult - your professor could check the results without committing any errors!".

Chapter 1

Vectors

1.1 Cylindrical and Spherical Coordinates

Express the point $(x, y, z) = (-\sqrt{2}, \sqrt{2}, 2\sqrt{3})$ in cylindrical and spherical coordinates.

Resources: [Coordinate Systems](#)

Problem Variants

■ $(x, y, z) = (3, 3, \sqrt{6})$

$$\varrho = 3\sqrt{?}, \varphi = ?\pi, r = ?\sqrt{6}, \vartheta = ?\pi:$$

check

■ $(x, y, z) = (-\sqrt{2}, -\sqrt{6}, -2\sqrt{2})$

$$\varrho = 2\sqrt{?}, \varphi = -?\pi, r = ?, \vartheta = ?\pi:$$

check

■ $(x, y, z) = (1, -1, \sqrt{6})$

$$\varrho = \sqrt{?}, \varphi = -?\pi, r = \sqrt{?}, \vartheta = ?\pi:$$

check

Solution

application of the formulas for transforming the cartesian coordinates $P = (x, y, z) = (-\sqrt{2}, \sqrt{2}, 2\sqrt{3})$:

- cylindrical coordinates (ϱ, φ, z)

$$\varrho = \sqrt{x^2 + y^2} = 4$$

$$\varphi \stackrel{(\star)}{=} \arctan(y/x) + \pi = \arctan(-1) + \pi = -\pi/4 + \pi = 3\pi/4$$

- spherical coordinates (r, ϑ, φ)

$$r = \sqrt{x^2 + y^2 + z^2} = \sqrt{2 + 2 + 12} = 4$$

$$\vartheta = \arccos(z/r) = \arccos(\sqrt{3}/2) = \pi/6$$

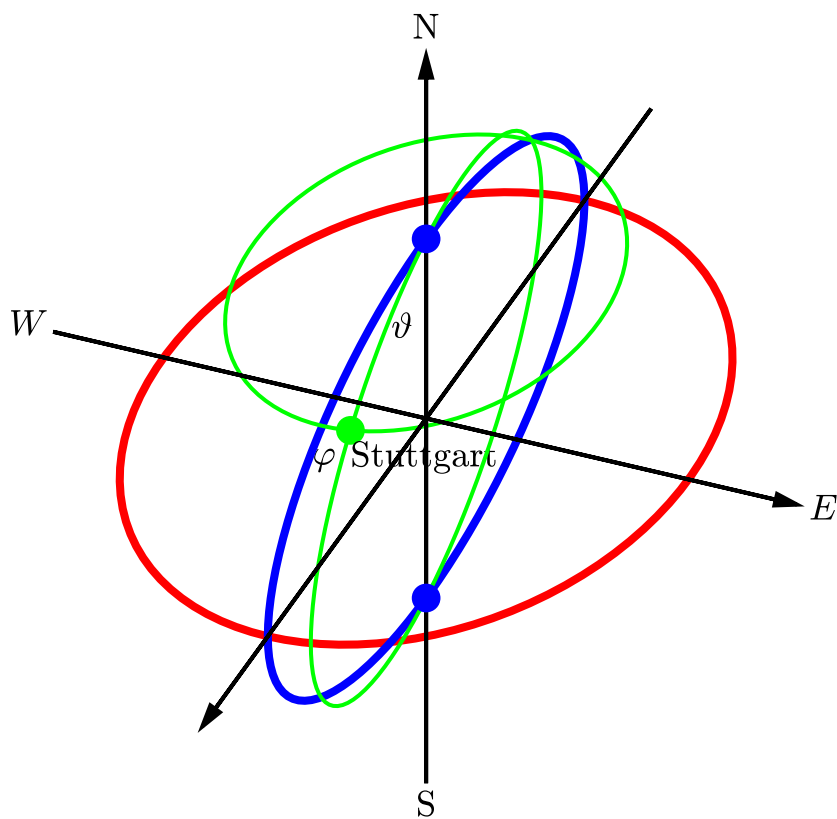
(\star) Adding π is necessary since (x, y) lies in the second quadrant ($x < 0$, $y \geq 0$) and the principal branch of \arctan yields the correct value only for points in the first and fourth quadrant.

1.2 GPS, Latitude, and Longitude

The figure illustrates the position of *Stuttgart* with GPS coordinates

$$(\text{latitude}, \text{longitude}) = (48.778449, 9.180013)$$

on the globe with the corresponding latitude and longitude circles (green). Also shown is the equator (red) and the prime meridian (blue) passing through the poles (blue dots) and the London Greenwich observatory.



Compute the angles ϑ and φ in radians, to describe the position of *Stuttgart* in spherical coordinates (z -axis pointing to the north, y -axis pointing to the east, and x -axis aligned with the prime meridian). Moreover, determine the distance d to *Baia do Sancho* $(-3.85460, -32.44369)$ where you can relax after a strenuous math exam.

Resources: [Coordinate Systems](#)

Problem Variants

■ New York (40.71576, -74.02231), Pink Sands Beach (25.50798, -76.63208)

$d = \text{????}$:

check

■ Paris (48.86199, 2.34394), Matira Beach (-16.54095, -151.73787)

$d = \text{????}$:

check

■ Tokio (35.72088, 139.77435), Whitehaven Beach (-20.28084, 149.03855)

$d = \text{????}$:

check

Solution

Spherical coordinates

The following table shows the correspondence of the GPS coordinates (latitude, longitude) to the spherical angles (ϑ, φ) :

northern latitude: $0 \dots 90$	$\vartheta : \pi/2 \dots 0$
southern latitude: $-0 \dots -90$	$\vartheta : \pi/2 \dots \pi$
eastern longitude: $0 \dots 180$	$\varphi : 0 \dots \pi$
western longitude: $-0 \dots -180$	$\varphi : -0 \dots -\pi$

Converting degrees to radians (radians = $(\pi/180) \cdot$ degrees) leads to the following spherical coordinates for *Stuttgart*:

$$\begin{aligned}\vartheta &= \pi/2 - (\pi/180) \cdot 48.778449 \approx 0.7195, \\ \varphi &= (\pi/180) \cdot 9.180013 \approx 0.1602.\end{aligned}$$

The angles for *Baia do Sanchos*, located on the southern hemisphere, are

$$\begin{aligned}\vartheta &= \pi/2 + (\pi/180) \cdot 3.85460 \approx 1.6381, \\ \varphi &= -(\pi/180) \cdot 32.44369 \approx -0.5662.\end{aligned}$$

Distance

The length of a circular segment connecting two points P and Q on a sphere with radius r is

$$d = r \sphericalangle(\vec{p}, \vec{q}) = r \arccos(\vec{p}^\circ \cdot \vec{q}^\circ), \quad \vec{v}^\circ = \vec{v}/|\vec{v}|.$$

normalized (unit length) position vectors, expressed in terms of spherical coordinates using the angles ϑ and φ computed above

- *Stuttgart*

$$\vec{p}^\circ = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)^t \approx (0.6505, 0.1051, 0.7522)^t$$

- *Baia do Sanchos*

$$\vec{q}^\circ \approx (0.8420, -0.5353 - 0.0672)^t$$

computing the angle between the two vectors via their scalar product and multiplying with the earth radius \rightsquigarrow distance

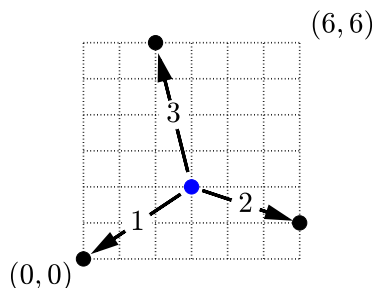
$$d = \arccos(\vec{p}^\circ \cdot \vec{q}^\circ) \cdot r \approx \arccos(0.4409) \cdot 6367 \text{ km} \approx 7093 \text{ km}$$

1.3 Equilibrium of Forces

Assume that the force \vec{f}_Q acting on a point P (blue dot) in the direction of a point Q is proportional to $|\vec{q} - \vec{p}|$, i.e.

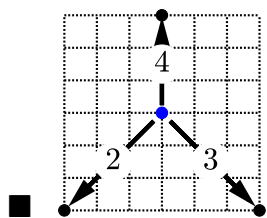
$$\vec{f}_Q = c_Q (\vec{q} - \vec{p}).$$

Compute the total force acting on P for the three points Q (black dots) and the constants $c_Q = 1, 2, 3$ depicted in the figure. Moreover, determine the equilibrium point P_\star where the total force is zero.



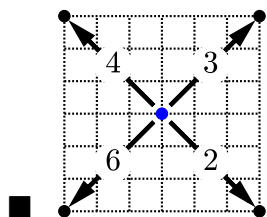
Resources: [Addition and Scalar Multiplication of Vectors](#)

Problem Variants



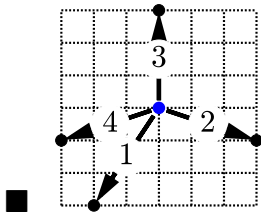
$P_\star = (?.??, ?.??):$

check



$P_\star = (?, ?.?):$

check



$P_{\star} = (?.?, ?)$:

check

Solution

Total force

$$\vec{f} = \sum_Q \vec{f}_Q = \sum_Q c_Q (\vec{q} - \vec{p}) \quad (1)$$

substituting the given data $P = (3, 2)$ and $Q = (0, 0), (6, 1), (2, 6)$ \rightsquigarrow

$$\begin{aligned} \vec{f} &= 1 \cdot \begin{pmatrix} 0-3 \\ 0-2 \end{pmatrix} + 2 \cdot \begin{pmatrix} 6-3 \\ 1-2 \end{pmatrix} + 3 \cdot \begin{pmatrix} 2-3 \\ 6-2 \end{pmatrix} \\ &= \begin{pmatrix} -3 \\ -2 \end{pmatrix} + \begin{pmatrix} 6 \\ -2 \end{pmatrix} + \begin{pmatrix} -3 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} \end{aligned}$$

Equilibrium point

setting $\vec{f} = (0, 0)^t$ and solving equation (1) for \vec{p} \implies

$$\vec{p} = \frac{1}{\sum_Q c_Q} \sum_Q c_Q \vec{q}$$

substituting the given data \rightsquigarrow

$$\begin{aligned} \vec{p} &= \frac{1}{1+2+3} \left(1 \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 6 \\ 1 \end{pmatrix} + 3 \cdot \begin{pmatrix} 2 \\ 6 \end{pmatrix} \right) \\ &= \frac{1}{6} \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 12 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ 18 \end{pmatrix} \right) = \frac{1}{6} \begin{pmatrix} 18 \\ 20 \end{pmatrix} = \begin{pmatrix} 3 \\ 10/3 \end{pmatrix} \end{aligned}$$

Solution

expressing \vec{f} as sum of multiples of \vec{u} and \vec{v} \rightsquigarrow linear system

$$\vec{f} = \begin{pmatrix} 9 \\ 1 \end{pmatrix} = s \underbrace{\begin{pmatrix} 3 \\ 2 \end{pmatrix}}_{\vec{u}} + t \underbrace{\begin{pmatrix} 2 \\ -2 \end{pmatrix}}_{\vec{v}},$$

i.e.

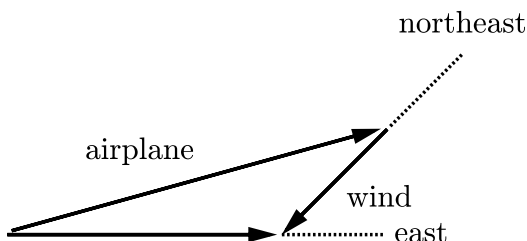
$$3s + 2t = 9, \quad 2s - 2t = 1$$

solving the second equation for s $\rightsquigarrow s = 1/2 + t$

substituting into the first equation $\rightsquigarrow 3/2 + 3t + 2t = 9$, i.e. $t = 3/2$ and $s = 2$

1.5 Drift of an Airplane

In which direction v does an airplane have to fly at 600 km/h with a headwind from the northeast at 60 km/h in order to reach a destination 6000 km to the east? Determine also the flight time T . Note that the velocity of the wind is magnified in the figure.



Resources: [Addition and Scalar Multiplication of Vectors](#)

Problem Variants

■ velocity of the airplane: 500 km/h, sidewind at 50 km/h from the north, distance 5000 km

$T = ???.??:$

check

■ velocity of the airplane: 400 km/h, tailwind at 40 km/h from the northwest, distance 1000 km

$T = ?.??:$

check

■ velocity of the airplane: 600 km/h, sidewind at 30 km/h from the southeast, distance 3000 km

$T = ???.??:$

check

Solution

Flight direction

adding the velocity vectors of the airplane and the wind \rightsquigarrow

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + 60 \begin{pmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} R \\ 0 \end{pmatrix}, \quad \sqrt{v_1^2 + v_2^2} = 600 \quad (1)$$

with R the resulting velocity in easterly direction (units in km/h)
second component of the equation \implies

$$v_2 = \frac{60}{\sqrt{2}} \approx 42.42$$

and

$$v_1 = \sqrt{600^2 - v_2^2} = \sqrt{360000 - 1800} \approx 598.49$$

Flight time

first component of the equation (1) \rightsquigarrow velocity in easterly direction

$$R = \sqrt{358200} - 60/\sqrt{2} \approx 556.07$$

flight time:

$$T = 6000/R \approx 10.78$$

1.6 Components of a Vector

Write $(5, -7, 3)^t$ as a sum of a vector orthogonal and a vector parallel to the plane $E : x - 2y + 3z = 0$.

Resources: [Addition and Scalar Multiplication of Vectors](#), [Orthogonal Basis](#)

Problem Variants

■ $\vec{v} = (7, -1, 3)^t$, $E : 2x - y = 0$

coordinates of the orthogonal and of the parallel component: $(?, -?, ?)$, $(?, ?, ?)$:

check

■ $\vec{v} = (-1, 5, -1)^t$, $E : x - 2y + z = 0$

coordinates of the orthogonal and of the parallel component: $(-?, ?, -?)$, $(?, ?, ?)$:

check

■ $\vec{v} = (5, 1, 7)^t$, $E : x - y + 2z = 0$

coordinates of the orthogonal and of the parallel component: $(?, -?, ?)$, $(?, ?, ?)$:

check

Solution

Orthogonal component

The component \vec{v}_\perp of $\vec{v} = (5, -7, 3)^t$ orthogonal to the plane $E : x - 2y + 3z = 0$ is the projection of \vec{v} on the normal vector $\vec{n} = (1, -2, 3)^t$ of E , i.e.

$$\vec{v}_\perp = \frac{\vec{v} \cdot \vec{n}}{\vec{n} \cdot \vec{n}} \vec{n} = \frac{\begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \frac{28}{14} \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix}$$

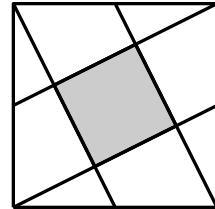
Parallel component

$(\vec{v} - \vec{v}_\perp) \perp \vec{n}$ (general property of a projection, immediately apparent from the definition) \implies

$$\vec{v}_\parallel = \vec{v} - \vec{v}_\perp = \begin{pmatrix} 5 \\ -7 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 6 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ -3 \end{pmatrix}$$

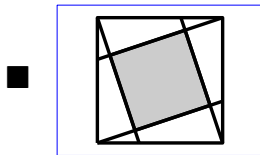
1.7 Area Bounded by Intersecting Lines

Determine the area of the gray square, bounded by line segments connecting the corners of a square with side length 1 with midpoints of its edges.



Resources: [Straight Line](#)

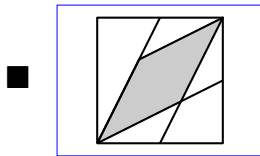
Problem Variants



The line intersections divide the edges of the unit square in the ratio 1 : 2.

?.?:

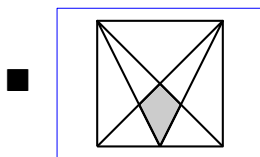
check



The line intersections divide the edges of the unit square in the ratio 1 : 1.

?.??:

check



The line intersections divide the edges of the unit square in the ratio 1 : 1.

?.???:

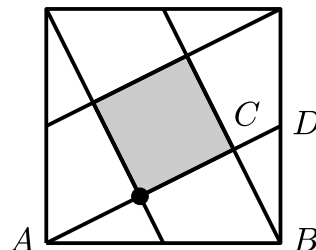
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Solution

Partition of the unit square

partitioning the square with side length 1 into the smaller gray square S and four right-angled triangles congruent to $\Delta(A, B, C) \rightsquigarrow$

$$\text{area } Q = 1 - 4 \cdot \text{area } \Delta(A, B, C) \quad (1)$$



Areas of the triangle and the square

similarity (equal angles) of the triangle $\Delta(A, B, C)$ to the larger right-angled triangle $\Delta(A, D, B)$, \implies

$$\frac{\text{area } \Delta(A, B, C)}{\text{area } \Delta(A, D, B)} = \left(\frac{|AB|}{|AD|} \right)^2 \quad (2)$$

(ratio of areas = ratio of lengths squared)

theorem of Pythagoras \implies

$$|AD|^2 = |AB|^2 + |BD|^2 = 1 + (1/2)^2 = 5/4$$

and, noting also that $\text{area } \Delta(A, D, B) = 1/4$, equation (2) \implies

$$\text{area } \Delta(A, B, C) = \frac{1}{4} \left(\frac{1}{\sqrt{5/4}} \right)^2 = \frac{1}{5}$$

$$(1) \implies \text{area } S = 1 - 4 \cdot (1/5) = 1/5$$

Alternative Solution

Placing the origin of the coordinate system at the point A , the point C is the intersection of the lines

$$g_{CD} : y = x/2 \quad \text{and} \quad g_{CB} : y = 2(1 - x).$$

Hence, the x -coordinate of C satisfies $x/2 = 2(1 - x)$ which yields $x = 4/5$ and $y = 2/5$.

By symmetry, the coordinates of the marked point are $(2/5, 1/5)$ and thus

$$\text{area } S = |(4/5, 2/5) - (2/5, 1/5)|^2 = |(2/5, 1/5)|^2 = 4/25 + 1/25 = 1/5.$$

1.8 Coefficients of a Vector with Respect to a Basis

Write the vector $\vec{x} = (-4, 2, 3)^t$ as a linear combination of the vectors¹ $\vec{u} = (3, 1, -2)^t$, $\vec{v} = (4, -1, -2)^t$, $\vec{w} = (4, 1, -3)^t$.

Resources: [Basis Coefficients](#)

Problem Variants

■ $\vec{x} = (4, -3, -4)^t$, $\vec{u} = (-3, 2, -3)^t$, $\vec{v} = (-1, 1, 1)^t$, $\vec{w} = (4, -3, -1)^t$

$$\vec{x} = ?\vec{u} + ?\vec{v} + ?\vec{w}:$$

check

■ $\vec{x} = (-3, 3, -4)^t$, $\vec{u} = (2, -1, 3)^t$, $\vec{v} = (-3, -2, -3)^t$, $\vec{w} = (3, 1, 4)^t$

$$\vec{x} = -?\vec{u} + ?\vec{v} + ?\vec{w}:$$

check

■ $\vec{x} = (-4, -3, 1)^t$, $\vec{u} = (-3, 2, 4)^t$, $\vec{v} = (1, 3, 1)^t$, $\vec{w} = (-2, -1, 1)^t$

$$\vec{x} = -?\vec{u} + ?\vec{v} + ?\vec{w}:$$

check

¹All numbers appearing in the problem, its variants, and the solution are nonzero with absolute value < 5 (You do not want to resort to a pocket calculator in an exam.). Also, each number within the chosen range appears (esthetic reasons ...). Time consuming to generate? NO, < 2 seconds, says MATLAB[®] !

Solution

comparing the x -, y -, z -components in the representation of the vector $\vec{x} = (-4, 2, 3)^t$ as linear combination of the three basis vectors \vec{u} , \vec{v} , \vec{w} ,

$$r \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix} + s \begin{pmatrix} 4 \\ -1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ -3 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}$$

\rightsquigarrow linear system for the coefficients r, s, t :

$$\underbrace{\begin{pmatrix} 3 & 4 & 4 \\ 1 & -1 & 1 \\ -2 & -2 & -3 \end{pmatrix}}_{(\vec{u}, \vec{v}, \vec{w})} \begin{pmatrix} r \\ s \\ t \end{pmatrix} = \underbrace{\begin{pmatrix} -4 \\ 2 \\ 3 \end{pmatrix}}_{\vec{x}}$$

Cramer's rule \rightsquigarrow coefficients r, s, t as quotients of determinants (or, equivalently, triple products; $\det(\vec{a}, \vec{b}, \vec{c}) = [\vec{a}, \vec{b}, \vec{c}]$):

$$r = \frac{\det(\vec{x}, \vec{v}, \vec{w})}{\det(\vec{u}, \vec{v}, \vec{w})} = \frac{\begin{vmatrix} -4 & 4 & 4 \\ 2 & -1 & 1 \\ 3 & -2 & -3 \end{vmatrix}}{\begin{vmatrix} 3 & 4 & 4 \\ 1 & -1 & 1 \\ -2 & -2 & -3 \end{vmatrix}}$$

computing the determinants with Sarrus' rule \rightsquigarrow

$$r = \frac{-12 + 12 - 16 - 8 + 24 + 12}{9 - 8 - 8 + 6 + 12 - 8} = \frac{12}{3} = 4$$

and, similarly,

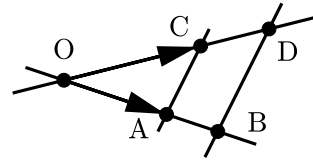
$$s = \frac{\det(\vec{u}, \vec{x}, \vec{w})}{\det(\vec{u}, \vec{v}, \vec{w})} = -1, \quad t = \frac{\det(\vec{u}, \vec{v}, \vec{x})}{\det(\vec{u}, \vec{v}, \vec{w})} = -3$$

1.9 Proving Elementary Geometric Theorems

Using the depicted vectors, show that

$$\vec{AC} \parallel \vec{BD} \implies |\overline{OA}| : |\overline{OB}| = |\overline{AC}| : |\overline{BD}|$$

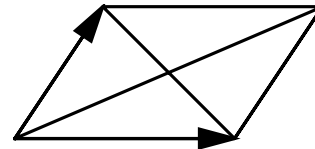
(proportionality of intersecting lines).



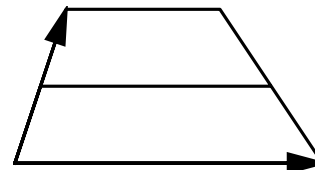
Resources: [Addition and Scalar Multiplication of Vectors](#)

Problem Variants

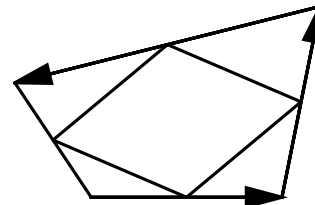
- Show that the diagonals of a parallelogram intersect at their midpoints.



- Show that the line segment, connecting the midpoints of the non-parallel sides of a trapezoid, is parallel to the base line.



- Show that connecting the midpoints of the sides of an arbitrary quadrilateral yields a parallelogram.



Solution

The position vectors $\vec{a} = \overrightarrow{OA}$ and $\vec{b} = \overrightarrow{OB}$ as well as the position vectors \vec{c} and \vec{d} are colinear:

$$\vec{b} = s\vec{a}, \quad \vec{d} = t\vec{c}. \quad (1)$$

Since

$$\overrightarrow{AC} = \vec{c} - \vec{a} \parallel \overrightarrow{BD} = \vec{d} - \vec{b}, \quad (2)$$

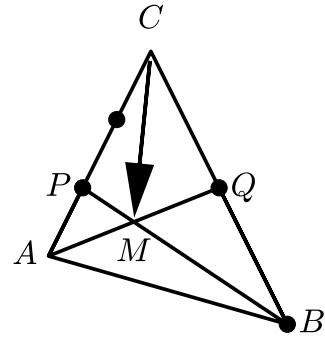
it follows that

$$r(\vec{c} - \vec{a}) \underset{(2)}{=} \vec{d} - \vec{b} \underset{(1)}{=} t\vec{c} - s\vec{a}.$$

comparing the coefficients of the linearly independent vectors \vec{a} and $\vec{c} \implies r = s$ and $r = t$ (equal ratios of the line segments)

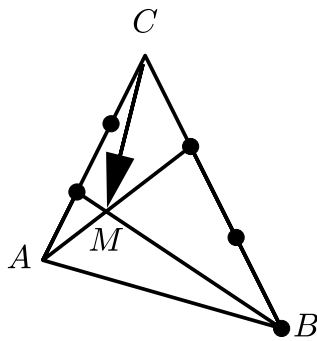
1.10 Linear Combination of Vectors

As indicated by the black dots, the point P divides the segment \overline{AC} in the ratio 1 : 2 and Q divides \overline{BC} in the ratio 1 : 1. Express the vector \overrightarrow{CM} as a linear combination of the vectors \overrightarrow{CA} and \overrightarrow{CB} .



Resources: [Addition and Scalar Multiplication of Vectors](#)

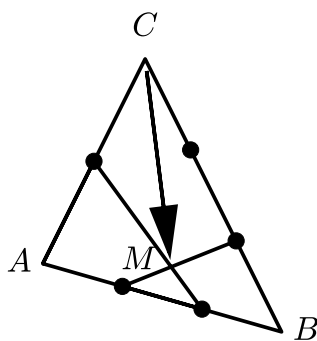
Problem Variants



■

$$\overrightarrow{CM} = \text{?.??}\overrightarrow{CA} + \text{?.??}\overrightarrow{CB}:$$

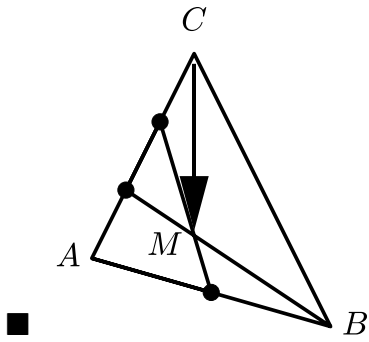
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■

$$\overrightarrow{CM} = \text{?.??}\overrightarrow{CA} + \text{?.??}\overrightarrow{CB}:$$

check

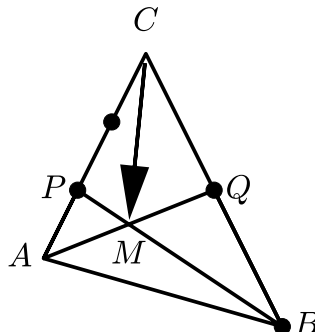


$$\vec{CM} = \text{?.??}\vec{CA} + \text{?.??}\vec{CB}:$$

check

Solution

The vector \overrightarrow{CM} is represented by two different linear combinations, corresponding to the paths $C \rightarrow A \rightarrow M$ and $C \rightarrow B \rightarrow M$ from C to M .



- $C \rightarrow A \rightarrow M$:

$$\overrightarrow{CM} = \overrightarrow{CA} + \overrightarrow{AM} = \overrightarrow{CA} + s\overrightarrow{AQ}$$

$$\overrightarrow{AQ} = \overrightarrow{AC} + \overrightarrow{CQ}, \overrightarrow{AC} = -\overrightarrow{CA}, \text{ and } \overrightarrow{CQ} = \frac{1}{2}\overrightarrow{CB} \text{ (ratio 1 : 1 on the edge } \overline{BC}) \rightsquigarrow$$

$$\overrightarrow{CM} = \overrightarrow{CA} + s(-\overrightarrow{CA} + \frac{1}{2}\overrightarrow{CB}) = (1-s)\overrightarrow{CA} + (s/2)\overrightarrow{CB} \quad (1)$$

- $C \rightarrow B \rightarrow M$:

$$\overrightarrow{CM} = \overrightarrow{CB} + t(\underbrace{\overrightarrow{BC} + \overrightarrow{CP}}_{\overline{BP}}) = (1-t)\overrightarrow{CB} + (2t/3)\overrightarrow{CA}, \quad (2)$$

the last term in view of the ratio 1 : 2 on the edge \overline{AC}

comparing the coefficients of \overrightarrow{CA} and \overrightarrow{CB} in equations (1) and (2) \rightsquigarrow

$$1-s = \frac{2}{3}t, \quad \frac{1}{2}s = 1-t$$

with the solution $s = 1/2, t = 3/4$

substituting in either of the equations (1) or (2) \rightsquigarrow

$$\overrightarrow{CM} = \frac{1}{2}\overrightarrow{CA} + \frac{1}{4}\overrightarrow{CB}$$

Chapter 2

Scalar Product

2.1 Angle Between Vectors

Determine the angle enclosed by the vectors $(-4, -2, 6)^t$ and $(-5, 1, 4)^t$.

Resources: [Scalar Product](#)

Problem Variants

■ $(1, 1, 2)^t, (2, -1, 1)^t$

???\(\pi\):

check

■ $(-4, 5, 7)^t, (2, 5, 4)^t$

???\(\pi\):

check

■ $(-7, -2, 5)^t, (4, 3, -1)^t$

???\(\pi\):

check

Solution

substituting $\vec{u} = (-4, -2, 6)^t$, $\vec{v} = (-5, 1, 4)^t$ into the formula for the cosine of the enclosed angle \rightsquigarrow

$$\cos \sphericalangle(\vec{u}, \vec{v}) = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} = \frac{20 - 2 + 24}{\sqrt{56} \sqrt{42}} = \frac{2 \cdot 3 \cdot 7}{\sqrt{(2^3 \cdot 7) \cdot (2 \cdot 3 \cdot 7)}} = \frac{\sqrt{3}}{2}$$

and $\sphericalangle(\vec{u}, \vec{v}) = \arccos(\sqrt{3}/2) = \pi/6$

2.2 Side Lengths and Angles of a Triangle

Determine the lengths of the sides and the angles of the triangle with the vertices

$$A = (1, 3, 5), \quad B = (2, 5, 6), \quad C = (1, 5, 7).$$

Resources: [Scalar Product](#), [Trigonometric Theorems](#)

Problem Variants

■ $A = (-1, 1, 2), B = (1, 3, 3), C = (4, 5, -1)$

smallest angle: $?.??\pi$:

check

■ $A = (1, 3, -2), B = (-1, 1, 2), C = (0, 5, -3)$

largest angle: $?.??\pi$:

check

■ $A = (-1, 2, -3), B = (4, 3, 1), C = (-4, 0, -4)$

largest angle: $?.??\pi$:

check

Solution

Side length

vectors corresponding to the sides of the triangle (oriented counterclockwise)

$$\vec{c} = \overrightarrow{AB} = \begin{pmatrix} 2 \\ 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$
$$\vec{a} = \overrightarrow{BC} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{b} = \overrightarrow{CA} = \begin{pmatrix} 0 \\ -2 \\ -2 \end{pmatrix}$$

↪ lengths

$$c = |\vec{c}| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}, \quad a = \sqrt{2}, \quad b = 2\sqrt{2}$$

Angles

$\vec{a} \cdot \vec{c} = -1 + 0 + 1 = 0 \implies \vec{a} \perp \vec{c} \implies \beta = \angle(\overrightarrow{BA}, \overrightarrow{BC}) = \angle(-\vec{c}, \vec{a}) = \pi/2$
($-\vec{c}$ instead of \vec{c} since for computing the angle at B both vectors should emanate from B)

applying the law of cosines, $2ab \cos \gamma = a^2 + b^2 - c^2$, with the computed side lengths \implies

$$2\sqrt{2}(2\sqrt{2}) \cos \gamma = 2 + 8 - 6,$$

i.e. $\cos \gamma = 1/2$ and $\gamma = \pi/3$

Since the angles of a triangle sum to π ,

$$\alpha = \pi - \beta - \gamma = \pi - \pi/2 - \pi/3 = \pi/6.$$

Alternative Solution

computation of γ with the law of sines:

$$c/b = \sin \gamma / \sin \beta$$

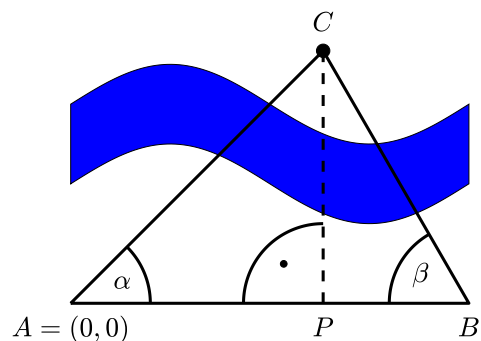
substituting the computed values \rightsquigarrow

$$\sqrt{6}/(2\sqrt{2}) = \sin \gamma / 1,$$

i.e. $\sin \gamma = \sqrt{3}/2$ and $\gamma = \pi/3$ ✓

2.3 Triangle $\Delta(A, B, C)$, determined by c , α , and β

Determine the coordinates of the point C , which is not easily accessible (e.g. on the other side of a river), from the measured distance $c = |\overline{AB}| = 100$ and the angles $\alpha = \pi/4$ and $\beta = \pi/3$.



Resources: [Trigonometric Theorems](#)

Problem Variants

■ $\alpha = \pi/6, \beta = \pi/3$

$C = (???, ???)$:

check

■ $\alpha = \pi/6, \beta = \pi/4$

$C = (???, ???)$:

check

■ $\alpha = 2\pi/3, \beta = \pi/4$

$C = (-???, ???)$:

check

Solution

$$x := c_1 = |\overline{AP}|, c - x = |\overline{BP}|, \text{ and } h := c_2 = |\overline{CP}| \rightsquigarrow$$

$$x \tan \alpha = h = (c - x) \tan \beta$$

$$\text{substituting } \alpha = \pi/4, \beta = \pi/3, \text{ and } c = 100 \rightsquigarrow$$

$$\begin{aligned} x &= \frac{c \tan \beta}{\tan \alpha + \tan \beta} = 100 \frac{\sqrt{3}}{1 + \sqrt{3}} = 100 \frac{\sqrt{3}(\sqrt{3} - 1)}{3 - 1} = \\ &= 100 \frac{3 - \sqrt{3}}{2} = 150 - 50\sqrt{3} \approx 63.3974 \end{aligned}$$

Therefore, since $h = x \underbrace{\tan \alpha}_{=1} = x$,

$$C = (x, h) = (150 - 50\sqrt{3}, 150 - 50\sqrt{3}).$$

Alternative Solution

application of the law of sines:

ratio of sides = ratio of corresponding angles

$$b : c = \sin \beta : \sin \gamma \text{ with } \gamma = \varphi + \psi = \pi - \alpha - \beta$$

$$\implies$$

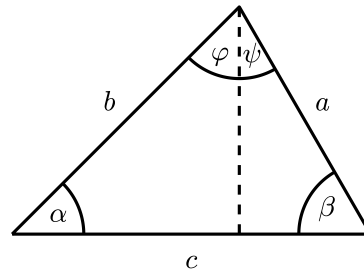
$$b = 100 \frac{\sin(\pi/3)}{\sin(5\pi/12)} = 100 \frac{\sqrt{3}/2}{\sqrt{2 + \sqrt{3}}/2} \quad (1)$$

$$h : b = \sin \alpha : \sin(\pi/2) \implies$$

$$h = b \frac{\sin(\pi/4)}{1} = b \frac{1}{\sqrt{2}} \quad (2)$$

combining (1) and (2) \rightsquigarrow

$$h = 100 \frac{\sqrt{3}}{\sqrt{2}\sqrt{2 + \sqrt{3}}} \approx 63.3974 \quad \checkmark$$



Of course, the numeric value coincides with the previous result. But why do the algebraic expressions coincide, and how can one obtain the algebraic expression for $\sin(5\pi/6)$?

Algebraic manipulations

• $\frac{3 - \sqrt{3}}{2} \stackrel{?}{=} \frac{\sqrt{3}}{\sqrt{2}\sqrt{2 + \sqrt{3}}}$
multiplying by 2 and squaring \rightsquigarrow

$$12 - 6\sqrt{3} = \frac{6}{2 + \sqrt{3}}$$

multiplying by $2 + \sqrt{3}$ \rightsquigarrow

$$24 - 12\sqrt{3} + 12\sqrt{3} - 18 = 6 \quad \checkmark$$

• $\sin(5\pi/12) \stackrel{?}{=} \sqrt{2 + \sqrt{3}}/2$
Since $5\pi/12 = (\pi/2 + \pi/3)/2$, the geometric interpretation of complex numbers and Euler's formula \implies

$$\sin(5\pi/12) = \operatorname{Im} e^{i5\pi/12} = \operatorname{Im}(z/|z|), \quad z = e^{i\pi/2} + e^{i\pi/3}$$

substituting $e^{i\pi/2} = i$ and $e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ and taking the imaginary part $\operatorname{Im} z$ of the numerator \rightsquigarrow

$$\begin{aligned} \sin(5\pi/12) &= \frac{1 + \sqrt{3}/2}{\sqrt{(1/2)^2 + (1 + \sqrt{3}/2)^2}} = \frac{(2 + \sqrt{3})/2}{\sqrt{1/4 + 1 + \sqrt{3} + 3/4}} \\ &= \frac{\sqrt{2 + \sqrt{3}}}{2} \quad \checkmark \end{aligned}$$

A beautiful argument based on the famous formula $e^{it} = \cos t + i \sin t$ of Euler!

2.4 Circumcircle of a Triangle

Determine the radius r and the midpoint M of the circumcircle for the triangle with vertices

$$A = (-2, 4), \quad B = (-1, -3), \quad C = (7, 1).$$

Resources: [Barycentric Coordinates](#), [Trigonometric Theorems](#)

Problem Variants

■ $A = (1, -1), B = (5, 1), C = (3, 5)$

$M = (?, ?), r = ?.??:$

check

■ $A = (-5, 0), B = (3, 8), C = (1, 2)$

$M = (-?, ?), r = ?.??:$

check

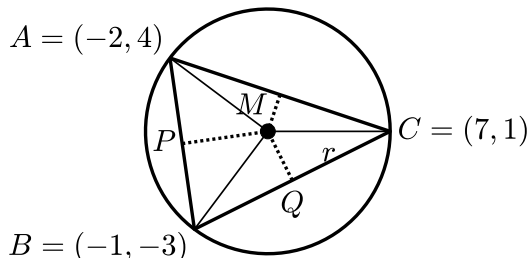
■ $A = (4, 5), B = (-4, 1), C = (4, -7)$

$M = (?, -?), r = ?.??:$

check

Solution

construction of the midpoint of the triangle by intersecting the bisectors of the edges (dotted lines)



$$g : \vec{p} + s\vec{u}, \quad h : \vec{q} + t\vec{v}$$

where

$$\vec{p} = (\vec{a} + \vec{b})/2 = (-3/2, 1/2)^t, \quad \vec{q} = (\vec{b} + \vec{c})/2 = (3, -1)^t$$

and

$$\vec{u} = (7, 1)^t \parallel \overrightarrow{PM}, \quad \vec{v} = (-4, 8)^t \parallel \overrightarrow{QM}$$

are vectors orthogonal to $\overrightarrow{AB} = (1, -7)^t$ and $\overrightarrow{BC} = (8, 4)^t$, respectively equating the parametrizations of the bisectors g and $h \rightsquigarrow$ linear system

$$\begin{pmatrix} -3/2 \\ 1/2 \end{pmatrix} + s \begin{pmatrix} 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

with the solution $s = 1/2, t = 1/4$

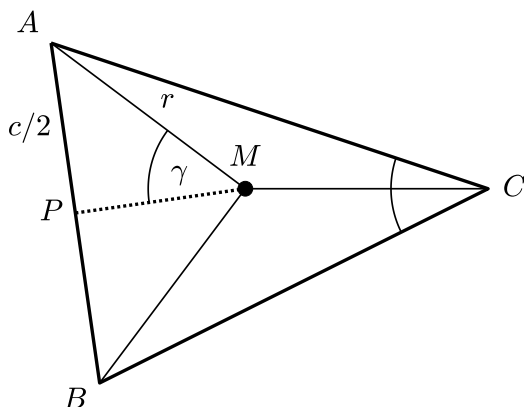
substituting in either of the parametrizations \rightsquigarrow

$$M = g \cap h = (-3/2 + 7/2, 1/2 + 1/2) = (2, 1)$$

and

$$r = |\vec{m} - \vec{a}| = |(2 - (-2), 1 - 4)^t| = \sqrt{4^2 + 3^2} = 5$$

Alternative computation of the radius



Since

$$\sphericalangle(AMP) \stackrel{*}{=} \gamma = \sphericalangle(ACB)$$

(cf. the proof of \star below), the law of sines implies

$$r : \frac{c}{2} = \sin(\pi/2) : \sin \gamma, \text{ i.e. } r = \frac{c}{2 \sin \gamma}$$

substituting the given data \rightsquigarrow

$$\begin{aligned}\cos \gamma &= \frac{\overrightarrow{CA} \cdot \overrightarrow{CB}}{|\overrightarrow{CA}| |\overrightarrow{CB}|} = \frac{(-9, 3)^t \cdot (-8, -4)^t}{|(-9, 3)^t| |(-8, -4)^t|} = \frac{60}{\sqrt{90} \sqrt{80}} = \frac{1}{\sqrt{2}} \\ c &= |\overrightarrow{AB}| = |(-1, -7)| = \sqrt{1 + 49} = 5\sqrt{2}\end{aligned}$$

substituting into the formula for the radius with $\gamma = \arccos(1/\sqrt{2}) = \pi/4$
 \implies

$$r = \frac{5\sqrt{2}}{2 \sin(\pi/4)} = \frac{5\sqrt{2}}{2(1/\sqrt{2})} = 5 \quad \checkmark$$

Proof of \star : Since $|\overline{MC}| = |\overline{MA}|$, $\sphericalangle(ACM) = \sphericalangle(CAM)$ and therefore $\sphericalangle(AMC) = \pi - 2\sphericalangle(ACM)$. Similarly, $\sphericalangle(BMC) = \pi - 2\sphericalangle(BCM)$. Therefore,

$$\begin{aligned}\underbrace{\sphericalangle(AMB)}_{2\sphericalangle(AMP)} &= 2\pi - (\pi - 2\sphericalangle(ACM)) - (\pi - 2\sphericalangle(BCM)) \\ &= 2(\sphericalangle(ACM) + \sphericalangle(BCM)) = 2\sphericalangle(ACB) = 2\gamma.\end{aligned}$$

Remark: Using the formula $\text{area } \Delta(A, B, C) = \frac{1}{2} ab \sin \gamma$ for the area of the triangle, the formula for the radius can also be written as

$$r = \frac{abc}{4 \text{ area } \Delta(A, B, C)}.$$

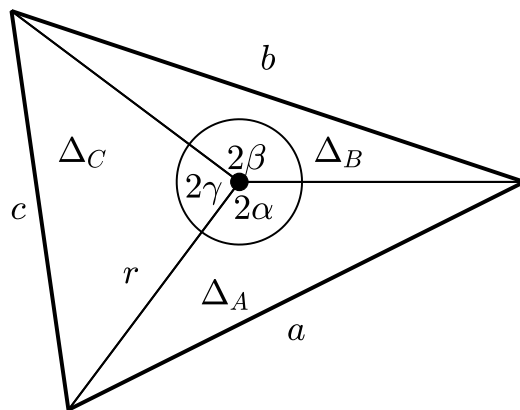
This allows to compute r without using any trigonometric functions.

Alternative computation of the midpoint

representation of the midpoint M , marked by a dot, in barycentric coordinates:

$$M = \frac{\Delta_A A + \Delta_B B + \Delta_C C}{\Delta_A + \Delta_B + \Delta_C} \quad (1)$$

where Δ_\star are the areas of the triangles determined by M



formula for the area of a triangle (cf. also the definition of the cross product)
 \implies

$$\Delta_C = \frac{1}{2} r^2 \sin(2\gamma) = r^2 \sin \gamma \cos \gamma$$

replacing $\sin \gamma$ by $c/(2r)$ and computing $\cos \gamma$ with the law of cosines \rightsquigarrow

$$\Delta_C = \frac{c}{2r} \frac{a^2 + b^2 - c^2}{2ab} = \frac{c(a^2 + b^2 - c^2)}{4abr}$$

Since the formula (1) is invariant under scaling, i.e., if Δ_* is replaced by $\Delta'_* := \lambda \Delta_*$, it is convenient to multiply by $\lambda = 4abc r$ which yields

$$\Delta'_A = a^2(b^2 + c^2 - a^2), \Delta'_B = b^2(c^2 + a^2 - b^2), \Delta'_C = c^2(a^2 + b^2 - c^2).$$

For the given data, $a^2 = |\overline{BC}|^2 = 8^2 + 4^2 = 80$, $b^2 = 90$, $c^2 = 50$, and therefore

$$\Delta'_A = 80(90+50-80) = 4800, \Delta'_B = 3600, \Delta'_C = 6000, \Delta'_A + \Delta'_B + \Delta'_C = 144000.$$

(1) with Δ_* replaced by Δ'_* \implies

$$M = \underbrace{\frac{4800}{144000}}_{1/3} (-2, 4) + \frac{1}{4} (-1, -3) + \frac{5}{12} (7, 1) = (2, 1) \quad \checkmark$$

2.5 Side Length and Angles of Isosceles Triangles

Determine the missing side lengths and angles for the data in the table.

	$a = b$	c	$\alpha = \beta$	γ
triangle 1	3	2		
triangle 2	3		$\pi/6$	
triangle 3		2		$\pi/2$

Resources: [Trigonometric Theorems](#)

Problem Variants

■ $a = b = 3, c = 4$

$\alpha = \beta = ?$, $\gamma = ?$:

check

■ $a = b = 3, \alpha = \beta = \pi/4$

$c = ?$, $\gamma = ?$:

check

■ $c = 2, \gamma = 2\pi/3$

$a = b = ?$, $\alpha = \beta = ?$:

check

Solution

Triangle 1 with $a = b = 3, c = 2$

law of cosines, $c^2 = a^2 + b^2 - 2ab \cos \gamma \implies$

$$\cos \gamma = \frac{a^2 + b^2 - c^2}{2ab} = \frac{9 + 9 - 4}{2 \cdot 3 \cdot 3} = \frac{7}{9}, \quad \gamma = \arccos(7/9) \approx 0.67$$

$\alpha + \beta + \gamma = \pi, \alpha = \beta \implies$

$$\alpha = \beta = (\pi - \gamma)/2 \approx 1.23$$

Triangle 2 with $a = b = 3, \alpha = \beta = \pi/6$

$\alpha + \beta + \gamma = \pi, \alpha = \beta \implies$

$$\gamma = \pi - 2\alpha = 2\pi/3 \approx 2.09$$

law of sines, $c : a = \sin \gamma : \sin \alpha \implies$

$$c = a \sin \gamma / \sin \alpha = 3 \cdot (\sqrt{3}/2) / (1/2) = 3\sqrt{3} \approx 5.20$$

Triangle 3 with $a = b, c = 2, \gamma = \pi/2$

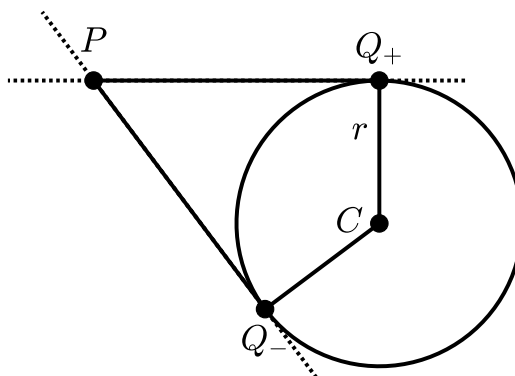
right angle at $C \implies$

$$a^2 + b^2 = c^2, \quad \text{d.h. } a = b = \sqrt{2^2/2} \approx 1.41$$

$\alpha = \beta = (\pi - \pi/2)/2 = \pi/4 \approx 0.79$

2.6 Tangents of a Circle

Determine the points Q_{\pm} of contact for the tangents of a circle with midpoint $C = (7, 4)$ and radius 5 which pass through the point $P = (-3, 9)$.



Resources: [Scalar Product](#)

Problem Variants

■ $C = (2, -3), P = (1, 4), r = 5$

largest coordinate of Q_{\pm} : ?:

check

■ $C = (1, 2), P = (0, 9), r = \sqrt{5}$

largest coordinate of Q_{\pm} : ?:

check

■ $C = (4, -2), P = (3, 1), r = \sqrt{2}$

largest coordinate of Q_{\pm} : ?:

check

Solution

At the points Q_{\pm} of tangency, the direction vector $\vec{q} - \vec{p}$ of a tangent is perpendicular to the radius vector $\vec{q} - \vec{c}$:

$$(\vec{q} - \vec{p}) \cdot (\vec{q} - \vec{c}) = 0$$

Substituting the given data \rightsquigarrow

$$0 = \begin{pmatrix} x + 3 \\ y - 9 \end{pmatrix} \cdot \begin{pmatrix} x - 7 \\ y - 4 \end{pmatrix} = x^2 - 4x - 21 + y^2 - 13y + 36$$

subtracting the equation

$$\begin{aligned} 5^2 &= |\vec{q} - \vec{c}|^2 = (x - 7)^2 + (y - 4)^2 \\ &= x^2 - 14x + 49 + y^2 - 8y + 16 \end{aligned} \quad (1)$$

\rightsquigarrow elimination of the quadratic terms:

$$-25 = 10x - 70 - 5y + 20 \iff y = 2x - 5$$

substituting into (1) \implies

$$25 = x^2 - 14x + 49 + (2x - 5)^2 - 8(2x - 5) + 16$$

simplifying and dividing by 5 \rightsquigarrow

$$0 = x^2 - 10x + 21$$

formula for the solution of quadratic equations \implies

$$x_{\pm} = 5 \pm \sqrt{5^2 - 21} = 5 \pm 2$$

and

$$y_{\pm} = 2x_{\pm} - 5 = 10 \pm 4 - 5 = 5 \pm 4$$

\rightsquigarrow two tangents with points of contact

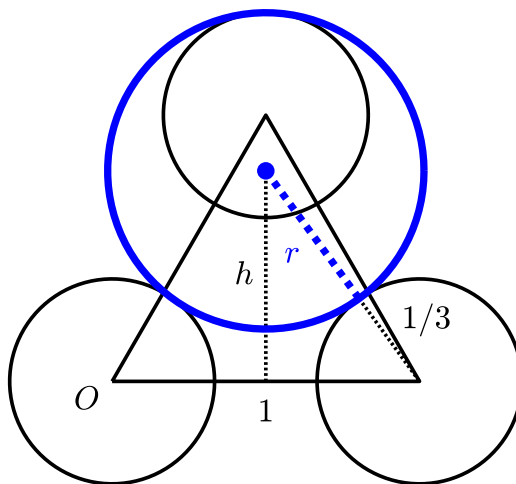
$$Q_+ = (7, 9), \quad Q_- = (3, 1)$$

Alternative solution

geometric construction of the points of tangency Q_{\pm} as intersections of the circle with the Thales circle with midpoint $(C + P)/2$ and radius $|\overline{CP}|/2$

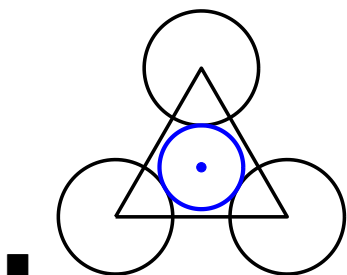
2.7 Touching Circles

The figure shows an equilateral triangle Δ with side length 1 and three black circles with radius $1/3$ and midpoints at the vertices of Δ . Determine the radius r and the y -coordinate h (signed distance to the horizontal side of Δ) of the midpoint of the blue circle which touches the three black circles as shown in the figure.



Resources: [Trigonometric Theorems](#)

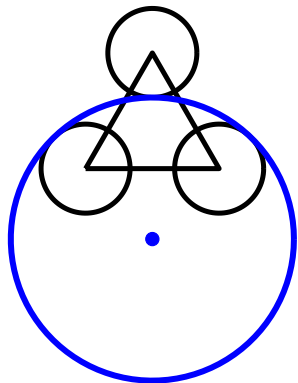
Problem Variants



■

$r = ?$, $h = ?$:

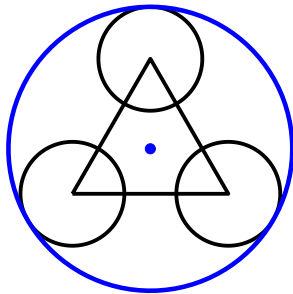
check



■

$r = ?$, $h = -?$:

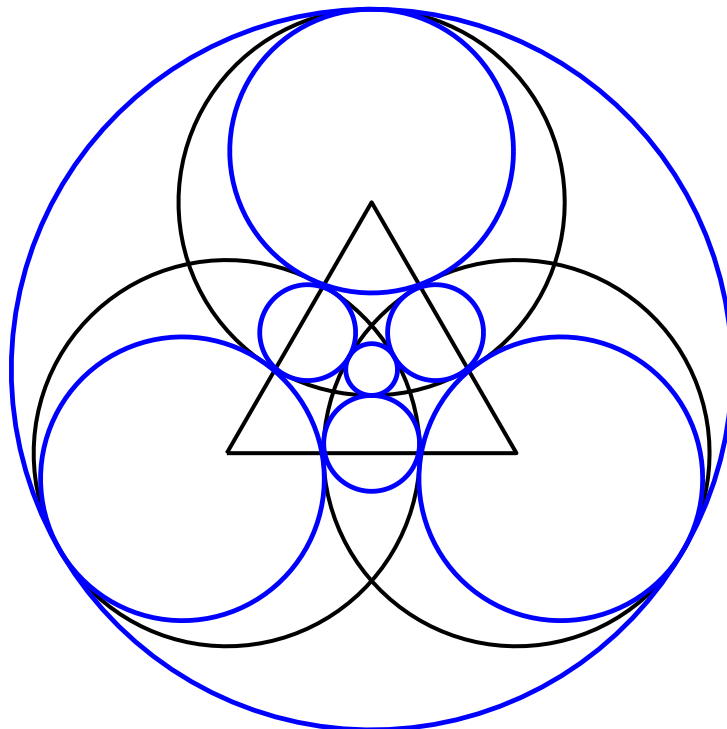
check



$r = ?$, $h = ?$:

check

Varying the radius R of the three black circles yields more variants as the following figure illustrates for $R = 2/3$.



Just as an incentive: One of the radii equals $(6 + 7\sqrt{3})/37$.

Solution

Two circles are tangent to each other if the distance of their midpoints is equal to the sum or the difference of their radii. Which case applies depends on whether the smaller circle lies outside or inside of the larger circle.

With the left vertex of the equilateral triangle Δ chosen as the origin $O = (0, 0)$, the coordinates of the other two vertices are

$$B = (1, 0), \quad C = (1/2, \sqrt{3}/2).$$

By symmetry, the midpoint M of the touching blue circle has coordinates $(1/2, h)$.

The characterization of tangency and the theorem of Pythagoras yield the following equations for the coordinate h of the midpoint and for the radius r of the blue circle:

$$\begin{aligned} |\overline{MO}| = |\overline{MB}| &= \sqrt{(1/2)^2 + h^2} = r + 1/3, \\ |\overline{MC}| &= \sqrt{3}/2 - h = r - 1/3. \end{aligned}$$

subtracting the two equations, rearranging terms, and squaring \rightsquigarrow

$$\begin{aligned} (1/2)^2 + h^2 &= (2/3 + \sqrt{3}/2 - h)^2 \\ \iff 1/4 + h^2 &= 4/9 + 3/4 + h^2 + 2\sqrt{3}/3 - (4/3)h - \sqrt{3}h \\ \iff (4/3 + \sqrt{3})h &= 17/18 + 2\sqrt{3}/3, \end{aligned}$$

i.e.

$$h = \frac{17 + 12\sqrt{3}}{24 + 18\sqrt{3}} = \dots \text{Maple}^{\text{TM}} \text{ } \dots = \frac{20}{33} + \frac{\sqrt{3}}{22} \approx 0.6847$$

and

$$r = \frac{1}{3} + \frac{\sqrt{3}}{2} - h = \frac{5\sqrt{3} - 3}{11} \approx 0.5145$$

¹Note that you can make the denominator of a fraction $a/(b+\sqrt{c})$ rational by expanding with $b - \sqrt{c}$.

2.8 Coordinates of a Vector with respect to an Orthonormal Basis

Normalize the vectors $\vec{u} = (3, 4)^t$ and $\vec{v} = (-4, 3)^t$ and determine the coordinates of $\vec{a} = (2, 1)^t$ with respect to the basis $\{\vec{u}^\circ, \vec{v}^\circ\}$.

Resources: [Scalar Product](#), [Orthogonal Basis](#), [Norm](#)

Problem Variants

■ $\vec{u} = (5, 12)^t$, $\vec{v} = (-12, 5)^t$, $\vec{a} = (2, 10)^t$

$$\vec{a} = ??\vec{u}^\circ + ?\vec{v}^\circ:$$

check

■ $\vec{u} = (8, 15)^t$, $\vec{v} = (-15, 8)^t$, $\vec{a} = (1, 4)^t$

$$\vec{a} = ?\vec{u}^\circ + ?\vec{v}^\circ:$$

check

■ $\vec{u} = (7, 24)^t$, $\vec{v} = (-24, 7)^t$, $\vec{a} = (1, 7)^t$

$$\vec{a} = ?\vec{u}^\circ + ?\vec{v}^\circ:$$

check

Solution

Normalization

$$|\vec{u}| = |(3, 4)^t| = \sqrt{9 + 16} = 5 \quad \rightsquigarrow$$

$$\vec{u}^\circ = \vec{u}/|\vec{u}| = \begin{pmatrix} 3 \\ 4 \end{pmatrix} / 5 = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}$$

$$\vec{v}^\circ = (-4, 3)^t / 5 = (-4/5, 3/5)^t$$

Coordinates

$$\vec{u}^\circ \perp \vec{v}^\circ, \quad |\vec{u}^\circ| = 1 = |\vec{v}^\circ| \quad \implies$$

$$\vec{a} = c_u \vec{u}^\circ + c_v \vec{v}^\circ, \quad c_w = \vec{a} \cdot \vec{w}^\circ \quad (w = u, v) \quad (1)$$

applying the formula for $\vec{a} = (2, 1)^t \rightsquigarrow$

$$c_u = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} = 6/5 + 4/5 = 2$$

$$c_v = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix} = -8/5 + 3/5 = -1$$

Remark: Without normalization, formula (1) and the representation of \vec{a} becomes

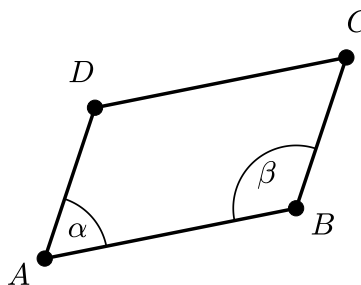
$$\vec{a} = \frac{\vec{a} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} + \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v}, \quad \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{10}{25} \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{5}{25} \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

2.9 Side Lengths, Angles, and Area of a Parallelogram

Determine, for the parallelogram with vertices

$$A = (1, 5, 3), B = (0, 1, 2), C = (2, 3, 1)$$

the fourth vertex D , the lengths of the sides, the circumference, the angles, and the area.



Resources: [Addition and Multiplication of Vectors](#), [Scalar Product](#)

Problem Variants

■ $A = (1, 2, 3), B = (2, 2, 4), D = (0, 3, 7)$

circumference: $\sqrt{???}$, area: $\sqrt{??}$:

check

■ planar parallelogram with $A = (-1, 3), B = (6, 4), |\overline{AD}| = 5, \alpha = \pi/4$

circumference: $??$, area: $??$:

check

■ planar parallelogram with $A = (-2, -3), C = (7, 6), |\overline{AB}| = 5, |\overline{BC}| = 13$

circumference: $??$, area: $??$:

check

Solution

Fourth vertex

$$\overrightarrow{AD} = \overrightarrow{BC} \text{ with } A = (1, 5, 3), B = (0, 1, 2), C = (2, 3, 1) \rightsquigarrow$$

$$\vec{d} = \vec{a} + \overrightarrow{AD} = \vec{a} + \overrightarrow{BC} = (1, 5, 3)^t + (2, 2, -1)^t = (3, 7, 2)^t$$

Side lengths

$$\overrightarrow{AB} = (0 - 1, 1 - 5, 2 - 3)^t = (-1, -4, -1)^t, \overrightarrow{BC} = (2 - 0, 3 - 1, 1 - 2)^t = (2, 2, -1)^t \implies$$

$$|\overrightarrow{AB}| = \sqrt{1 + 16 + 1} = 3\sqrt{2}, \quad |\overrightarrow{BC}| = \sqrt{4 + 4 + 1} = 3$$

$$\text{circumference: } 2(3\sqrt{2} + 3) = 6\sqrt{2} + 6 \approx 14.4853$$

Angles

law of cosines \implies

$$\cos \underbrace{\sphericalangle(D, A, B)}_{\alpha} = \frac{\overrightarrow{AD} \cdot \overrightarrow{AB}}{|\overrightarrow{AD}| \cdot |\overrightarrow{AB}|} = \frac{(2, 2, -1)^t \cdot (-1, -4, -1)^t}{3 \cdot 3\sqrt{2}} = -\frac{1}{\sqrt{2}}$$

$$\text{and } \alpha = \arccos(-1/\sqrt{2}) = 3\pi/4$$

Since the angles of a quadrilateral sum to 2π and for a parallelogram opposite angles coincide,

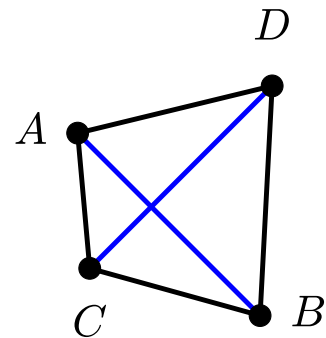
$$\beta = \sphericalangle(A, B, C) = (2\pi - 2\alpha)/2 = \pi/4$$

Area

$$\text{area } \square(A, B, C, D) = |\overrightarrow{BA}| |\overrightarrow{BC}| \sin \beta = 3\sqrt{2} \cdot 3 \cdot \frac{1}{\sqrt{2}} = 9$$

2.10 Orthogonal Edges of a Regular Tetrahedron

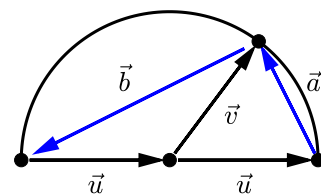
Show that two edges of a regular tetrahedron, which do not have a common vertex, are orthogonal.



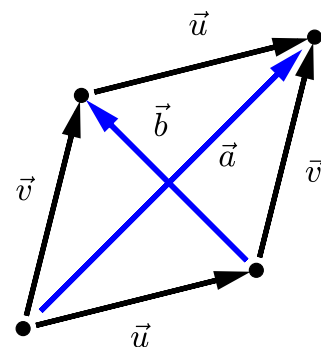
Resources: [Scalar Product](#), [Norm](#)

Problem Variants

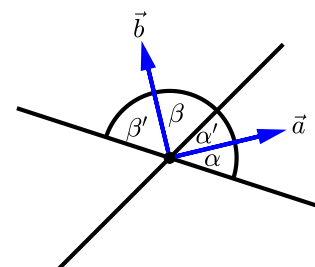
■ Prove (Theorem of Thales): $|\vec{u}| = |\vec{v}| \implies \vec{a} \perp \vec{b}$



■ Prove: $|\vec{u}| = |\vec{v}| \implies \vec{a} \perp \vec{b}$



■ Prove: $\alpha = \alpha', \beta = \beta' \implies \vec{a} \perp \vec{b}$



Solution

symmetry \rightsquigarrow consider only one pair of edges, e.g. \overline{AB} and \overline{CD}

To show

$$\overrightarrow{AB} \perp \overrightarrow{CD} \iff \overrightarrow{AB} \cdot \overrightarrow{CD} = 0$$

it is used that, for a regular tetrahedron, the lengths of all edges are equal and all angles coincide:

$$\begin{aligned} L &= |\overrightarrow{AB}| = |\overrightarrow{AC}| = |\overrightarrow{AD}| = \dots, \\ \pi/3 &= \sphericalangle(BAC) = \sphericalangle(BAD) = \dots. \end{aligned}$$

$$\vec{u} \cdot \vec{v} = \cos \sphericalangle(\vec{u}, \vec{v}) |\vec{u}| |\vec{v}| \implies$$

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{CD} &= \overrightarrow{AB} \cdot (\overrightarrow{CA} + \overrightarrow{AD}) = -\overrightarrow{AB} \cdot \overrightarrow{AC} + \overrightarrow{AB} \cdot \overrightarrow{AD} \\ &= -L^2 \underbrace{\cos(\pi/3)}_{\sphericalangle(BAC)} + L^2 \underbrace{\cos(\pi/3)}_{\sphericalangle(BAD)} = 0 \quad \checkmark \end{aligned}$$

2.11 Epsilon Tensor and Kronecker Symbol

Compute

$$p_{j,m} = \sum_{k=1}^3 \sum_{\ell=1}^3 \varepsilon_{j,k,\ell} k \varepsilon_{\ell,k,m}$$

for all indices $j, m \in \{1, 2, 3\}$.

Resources: [Epsilon Tensor](#)

Problem Variants

$$p_{j,m} = \sum_{k,\ell} \varepsilon_{j,k,\ell} \ell \delta_{\ell,m} \quad \blacksquare$$

$$\sum_{j,m} |p_{j,m}| = ??:$$

check

$$\blacksquare \quad p_{j,m} = \sum_{k,\ell} \delta_{j,k} k \ell \delta_{\ell,m}$$

$$\sum_{j,m} |p_{j,m}| = ??:$$

check

$$\blacksquare \quad p_{j,m} = \sum_{k,\ell} \varepsilon_{j,k,\ell} \delta_{j,m} \varepsilon_{k,\ell,m}$$

$$\sum_{j,m} |p_{j,m}| = ?:$$

check

Solution

$$\varepsilon_{j,k,\ell} = \begin{cases} 1 & \text{for } (j,k,\ell) = (1,2,3), (2,3,1), (3,1,2) \\ -1 & \text{for } (j,k,\ell) = (1,3,2), (2,1,3), (3,2,1) \\ 0 & \text{otherwise} \end{cases}$$

$\delta_{j,k} = 1$ for $j = k$ and $= 0$ otherwise

applying these definitions to $p_{j,m} = \sum_{k,\ell} \varepsilon_{j,k,\ell} k \varepsilon_{\ell,k,m} \rightsquigarrow$

- $p_{j,m} = 0$ for $j \neq m$, since either $\{j, k, \ell\}$ or $\{\ell, k, m\}$ contains two equal indices
- For $j = m$, only two summands with the complementary indices $\{k, \ell\} = \{1, 2, 3\} \setminus \{j\}$ are nonzero, and $\varepsilon_{j,k,\ell} = -\varepsilon_{\ell,k,j} \implies$

$$p_{1,1} = \varepsilon_{1,2,3} \cdot 2 \cdot \varepsilon_{3,2,1} + \varepsilon_{1,3,2} \cdot 3 \cdot \varepsilon_{2,3,1} = 2 - 3 = -5$$

and, similarly, $p_{2,2} = -1 - 3 = -4$, $p_{3,3} = -1 - 2 = -3$

Chapter 3

Cross Product and Triple Product

3.1 Computing Cross Products

Determine for $\vec{a} = (2, 1, 3)^t$ and $\vec{b} = (5, -1, 4)^t$

$$\vec{c} = \vec{a} \times \vec{b}, \quad \vec{d} = (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}), \quad \gamma = \sphericalangle(\vec{a}, \vec{b}).$$

Resources: [Cross Product](#)

Problem Variants

■ $\vec{a} = (1, -4, 3)^t, \vec{b} = (-1, 5, -4)^t, \vec{d} = (2\vec{a} + \vec{b}) \times (\vec{a} + 2\vec{b}), s = \vec{c} \cdot (\vec{a} + \vec{b} + \vec{c})$

$\max_k c_k = ?, \min_k d_k = ?, s = ?:$

check

■ $\vec{a} = (3, 2, 1)^t, \vec{b} = (4, 3, 1)^t, \vec{d} = \vec{a} \times \vec{c}, s = |\vec{b} \times \vec{c}|^2$

$\max_k c_k = ?, \min_k d_k = -?, s = ??:$

check

■ $\vec{a} = (1, -1, 2)^t, \vec{b} = (4, 3, 1)^t, \vec{d} = (\vec{a} - \vec{b}) \times \vec{c}, s = (\vec{a} + \vec{c}) \times (\vec{b} + \vec{c})$

$\max_k c_k = ?, \min_k d_k = -?, s = ??:$

check

Solution

Cross product

$$\vec{c} = \vec{a} \times \vec{b} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 5 \\ -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \cdot 4 - 3 \cdot (-1) \\ 3 \cdot 5 - 2 \cdot 4 \\ 2 \cdot (-1) - 1 \cdot 5 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \\ -7 \end{pmatrix}$$

Cross product of sums

linearity and antisymmetry of the vector product ($\vec{a} \times \vec{a} = \vec{b} \times \vec{b} = \vec{o}$, $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$) \implies

$$\begin{aligned} \vec{d} &= (\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) \\ &= \vec{a} \times \vec{a} - \vec{a} \times \vec{b} + \vec{b} \times \vec{a} - \vec{b} \times \vec{b} = -2\vec{a} \times \vec{b} = (-14, -14, 14)^t \end{aligned}$$

Angle

applying the formula $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \sphericalangle(\vec{a}, \vec{b}) \rightsquigarrow$

$$\sin \sphericalangle(\vec{a}, \vec{b}) = \frac{|\vec{c}|}{|\vec{a}| |\vec{b}|} = \frac{7\sqrt{3}}{\sqrt{14} \sqrt{42}} = \frac{1}{2}$$

Alternative computation of the angle

$$\cos \sphericalangle(\vec{a}, \vec{b}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}| |\vec{b}|} = \frac{21}{\sqrt{14} \sqrt{42}} = \frac{\sqrt{3}}{2}$$

$$\implies \sin \sphericalangle(\vec{a}, \vec{b}) = \sqrt{1 - \cos^2 \sphericalangle(\vec{a}, \vec{b})} = \sqrt{1 - 3/4} = 1/2 \quad \checkmark$$

3.2 Coordinates with Respect to an Orthogonal Basis

Augment $\vec{u} = (0, 1, 2)^t$ and $\vec{v} = (3, -2, 1)^t$ with $\vec{w} = \vec{u} \times \vec{v}$ to an orthogonal basis and compute the coordinates of $\vec{a} = (1, -8, 9)^t$.

Resources: [Cross Product](#), [Orthogonal Basis](#)

Problem Variants

■ $\vec{u} = (1, -4, -3)^t$, $\vec{v} = (-1, -1, 1)^t$, $\vec{a} = (8, 9, 8)^t$

$$\vec{a} = -? \vec{u} - ? \vec{v} - ? \vec{w}:$$

check

■ $\vec{u} = (1, -1, -1)^t$, $\vec{v} = (-2, -3, 1)^t$, $\vec{a} = (5, -7, 3)^t$

$$\vec{a} = ? \vec{u} + ? \vec{v} - ? \vec{w}:$$

check

■ $\vec{u} = (1, -1, 3)^t$, $\vec{v} = (2, -1, -1)^t$, $\vec{a} = (3, 9, -9)^t$

$$\vec{a} = -? \vec{u} + ? \vec{v} + ? \vec{w}:$$

check

Solution

Completing the basis

augment $\vec{u} = (0, 1, 2)^t$ and $\vec{v} = (3, -2, 1)^t$ by a third basis vector

$$\vec{w} = \vec{u} \times \vec{v} = \begin{pmatrix} 1 \cdot 1 - 2 \cdot (-2) \\ 2 \cdot 3 - 0 \cdot 1 \\ 0 \cdot (-2) - 1 \cdot 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix}$$

Coordinates

application of the formula

$$\vec{a} = \frac{\vec{a} \cdot \vec{u}}{|\vec{u}|^2} \vec{u} + \frac{\vec{a} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} + \frac{\vec{a} \cdot \vec{w}}{|\vec{w}|^2} \vec{w} \quad (1)$$

for the representation of the vector $\vec{a} = (1, -8, 9)^t$ with respect to the orthogonal basis $\{\vec{u}, \vec{v}, \vec{w}\}$:

- coefficient of $\vec{u} = (0, 1, 2)^t$

$$\frac{(1, -8, 9)^t \cdot \vec{u}}{\vec{u} \cdot \vec{u}} = \frac{0 - 8 + 18}{0 + 1 + 4} = 2$$

- coefficients of \vec{v} and \vec{w}

$$\frac{(1, -8, 9)^t \cdot (3, -2, 1)^t}{(3, -2, 1)^t \cdot (3, -2, 1)^t} = 2, \quad \frac{(1, -8, 9)^t \cdot (5, 6, -3)^t}{(5, 6, -3)^t \cdot (5, 6, -3)^t} = -1$$

substituting into (1) \rightsquigarrow

$$\begin{pmatrix} 1 \\ -8 \\ 9 \end{pmatrix} = 2 \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix}$$

3.3 Simplification Using Grassmann's Identity

Compute

$$(\vec{u} \times \vec{v}) \times (\vec{v} \times \vec{w})$$

for the vectors $\vec{u} = (1, -1, 0)^t$, $\vec{v} = (0, -1, 1)^t$, and $\vec{w} = (1, 1, 1)^t$.

Resources: [Cross Product](#)

Problem Variants

■ $\vec{u} = (-2, -7, 1)^t$, $\vec{v} = (-6, 7, 9)^t$, $\vec{w} = (-3, 1, 4)^t$

$$(\vec{u} \times \vec{w}) \times (\vec{v} \times \vec{w}) = (?, -?, -?)^t:$$

check

■ $\vec{u} = (-6, 5, -6)^t$, $\vec{v} = (-1, 1, -1)^t$, $\vec{w} = (4, 5, 7)^t$

$$((\vec{u} \times \vec{v}) \times \vec{v}) \times \vec{w} = (?, -?, -?)^t:$$

check

■ $\vec{u} = (4, 7, 1)^t$, $\vec{v} = (3, 5, 2)^t$, $\vec{w} = (-5, 4, -2)^t$

$$\vec{v} \times ((\vec{u} \times \vec{v}) \times \vec{w}) = (?, -?, -?)^t:$$

check

Solution

applying Grassmann's identity

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c}) \vec{b} - (\vec{b} \cdot \vec{c}) \vec{a}$$

with $\vec{a} = \vec{u}$, $\vec{b} = \vec{v}$, and $\vec{c} = \vec{v} \times \vec{w} \implies$

$$\begin{aligned} \vec{d} &= (\vec{u} \times \vec{v}) \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot (\vec{v} \times \vec{w})) \vec{v} - (\vec{v} \cdot \underbrace{(\vec{v} \times \vec{w})}_{\perp \vec{v}}) \vec{u} \\ &= (\vec{u} \cdot (\vec{v} \times \vec{w})) \vec{v} - 0 \vec{u} \end{aligned}$$

substituting $\vec{u} = (1, -1, 0)^t$, $\vec{v} = (0, -1, 1)^t$, $\vec{w} = (1, 1, 1)^t \rightsquigarrow$

$$\vec{v} \times \vec{w} = \begin{pmatrix} 1+1 \\ 0-1 \\ -1-0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

and

$$\vec{d} = \left(\left(\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right) \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right) = -3 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix}$$

3.4 Simplification Using Lagrange's Identity

Determine

$$(\vec{u} \times \vec{v}) \cdot ((\vec{w} \times \vec{v}) \times \vec{u})$$

for the vectors $\vec{u} = (9, 4, 1)^t$, $\vec{v} = (2, 5, 7)^t$, and $\vec{w} = (3, 6, 8)^t$.

Resources: [Cross Product](#)

Problem Variants

■ $\vec{u} = (4, 1, 2)^t$, $\vec{v} = (9, 3, 5)^t$, $\vec{w} = (8, 6, 7)^t$

$$(\vec{u} \times \vec{v}) \cdot (\vec{u} \times (\vec{w} \times \vec{u})) = ??:$$

check

■ $\vec{u} = (3, 7, 6)^t$, $\vec{v} = (1, 5, 9)^t$, $\vec{w} = (2, 8, 4)^t$

$$(\vec{u} \times (\vec{v} + \vec{w})) \cdot (\vec{v} \times (\vec{w} - \vec{u})) = ?:$$

check

■ $\vec{u} = (3, 4, 5)^t$, $\vec{v} = (2, 1, 8)^t$, $\vec{w} = (6, 9, 7)^t$

$$(\vec{u} \times \vec{v}) \cdot (\vec{v} \times (\vec{w} \times \vec{u})) = ??:$$

check

Solution

Lagrange's identity

$$(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$$

with $\vec{a} = \vec{u}$, $\vec{b} = \vec{v}$, $\vec{c} = \vec{w} \times \vec{v}$, and $\vec{d} = \vec{u} \rightsquigarrow$

$$\begin{aligned} s &= (\vec{u} \times \vec{v}) \cdot ((\vec{w} \times \vec{v}) \times \vec{u}) \\ &= (\vec{u} \cdot (\vec{w} \times \vec{v}))(\vec{v} \cdot \vec{u}) - (\vec{u} \cdot \vec{u})(\vec{v} \cdot \underbrace{(\vec{w} \times \vec{v})}_{\perp \vec{v}}) \\ &= (\vec{u} \cdot (\vec{w} \times \vec{v}))(\vec{v} \cdot \vec{u}) \end{aligned}$$

substituting $\vec{u} = (9, 4, 1)^t$, $\vec{v} = (2, 5, 7)^t$, and $\vec{w} = (3, 6, 8)^t \rightsquigarrow$

$$\vec{w} \times \vec{v} = \begin{pmatrix} 6 \cdot 7 - 8 \cdot 5 \\ 8 \cdot 2 - 3 \cdot 7 \\ 3 \cdot 5 - 6 \cdot 2 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix}$$

and

$$\begin{aligned} s &= \left(\begin{pmatrix} 9 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 3 \end{pmatrix} \right) \cdot \left(\begin{pmatrix} 2 \\ 5 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} 9 \\ 4 \\ 1 \end{pmatrix} \right) \\ &= (18 - 20 + 3) \cdot (18 + 20 + 7) = 45 \end{aligned}$$

3.5 Inscribed Circle of a Triangle

Determine the midpoint M and the radius r of the inscribed circle for the triangle with vertices

$$A = (-12, -3), \quad B = (13, -3), \quad C = (4, 9).$$

Resources: [Barycentric Coordinates](#)

Problem Variants

■ $A = (0, 0), B = (4, 0), C = (0, 3)$

$M = (?, ?), r = ?:$

check

■ $A = (1, 2), B = (5, 5), C = (4, 6)$

$M = (?.??, ?.??), r = ?.??:$

check

■ $A = (3, 2), B = (5, 3), C = (2, 4)$

$M = (?.??, ?.??), r = ?.??:$

check

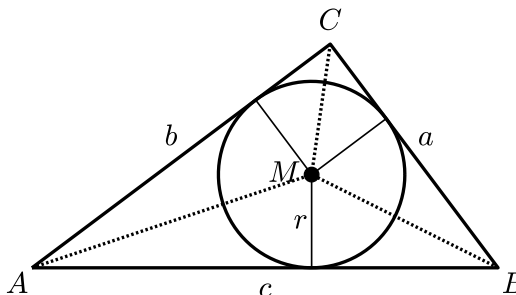
Solution

Midpoint

representation of the midpoint M in barycentric coordinates:

$$M = \frac{\Delta_A A + \Delta_B B + \Delta_C C}{\Delta_A + \Delta_B + \Delta_C}$$

where Δ_A , Δ_B , Δ_C are the areas of the triangles $\Delta(B, C, M)$, $\Delta(C, A, M)$, $\Delta(A, B, M)$, respectively



$$\Delta_A = ar/2, \Delta_B = br/2, \Delta_C = cr/2 \quad \implies$$

$$M = \frac{aA + bB + cC}{a + b + c}$$

where the common factor $r/2$ in the numerator and denominator has been cancelled

substituting the given data $A = (-12, -3)$, $B = (13, -3)$, $C = (4, 9) \rightsquigarrow$

$$a = |\overline{BC}| = |(4 - 13, 9 + 3)| = \sqrt{9^2 + 12^2} = 15, \quad b = 20, \quad c = 25$$

and

$$M = \frac{15(-12, -3) + 20(13, -3) + 25(4, 9)}{15 + 20 + 25} = \frac{(180, 120)}{60} = (3, 2)$$

Radius

adding the areas of the triangles $\Delta(A, B, M)$, $\Delta(B, C, M)$, $\Delta(C, A, M) \implies$

$$\text{area } \Delta(A, B, C) = \frac{ar + br + cr}{2} \iff r = \frac{2 \text{ area } \Delta(A, B, C)}{a + b + c}$$

substituting the given data \rightsquigarrow

$$\begin{aligned} 2 \text{ area } \Delta(A, B, C) &= \left| \det \begin{pmatrix} -12 & -3 & 1 \\ 13 & -3 & 1 \\ 4 & 9 & 1 \end{pmatrix} \right| \\ &= \underset{\text{Sarrus}}{|36 - 12 + 117 + 108 + 39 + 12|} = 300 \end{aligned}$$

and

$$r = \frac{300}{15 + 20 + 25} = 5$$

3.6 Computing Triple Products

Compute

$$[\vec{a} + 2\vec{b}, 3\vec{b} + 4\vec{c}, 5\vec{b} + 6\vec{c}]$$

for the vectors $\vec{a} = (5, 4, 8)^t$, $\vec{b} = (1, 7, 3)^t$, and $\vec{c} = (6, 2, 9)^t$.

Resources: [Triple Product](#)

Problem Variants

■ $\vec{a} = (2, 7, 1)^t$, $\vec{b} = (9, 6, 5)^t$, $\vec{c} = (3, 8, 4)^t$

$$[\vec{a} + 2\vec{b}, 4\vec{c} - 2\vec{b}, 4\vec{c} - \vec{a}] = \text{?????}:$$

check

■ $\vec{a} = (0, 1, 1)^t$, $\vec{b} = (1, 0, 1)^t$, $\vec{c} = (1, 1, 0)^t$

$$[\vec{a}, \vec{b} + \vec{c}, \vec{a} - \vec{b} + \vec{c}] = ?:$$

check

■ $\vec{a} = (-1, 2, 2)^t$, $\vec{b} = (2, -1, 2)^t$, $\vec{c} = (2, 2, -1)^t$

$$[\vec{a} - 2\vec{b} + 3\vec{c}, 4\vec{b} - 5\vec{c}, \vec{c}] = \text{?????}:$$

check

Solution

Expanding the triple product

multilinearity \rightsquigarrow 8 triple products, 6 of which are zero because arguments are parallel (e.g. $[\vec{a}, 3\vec{b}, 5\vec{b}] = 0$):

$$\begin{aligned}s &= [\vec{a} + 2\vec{b}, 3\vec{b} + 4\vec{c}, 5\vec{b} + 6\vec{c}] = [\vec{a}, 3\vec{b}, 6\vec{c}] + [\vec{a}, 4\vec{c}, 5\vec{b}] \\ &= 3 \cdot 6[\vec{a}, \vec{b}, \vec{c}] + 4 \cdot 5[\vec{a}, \vec{c}, \vec{b}] = (18 - 20) [\vec{a}, \vec{b}, \vec{c}];\end{aligned}$$

the last equality because the triple product changes sign if arguments are permuted ($[\vec{a}, \vec{b}, \vec{c}] = -[\vec{a}, \vec{c}, \vec{b}]$)

Computation of the triple product

definition of the triple product and substitution of $\vec{a} = (5, 4, 8)^t$, $\vec{b} = (1, 7, 3)^t$, $\vec{c} = (6, 2, 9)^t \rightsquigarrow$

$$\begin{aligned}[\vec{a}, \vec{b}, \vec{c}] &= \vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 7 \\ 3 \end{pmatrix} \times \begin{pmatrix} 6 \\ 2 \\ 9 \end{pmatrix} \right) \\ &= \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 7 \cdot 9 - 3 \cdot 2 \\ 3 \cdot 6 - 1 \cdot 9 \\ 1 \cdot 2 - 7 \cdot 6 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 8 \end{pmatrix} \cdot \begin{pmatrix} 57 \\ 9 \\ -40 \end{pmatrix} \\ &= 285 + 36 - 320 = 1\end{aligned}$$

and $s = -2 [\vec{a}, \vec{b}, \vec{c}] = -2$

3.7 Equations with Scalar, Cross, and Triple Products

Determine the solution \vec{v} of the equations

a) $\vec{a} \times \vec{v} = \vec{a} \times \vec{b}, \quad \vec{a} \cdot \vec{v} = -4$

b) $[\vec{a}, \vec{b}, \vec{v}] = 6, \quad (\vec{a} \times \vec{b}) \times \vec{v} = \vec{o}$

for $\vec{a} = (0, 1, 2)^t$ and $\vec{b} = (2, 1, 0)^t$, and give a geometric interpretation.

Resources: [Scalar Product](#), [Cross Product](#), [Triple Product](#)

Problem Variants

■ $\vec{a} \times \vec{v} = (5, 4, -2)^t, \quad \vec{a} \cdot \vec{v} = 5$

$\vec{v} = (?, -?, ?)^t:$

check

■ $(\vec{a} \times \vec{v}) \times \vec{b} = \vec{o}, \quad \vec{v} \cdot \vec{b} = 3$

$\vec{v} = (?, ?, ?)^t:$

check

■ $(\vec{a} + \vec{v}) \times (\vec{b} + \vec{a}) = \vec{o}, \quad [a, b, v] = 0$

$\vec{v} = (?.??. -?, -?.??)^t:$

check

Solution

a) $\vec{a} \times \vec{v} = \vec{a} \times \vec{b}$, $\vec{a} \cdot \vec{v} = -4$, $\vec{a} = (0, 1, 2)^t$, $\vec{b} = (2, 1, 0)^t$

$\vec{a} \times (\vec{v} - \vec{b}) = \vec{0} \iff \vec{a} \parallel (\vec{v} - \vec{b})$, i.e. V lies on a straight line:

$$\vec{v} = \vec{b} + t\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

substituting into the second equation, which describes a plane \rightsquigarrow

$$-4 = \vec{a} \cdot \vec{b} + t\vec{a} \cdot \vec{a} = 1 + 5t,$$

i.e. $t = -1$ and $\vec{v} = (2, 1, 0)^t + (-1)(0, 1, 2)^t = (2, 0, -2)^t$

b) $[\vec{a}, \vec{b}, \vec{v}] = 6$, $(\vec{a} \times \vec{b}) \times \vec{v} = \vec{0}$, $\vec{a} = (0, 1, 2)^t$, $\vec{b} = (2, 1, 0)^t$

second equation \iff

$$\vec{v} \parallel \vec{a} \times \vec{b} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix}$$

$\implies \vec{v} = t(-2, 4, -2)^t$, i.e. \vec{v} lies on a straight line passing through the origin

substituting into the first equation, which represents a plane \rightsquigarrow

$$\begin{aligned} 6 &= [\vec{a}, \vec{b}, t(-2, 4, -2)^t] = t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \left(\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \times \begin{pmatrix} -2 \\ 4 \\ -2 \end{pmatrix} \right) \\ &= t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 4 \\ 10 \end{pmatrix} = 24t, \end{aligned}$$

i.e. $t = 1/4$ and $\vec{v} = (-1/2, 1, -1/2)^t$

3.8 Volume and Surface Area of a Parallelepiped

Determine the volume and the surface area of the parallelepiped spanned by the vectors $\vec{u} = (1, 0, 0)^t$, $\vec{v} = (1, 2, 0)^t$, and $\vec{w} = (1, 3, 4)^t$.

Resources: [Triple Product](#)

Problem Variants

■ $\vec{u} = (1, 0, 0)^t$, $\vec{v} = (0, 1, 0)^t$, $\vec{w} = (0, 3, 4)^t$

volume: ?, area: ??:

check

■ $\vec{u} = (0, 1, 1)^t$, $\vec{v} = (1, 0, 1)^t$, $\vec{w} = (1, 1, 0)^t$

volume: ?, area: ??.??:

check

■ $\vec{u} = (-1, 2, 2)^t$, $\vec{v} = (0, 1, 0)^t$, $\vec{w} = (2, 2, -1)^t$

volume: ?, area: ??.??:

check

Solution

Volume

absolut value of the triple product:

$$\begin{aligned} V &= \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right) \right| \\ &= \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2 \cdot 4 - 0 \cdot 3 \\ ? \\ ? \end{pmatrix} \right| = 8 + 0 + 0 = 8 \end{aligned}$$

Surface area

sum of the areas of the three pairs (factor 2) of bounding parallelograms, the area of which is the absolute value of the cross product of the spanning vectors

$$\begin{aligned} A &= 2 \left| \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right| + 2 \left| \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \right| + 2 \left| \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right| \\ &= 2 \left| \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right| + 2 \left| \begin{pmatrix} 8 \\ -4 \\ 1 \end{pmatrix} \right| + 2 \left| \begin{pmatrix} 0 \\ 4 \\ -3 \end{pmatrix} \right| = 4 + 18 + 10 = 32 \end{aligned}$$

3.9 Volume and Surface Area of a Tetrahedron

Determine the volume and the surface area of the tetrahedron with vertices

$$(1, 1, 1), \quad (3, 3, 2), \quad (2, 3, 3), \quad (3, 2, 3).$$

Resources: [Triple Product](#)

Problem Variants

■ $(1, 2, 3), (5, 2, 3), (1, 3, 3), (1, 2, 8)$

volume: ???, area: ??:

check

■ $(1, -1, -1), (-1, 1, -1), (-1, -1, 1), (1, 1, 1)$

volume: ???, area: ????:

check

■ $(0, -2, 1), (0, -1, 2), (1, -2, 2), (1, -1, 1)$

volume: ???, area: ???:

check

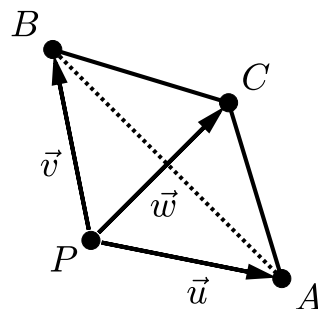
Solution

Spanning vectors

choosing one of the vertices, e.g. $P = (1, 1, 1)$, and forming the differences to the other vertices $A = (3, 3, 2)$, $B = (2, 3, 3)$, and $C = (3, 2, 3)$ \rightsquigarrow

$$\vec{u} = \overrightarrow{PA} = \begin{pmatrix} 3 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix},$$

$$\vec{v} = \overrightarrow{PB} = (1, 2, 2)^t, \quad \vec{w} = \overrightarrow{PC} = (2, 1, 2)^t$$



Volume

$V = |[\vec{u}, \vec{v}, \vec{w}]|/6 = |\vec{u} \cdot (\vec{v} \times \vec{w})|/6$, i.e.

$$V = \frac{1}{6} \left| \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right) \right| = \frac{1}{6} \left| \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 2 \\ -3 \end{pmatrix} \right| = \frac{5}{6}$$

Area

sum of the areas of the three congruent triangles (spanned by the vectors \vec{u} , \vec{v} , and \vec{w} , which have equal length and form equal angles) and the base triangle, which is spanned by the vectors

$$\vec{a} = \overrightarrow{AB} = \vec{v} - \vec{u} = (-1, 0, 1)^t, \quad \vec{b} = \overrightarrow{AC} = \vec{w} - \vec{v} = (1, -1, 0)^t$$

applying the formula

$$\text{area } \Delta(P, Q, R) = |\overrightarrow{PQ} \times \overrightarrow{PR}|/2,$$

the surface area of the tetrahedron equals

$$\begin{aligned} A &= 3|\vec{u} \times \vec{v}|/2 + |\vec{a} \times \vec{b}|/2 \\ &= \frac{3}{2} \left| \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right| + \frac{1}{2} \left| \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right| \\ &= \frac{3}{2} \left| \begin{pmatrix} 2 \\ -3 \\ 2 \end{pmatrix} \right| + \frac{1}{2} \left| \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right| = \frac{3}{2}\sqrt{17} + \frac{1}{2}\sqrt{3} \approx 7.0507 \end{aligned}$$

3.10 Volumina Resulting from an Intersection of a Tetrahedron With a Plane

The plane

$$E : f(x, y, z) = 2x - 5y + 4z = 0$$

intersects the tetrahedron T with vertices $O = (0, 0, 0)$, $A = (3, -2, 0)$, $B = (0, 4, 3)$, and $C = (-2, 0, 3)$. Compute the volumina of the resulting two parts T_1 and T_2 of T ¹.

Resources: [Triple Product](#)

Problem Variants

■ vertices: $(3, 3, -3)$, $(0, 3, -4)$, $(-1, -1, 1)$, $(5, 5, 1)$, $f(x, y, z) = -2x + 2y + 2z + 2$

smaller volume: ?:

check

■ vertices: $(-2, -4, 0)$, $(-2, -5, -5)$, $(2, -4, 4)$, $(2, 0, 0)$, $f(x, y, z) = 3x - y - z - 4$

larger volume: ?:

check

■ vertices: $(-3, -5, -5)$, $(2, 0, 0)$, $(-1, 3, -1)$, $(4, 2, -5)$, $f(x, y, z) = -4x - 2y + 3z + 5$

smaller volume: ?:

check

¹Note that one of the two parts T_k is a tetrahedron since the plane E contains one of the vertices of the tetrahedron (as is the case for the problem variants). In general, this is not necessarily the case, leading to a slightly more difficult problem.

Solution

Edge intersections

\exists intersection of E with \overline{PQ} \iff different sign of $f(P)$ and $f(Q)$

$$f(\underbrace{(3, -2, 0)}_A) = 2 \cdot 3 - 5 \cdot (-2) + 4 \cdot 0 = 16, \quad f(B) = -8, \quad f(C) = 8$$

$\implies \exists$ intersections $S \in \overline{AB}$ and $R \in \overline{BC}$

substituting the parametrization

$$\vec{s} = \vec{a} + t(\vec{b} - \vec{a}) = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 - 3 \\ 4 - (-2) \\ 3 - 0 \end{pmatrix}$$

of the edge \overline{AB} into the equation of the plane $E \implies$

$$0 = 2(3 - 3t) - 5(-2 + 6t) + 4(3t) = 16 - 24t, \quad \text{i.e. } t = 2/3, \quad S = (1, 2, 2)$$

analogous computation \rightsquigarrow intersection $R = (-1, 2, 3) \in \overline{BC}$

Volumina of the two parts T_k of T

T_k : polyhedron with vertices P_j on one side of E (equal sign of $f(P_j)$ including 0)

$\implies T_1$: tetrahedron with vertices O, B, S, R ($f \leq 0$)

computation of the volume with the triple product $[\vec{b}, \vec{s}, \vec{r}] = \vec{b} \cdot (\vec{s} \times \vec{r})$ of the spanning vectors $\overrightarrow{OB}, \overrightarrow{OS}, \overrightarrow{OR}$

$$\text{vol } T_1 = \frac{1}{6} \left| \left[\begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} \right] \right| = \frac{1}{6} \left| \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -5 \\ 4 \end{pmatrix} \right| = \frac{4}{3}$$

difference to $\text{vol } T = |[\vec{a}, \vec{b}, \vec{c}]|/6 \rightsquigarrow$

$$\text{vol } T_2 = \text{vol } T - \frac{4}{3} = \frac{1}{6} \left| \left[\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 3 \end{pmatrix} \right] \right| - \frac{4}{3} = 8 - \frac{4}{3} = \frac{20}{3}$$

3.11 Basis Coefficients of a Vector

Represent $\vec{a} = (5, 9, 7)^t$ as linear combination of the vectors

$$\vec{u} = (2, 3, 4)^t, \quad \vec{v} = (1, 0, 1)^t, \quad \vec{w} = (4, 3, 2)^t.$$

Resources: [Basis Coefficients](#)

Problem Variants

■ $\vec{a} = (-5, 4, -1)^t, \vec{u} = (-7, 9, -2)^t, \vec{v} = (2, -9, -3)^t, \vec{w} = (7, -8, 3)^t$

$$\vec{a} = ?\vec{u} + ?\vec{v} + ?\vec{w}:$$

check

■ $\vec{a} = (7, 6, 5)^t, \vec{u} = (4, -6, -2)^t, \vec{v} = (-5, 9, 1)^t, \vec{w} = (3, 0, -1)^t$

$$\vec{a} = -?\vec{u} - ?\vec{v} + ?\vec{w}:$$

check

■ $\vec{a} = (-1, 4, -2)^t, \vec{u} = (6, 0, -5)^t, \vec{v} = (2, 1, -6)^t, \vec{w} = (9, -4, 5)^t$

$$\vec{a} = ?\vec{u} - ?\vec{v} - ?\vec{w}:$$

check

Solution

computation of the coefficients using the triple product:

$$\vec{a} = r\vec{u} + s\vec{v} + t\vec{w} \quad \Longrightarrow \quad r = \frac{[\vec{a}, \vec{v}, \vec{w}]}{[\vec{u}, \vec{v}, \vec{w}]}, \quad s = \frac{[\vec{u}, \vec{a}, \vec{w}]}{[\vec{u}, \vec{v}, \vec{w}]}, \quad t = \frac{[\vec{u}, \vec{v}, \vec{a}]}{[\vec{u}, \vec{v}, \vec{w}]},$$

substituting $\vec{a} = (5, 9, 7)^t$, $\vec{u} = (2, 3, 4)^t$, $\vec{v} = (1, 0, 1)^t$, $\vec{w} = (4, 3, 2)^t \rightsquigarrow$

$$\begin{aligned} r &= \begin{pmatrix} 5 \\ 9 \\ 7 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \right) / \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \right) \\ &= \begin{pmatrix} 5 \\ 9 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} / \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix} = \frac{24}{12} = 2 \end{aligned}$$

and

$$\begin{aligned} s &= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \left(\begin{pmatrix} 5 \\ 9 \\ 7 \end{pmatrix} \times \begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} \right) / 12 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 18 \\ -21 \end{pmatrix} / 12 \\ &= -3 \\ t &= \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 5 \\ 9 \\ 7 \end{pmatrix} \right) / 12 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -9 \\ -2 \\ 9 \end{pmatrix} / 12 \\ &= 1 \end{aligned}$$

Chapter 4

Straight Lines and Planes

4.1 Intersection of Two Straight Lines

Determine the intersection X of the straight lines

$$g: \begin{pmatrix} -1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \quad h: \begin{pmatrix} 5 \\ -4 \end{pmatrix} + t \begin{pmatrix} -4 \\ 3 \end{pmatrix}.$$

Resources: [Straight Line](#)

Problem Variants

■ $g: \begin{pmatrix} 0 \\ -4 \end{pmatrix} + s \begin{pmatrix} -1 \\ 8 \end{pmatrix}, \quad h: \begin{pmatrix} 9 \\ 2 \end{pmatrix} + t \begin{pmatrix} -5 \\ 1 \end{pmatrix}$

$X = (-?, ?)$:

check

■ $g: \begin{pmatrix} 6 \\ 3 \end{pmatrix} + s \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad h: \begin{pmatrix} -8 \\ 1 \end{pmatrix} + t \begin{pmatrix} -7 \\ -3 \end{pmatrix}$

$X = (?, ?)$:

check

■ $g: \begin{pmatrix} -3 \\ -4 \end{pmatrix} + s \begin{pmatrix} 7 \\ 9 \end{pmatrix}, \quad h: \begin{pmatrix} 4 \\ -5 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

$X = (?, ?)$:

check

Solution

Equating the parametrizations

$$g : \begin{pmatrix} -1 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \text{and} \quad h : \begin{pmatrix} 5 \\ -4 \end{pmatrix} + t \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

\rightsquigarrow linear system for the line parameters s and t :

$$-1 + 2s = p_1 = 5 - 4t, \quad 1 - s = p_2 = -4 + 3t$$

with the solution $s = -1, t = 2$

substituting into either of the parametrizations, e.g. $s = -1$ into the parametrization of g \rightsquigarrow

$$\vec{x} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$$

4.2 Distance of a Point from a Straight Line

Determine the distance d of the point $Q = (6, 1, 7)$ from the straight line

$$g : (5, 6, -3)^t + t(0, 1, 2)^t$$

and the projection X of Q on g .

Resources: [Distance Point-Line](#)

Problem Variants

■ $Q = (-3, 1, -4), \quad g : (0, 1, 2)^t + t(2, 1, 2)^t$

$d = ?, x_3 = -?$:

check

■ $Q = (7, 6, -1), \quad g : (0, 1, 0)^t + t(3, 0, -4)^t$

$d = ?.??, x_1 = ?$:

check

■ $Q = (0, 0, 1), \quad g : (1, 0, 0)^t + t(-1, 1, 1)^t$

$d = ?.??, x_1 = ?.??$:

check

Solution

Distance

applying the formula $d = |(\vec{q} - \vec{p}) \times \vec{u}|/|\vec{u}|$ for the distance of a point Q from a straight line $g : \vec{p} + t\vec{u}$ with $Q = (6, 1, 7)$, $P = (5, 6 - 3)$, $\vec{u} = (0, 1, 2)^t \rightsquigarrow$

$$\vec{q} - \vec{p} = \begin{pmatrix} 1 \\ -5 \\ 10 \end{pmatrix}, \quad d = \left| \begin{pmatrix} 1 \\ -5 \\ 10 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right| / \left| \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right|$$

The cross product in the numerator equals $(-10 - 10, 0 - 2, 1 + 0)^t = (-20, -2, 1)^t$ and hence

$$d = |\overrightarrow{XQ}| = \frac{\sqrt{400 + 4 + 1}}{\sqrt{0 + 1 + 4}} = 9.$$

Projection

Since $\overrightarrow{PX} \perp \overrightarrow{XQ}$ and $\overrightarrow{PX} + \overrightarrow{XQ} = \overrightarrow{PQ}$, $|\overrightarrow{PX}|$ can be computed with the theorem of Pythagoras:

$$|\overrightarrow{PX}|^2 = |\overrightarrow{PQ}|^2 - |\overrightarrow{XQ}|^2 = |(1, -5, 10)^t|^2 - d^2 = 126 - 81 = 45$$

i.e. $|\overrightarrow{PX}| = \sqrt{45} = 3\sqrt{5}$.

\rightsquigarrow position of X on g (adding a multiple of the normalized direction vector $\vec{u}^\circ = \vec{u}/|\vec{u}|$ to \vec{p}):

$$\vec{x} = \vec{p} + |\overrightarrow{PX}| \vec{u}^\circ = (5, 6, -3)^t + (3\sqrt{5})(0, 1, 2)^t/\sqrt{5} = (5, 9, 3)^t$$

Alternative solution

applying the formula for the component of the vector $\vec{q} - \vec{p}$ in the direction of $\vec{u} \rightsquigarrow$

$$\begin{aligned} \vec{x} &= \vec{p} + \frac{(\vec{q} - \vec{p}) \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \\ &= \begin{pmatrix} 5 \\ 6 \\ -3 \end{pmatrix} + \underbrace{\frac{(1, -5, 10)^t \cdot (0, 1, 2)^t}{(0, 1, 2)^t \cdot (0, 1, 2)^t}}_{=15/5} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 9 \\ 3 \end{pmatrix} \end{aligned}$$

distance:

$$d = |\vec{x} - \vec{q}| = |(5, 9, 3)^t - (6, 1, 7)^t| = |(-1, 8, -4)^t| = 9$$

4.3 Existence of an Intersection for Straight Lines

For which value of the parameter a do the straight lines

$$g: \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad h: \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ a \end{pmatrix}$$

possess an intersection X ? Determine the coordinates x_k .

Resources: [Straight Lines](#)

Problem Variants

■ $g: \vec{x} = \begin{pmatrix} -8 \\ -6 \\ 1 \end{pmatrix} + s \begin{pmatrix} -a \\ 0 \\ 3 \end{pmatrix}, \quad h: \vec{x} = \begin{pmatrix} -2 \\ -4 \\ -3 \end{pmatrix} + t \begin{pmatrix} -9 \\ 2 \\ a \end{pmatrix}$

$X = (?, -?, -?)$:

check

■ $g: \vec{x} = \begin{pmatrix} 6 \\ 4 \\ -1 \end{pmatrix} + s \begin{pmatrix} -3 \\ -a \\ 1 \end{pmatrix}, \quad h: \vec{x} = \begin{pmatrix} -7 \\ a \\ -2 \end{pmatrix} + t \begin{pmatrix} 8 \\ 2 \\ 0 \end{pmatrix}$

$X = (?, ?, -?)$:

check

■ $g: \vec{x} = \begin{pmatrix} -5 \\ 7 \\ -4 \end{pmatrix} + s \begin{pmatrix} 5 \\ -7 \\ 1 \end{pmatrix}, \quad h: \vec{x} = \begin{pmatrix} -3 \\ -9 \\ -8 \end{pmatrix} + t \begin{pmatrix} 8 \\ a \\ 6 \end{pmatrix}$

$X = (?, -?, -?)$:

check

Solution

equating the parametrizations

$$g : \vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad h : \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 2 \\ a \end{pmatrix}$$

\rightsquigarrow linear system

$$2s = 1 - 3t, \quad 1 - s = 2t, \quad 3s = 1 + at$$

equations 1 and 2 $\implies t = 1, s = -1$

consistent with equation 3 if $-3 = 1 + a$, i.e. an intersection X exists for $a = -4$

substituting $s = -1$ into the parametrization of g \rightsquigarrow

$$\vec{x} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + (-1) \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -3 \end{pmatrix}$$

4.4 Distance of Two Straight Lines

Determine the distance d of the straight lines

$$g: \begin{pmatrix} 5 \\ 8 \\ 4 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad h: \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

as well as the closest points X and Y .

Resources: [Distance of Straight Lines](#)

Problem Variants

■ $g: \begin{pmatrix} 0 \\ -1 \\ 9 \end{pmatrix} + s \begin{pmatrix} 1 \\ -8 \\ 3 \end{pmatrix}, \quad h: \begin{pmatrix} -5 \\ 2 \\ 8 \end{pmatrix} + t \begin{pmatrix} -2 \\ -4 \\ 4 \end{pmatrix}$

$d = ?$, $X = (-?, ?, ?)$, $Y = (-?, ?, ?)$:

check

■ $g: \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix} + s \begin{pmatrix} -5 \\ -2 \\ 6 \end{pmatrix}, \quad h: \begin{pmatrix} -4 \\ 8 \\ 2 \end{pmatrix} + t \begin{pmatrix} -9 \\ 0 \\ 9 \end{pmatrix}$

$d = ?$, $X = (?, ?, -?)$, $Y = (?, ?, -?)$:

check

■ $g: \begin{pmatrix} -5 \\ 3 \\ -4 \end{pmatrix} + s \begin{pmatrix} -1 \\ 4 \\ -3 \end{pmatrix}, \quad h: \begin{pmatrix} -8 \\ -9 \\ 8 \end{pmatrix} + t \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

$d = ?$, $X = (-?, -?, -?)$, $Y = (-?, ?, ?)$:

check

Solution

Distance

application of the formula

$$d = |(\vec{p} - \vec{q}) \cdot \vec{n}^\circ|, \quad \vec{n} = \vec{p} \times \vec{q}, \quad \vec{n}^\circ = \vec{n}/|\vec{n}|$$

for the distance of two straight lines with parametrizations $g : \vec{p} + s\vec{u}$ and $h : \vec{q} + t\vec{v}$

substituting $\vec{p} = (5, 8, 4)^t$, $\vec{u} = (1, 2, 1)^t$ and $\vec{q} = (6, 3, 7)^t$, $\vec{v} = (2, 1, 2)^t \rightsquigarrow$

$$d = \begin{pmatrix} 5 - 6 \\ 8 - 3 \\ 4 - 7 \end{pmatrix} \cdot \left(\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right)^\circ = \begin{pmatrix} -1 \\ 5 \\ -3 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} / (3\sqrt{2}) = \sqrt{2}$$

Closest points

$(\vec{x} - \vec{y}) \perp \vec{u}, \vec{v} \rightsquigarrow$ linear system

$$\begin{aligned} \left[\underbrace{(5, 8, 4)^t + s(1, 2, 1)^t}_{\vec{x}} - \underbrace{((6, 3, 7)^t + t(2, 1, 2)^t)}_{\vec{y}} \right] \cdot (1, 2, 1)^t &= 0 \\ \left[\dots \right] \cdot (2, 1, 2)^t &= 0 \end{aligned}$$

simplification \rightsquigarrow

$$6 + 6s - 6t = 0, \quad -3 + 6s - 9t = 0$$

with the solution $s = -4$, $t = -3$

substituting into the parametrizations \rightsquigarrow

$$\vec{x} = \begin{pmatrix} 5 \\ 8 \\ 4 \end{pmatrix} + (-4) \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{y} = \begin{pmatrix} 6 \\ 3 \\ 7 \end{pmatrix} + (-3) \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

4.5 Points with Prescribed Distance from Two Lines

Determine the points P which have equal distance $d = 1$ from the straight lines

$$g_a : t \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad g_b : t \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Resources: [Distance Point-Line](#)

Problem Variants

■ $g_a : t \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad g_b : t \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad d = 1$

points closest to the origin: $\pm(-0.74, ??.?)$:

check

■ $g_a : t \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \quad g_b : t \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad d = 2$

largest coordinate of the 4 points: $?.??$:

check

■ $g_a : t \begin{pmatrix} 4 \\ 3 \end{pmatrix}, \quad g_b : t \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad d = 5$

largest coordinate of the 4 points: $?$:

check

Solution

General considerations

distance of a point P from a straight line $g : t \vec{u}$ with normalized direction vector \vec{u} ($|\vec{u}| = 1$):

$$\begin{aligned} d &= |\vec{p}| \sin \sphericalangle(\vec{p}, \vec{u}) = |\vec{p}| \sqrt{1 - \cos^2 \sphericalangle(\vec{p}, \vec{u})} \\ &= |\vec{p}| \sqrt{1 - (\vec{p} \cdot \vec{u} / |\vec{p}|)^2} = \sqrt{|\vec{p}|^2 - (\vec{p} \cdot \vec{u})^2} \end{aligned}$$

Hence, points P with distance 1 from two lines $g_a : t \vec{a}$ and $g_b : t \vec{b}$ with $|\vec{a}| = |\vec{b}| = 1$ satisfy

$$\sqrt{|\vec{p}|^2 - (\vec{p} \cdot \vec{a})^2} = 1 = \sqrt{|\vec{p}|^2 - (\vec{p} \cdot \vec{b})^2} \quad (1)$$

squaring and subtracting the equations \rightsquigarrow

$$(\vec{p} \cdot \vec{a})^2 = (\vec{p} \cdot \vec{b})^2 \iff \vec{p} \cdot \vec{a} = \sigma \vec{p} \cdot \vec{b} \iff \vec{p} \cdot (\vec{a} - \sigma \vec{b}) = 0$$

with $\sigma = \pm 1$

Hence, \vec{p} is either parallel to a vector $u \perp (\vec{a} + \vec{b})$ or to a vector $v \perp (\vec{a} - \vec{b})$. inserting $\vec{p} = s\vec{u}$ into one of the equations in (1), e.g. the first one \rightsquigarrow

$$1 = \sqrt{|s\vec{u}|^2 - (s\vec{u} \cdot \vec{a})^2} = \pm s \sqrt{|\vec{u}|^2 - (\vec{u} \cdot \vec{a})^2}$$

and, consequently,

$$\vec{p} = \pm \frac{1}{\sqrt{|\vec{u}|^2 - (\vec{u} \cdot \vec{a})^2}} \vec{u} \quad (2)$$

Similarly, substituting $\vec{p} = s\vec{v}$ \rightsquigarrow two more points with distance 1:

$$\vec{p} = \pm \frac{1}{\sqrt{|\vec{v}|^2 - (\vec{v} \cdot \vec{a})^2}} \vec{v} \quad (3)$$

Application to the given data

normalizing the direction vectors of the lines g_a and g_b \rightsquigarrow

$$\vec{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \vec{b} = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \quad \text{and} \quad \vec{a} + \vec{b} = \begin{pmatrix} 3/5 \\ 9/5 \end{pmatrix}, \vec{a} - \vec{b} = \begin{pmatrix} -3/5 \\ 1/5 \end{pmatrix}$$

Orthogonal vectors are $\vec{u} = (-3, 1)^t$ and $\vec{v} = (1, 3)^t$.

equations (2) and (3) \rightsquigarrow 4 points with distance 1 from g_a and g_b :

$$\vec{p} = \pm \frac{1}{\sqrt{10-1}} \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \pm \begin{pmatrix} -1 \\ 1/3 \end{pmatrix}, \quad \vec{p} = \pm \frac{1}{\sqrt{10-9}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \pm \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

4.6 Hesse Form of a Plane

Determine the Hesse Form of the plane E which contains the points $(3, 2, 1)$, $(7, 3, 2)$, $(7, 2, 0)$.

Resources: [Plane](#)

Problem Variants

■ $(1, -8, -2), (2, 4, 3), (3, -2, -1)$

$E : (?x - y + ?z) / ? = ? :$

check

■ $(1, -4, -5), (2, -8, 4), (0, -6, -5)$

$E : (?x - ?y - 2z) / ? = ? :$

check

■ $(2, -1, -2), (-2, 5, -3), (6, -3, 3)$

$E : (?x + ?y - 4z) / ? = ? :$

check

Solution

forming differences of the points $(3, 2, 1)$, $(7, 3, 2)$, $(7, 2, 0) \rightsquigarrow$ spanning vectors

$$\vec{u} = \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} 7-3 \\ 2-2 \\ 0-1 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$$

normal vector (perpendicular to the plane E):

$$\vec{n} = \vec{u} \times \vec{v} = \begin{pmatrix} -1 \\ 8 \\ -4 \end{pmatrix}, \quad |\vec{n}| = \sqrt{1 + 64 + 16} = 9, \quad \vec{n}^\circ = \vec{n}/|\vec{n}| = \begin{pmatrix} -1/9 \\ 8/9 \\ -4/9 \end{pmatrix}$$

\rightsquigarrow Hesse Form

$$E: \vec{x} \cdot \sigma \vec{n}^\circ = \vec{p} \cdot \sigma \vec{n}^\circ$$

with P a point in E and the sign $\sigma \in \{0, 1\}$ chosen so that the right hand side is non-negative

choosing $P = (3, 2, 1) \rightsquigarrow$

$$E: \sigma \left(-\frac{1}{9}x_1 + \frac{8}{9}x_2 - \frac{4}{9}x_3 \right) = \sigma \underbrace{\left(-\frac{1}{9} \cdot 3 + \frac{8}{9} \cdot 2 - \frac{4}{9} \cdot 1 \right)}_{=\sigma \cdot 1}$$

$\sigma \cdot 1 \stackrel{!}{\geq} 0 \implies \sigma = 1$, i.e. no correction of the direction of \vec{n} is necessary

4.7 Plane Containing a Point and a Straight Line

Determine the Hesse form of the plane E containing the point $P = (1, 3, 1)$ and the straight line

$$g : \begin{pmatrix} -4 \\ 7 \\ 4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}.$$

Resources: [Plane](#)

Problem Variants

■ $P = (3, 2, 7)$, $g : \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$

$E : (?x - ?y + z) / ? = ? :$

check

■ $P = (4, 5, 6)$, $g : \begin{pmatrix} 4 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

$E : (?x + ?y - 6z) / ?? = ? :$

check

■ $P = (1, 2, -5)$, $g : \begin{pmatrix} 3 \\ -2 \\ -2 \end{pmatrix} + t \begin{pmatrix} 4 \\ -5 \\ -3 \end{pmatrix}$

$E : (?x + ?y + 2z) / ?? = ? :$

check

Solution

spanning vectors for E : direction vector \vec{u} of g and a vector \overrightarrow{QP} with Q a point on g

$$\vec{u} = \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \quad \vec{v} = \vec{p} - \vec{q} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} - \begin{pmatrix} -4 \\ 7 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix}$$

normal of E

$$\vec{n} \parallel \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} 5 \\ -4 \\ -3 \end{pmatrix} = \begin{pmatrix} -8 \\ 2 \\ -16 \end{pmatrix}, \quad \vec{n}^\circ = \frac{1}{9} \begin{pmatrix} -4 \\ 1 \\ -8 \end{pmatrix}$$

Hesse form:

$$E : \sigma(x, y, z) \cdot \vec{n}^\circ = \sigma \vec{q} \cdot \vec{n}^\circ$$

with the sign σ chosen, so that the right-hand side is non-negative

$$\vec{q} \cdot \vec{n}^\circ = \begin{pmatrix} -4 \\ 7 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -4 \\ 1 \\ -8 \end{pmatrix} / 9 = -1 \quad \implies \quad \sigma = -1$$

and

$$E : \frac{4}{9}x - \frac{1}{9}y + \frac{8}{9}z = 1$$

4.8 Plane Determined by Two Straight Lines

Determine the intersection of the two straight lines

$$g: \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad h: \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

and the Hesse form of the plane E containing g and h .

Resources: [Straight Line](#), [Plane](#)

Problem Variants

$$\blacksquare g: \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} -1 \\ 4 \\ 1 \end{pmatrix}, \quad h: \begin{pmatrix} 5 \\ 2 \\ 0 \end{pmatrix} + s \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$

$$E: (?x - y + ?z)/? = ?:$$

check

$$\blacksquare g: \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 4 \\ -4 \end{pmatrix}, \quad h: \begin{pmatrix} 9 \\ -4 \\ 7 \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$E: (?x + ?y + 4z)/? = ?:$$

check

$$\blacksquare g: \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \quad h: \begin{pmatrix} 3 \\ 0 \\ 8 \end{pmatrix} + s \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$$

$$E: (?x - ?y - 2z)/?? = ?:$$

check

Solution

Intersection

The intersection $P = g \cap h$ is a common point of the two parametrizations:

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \vec{p} = \begin{pmatrix} 1 \\ 6 \\ 5 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}$$

\rightsquigarrow linear system

$$3 + t = 1, \quad 2 + 2t = 6 + 2s, \quad 1 = 5 + s$$

first and third equation $\implies t = -2, s = -4$, which is consistent with the second equation

substituting into either of the parametrizations, e.g. the representation of g

\rightsquigarrow

$$\vec{p} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} = \vec{p} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Plane determined by g and h

normal vector (orthogonal to the direction vectors of the lines)

$$\vec{n} \parallel \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}, \quad \vec{n}^\circ = \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix},$$

Hesse form:

$$E : \sigma(x, y, z) \cdot \vec{n}^\circ = \sigma \vec{q} \cdot \vec{n}^\circ$$

with the sign σ chosen, so that the right-hand side is non-negative

$$\vec{p} \cdot \vec{n}^\circ = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2/3 \\ -1/3 \\ 2/3 \end{pmatrix} = 2 \implies \sigma = 1$$

and

$$E : \frac{2}{3}x - \frac{1}{3}y + \frac{2}{3}z = 2$$

4.9 Distance of a Point from a Plane

Determine the distance d of the point $Q = (4, 1, 9)$ from the plane

$$E : 2x_1 - x_2 + 2x_3 = -2$$

as well as the closest point X .

Resources: [Distance Point-Plane](#)

Problem Variants

■ $E : 3x_1 - 6x_2 + 2x_3 = -6, Q = (8, -9, 7)$

$d = ??, X = (?, ?, ?):$

check

■ $E : 9x_1 + 6x_2 + 2x_3 = 2, Q = (7, 9, 3)$

$d = ??, X = (-?, ?, ?):$

check

■ $E : -4x_1 + 7x_2 - 4x_3 = 0, Q = (-3, 7, -5)$

$d = ?, X = (?, ?, -?):$

check

Solution

Closest point

normal vector of the plane $E : 2x_1 - x_2 + 2x_3 = -2$:

$$\vec{n} = (2, -1, 2)^t$$

$\vec{x} - \vec{q} \perp E \iff \vec{x} - \vec{q} \parallel \vec{n}$ for the point X closest to $Q = (4, 1, 9) \implies$

$$\vec{x} = \vec{q} + t\vec{n} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$$

substituting into the equation for the plane ($X \in E$) \rightsquigarrow

$$2(4 + 2t) - (1 - t) + 2(9 + 2t) = -2$$

with the solution $t = -3$ and hence

$$\vec{x} = \begin{pmatrix} 4 \\ 1 \\ 9 \end{pmatrix} - 3 \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2 \\ 4 \\ 3 \end{pmatrix}$$

Distance

$$d = |\vec{x} - \vec{q}| = \left| \begin{pmatrix} -2 - 4 \\ 4 - 1 \\ 3 - 9 \end{pmatrix} \right| = \sqrt{(-6)^2 + 3^2 + (-6)^2} = 9$$

Alternative solution

The distance d can be computed directly using the Hesse form

$$E : \underbrace{(\sigma \vec{n} / |\vec{n}|)}_{\vec{n}^\circ} \cdot \vec{x} = -\frac{2}{3}x_1 + \frac{1}{3}x_2 - \frac{2}{3}x_3 = \underbrace{\frac{2}{3}}_{d_E},$$

where the sign σ is chosen so that d_E is nonnegative. With this representation of the plane an alternative formula for the distance is

$$\begin{aligned} d &= |\vec{n}^\circ \cdot \vec{q} - d_E| = \left| \left(-\frac{2}{3} \cdot 4 + \frac{1}{3} \cdot 1 - \frac{2}{3} \cdot 9 \right) - \frac{2}{3} \right| \\ &= \left| -\frac{25}{3} - \frac{2}{3} \right| = 9 \quad \checkmark \end{aligned}$$

4.10 Intersection of a Straight Line with a Plane

Determine the intersection Q of the straight line

$$g : \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

with the plane $E : x - 3y + 2z = 4$.

Problem Variants

■ $g : (7, 7, 9)^t + t(0, -4, -6)^t, \quad E : -2x - 8y + 2z = 8$

$Q = (?, -?, -?):$

check

■ $g : (-4, -9, -8)^t + t(1, 3, 0)^t, \quad E : -6x + 6y - z = 2$

$Q = (-?, -?, -?):$

check

■ $g : (-4, 3, -9)^t + t(-2, 1, -3)^t, \quad E : -5x + 6z = -2$

$Q = (?, -?, ?):$

check

Solution

substituting

$$\vec{q} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

into the equation for the plane, $E : x - 3y + 2z = 4 \rightsquigarrow$

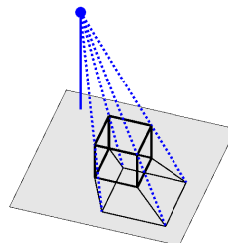
$$(-6 + 2t) - 3(7 - t) + 2(-2 + t) = 4$$

with the solution $t = 5$ and

$$\vec{q} = \begin{pmatrix} -6 \\ 7 \\ -2 \end{pmatrix} + 5 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

4.11 Perspective Projection of a Cube

Derive a formula for the perspective projection onto a plane $E : \vec{x} \cdot \vec{n} = d$. Apply this formula with the view point $V = (-1, 2, 3)$ to determine the images Q of the vertices P of the cube $[0, 1]^3$ on the x_1x_2 -plane.



Resources: [Plane](#)

Problem Variants

■ $P = (6, -1, 3), V = (4, 2, 5), \vec{n} = (1, 0, 0)^t, d = 0$

$Q = (?, ?, ?):$

check

■ $P = (-9, -7, 6), V = (3, -5, -4), \vec{n} = (2, -2, -6)^t, d = 0$

$Q = (-?, -?, ?):$

check

■ $P = (-4, 8, 1), V = (2, -1, -2), \vec{n} = (7, 6, 3)^t, d = 9$

$Q = (?, ?, -?):$

check

Solution

Perspective projection

The image $Q \in E$ of a point P is the intersection of the straight line g , passing through the point P and the view point V with the plane E , i.e.

$$Q \in g \iff \vec{q} = \vec{v} + t(\vec{p} - \vec{v}), \quad Q \in E \iff \vec{q} \cdot \vec{n} = d.$$

substituting the expression for \vec{q} into the equation of the plane and solving for the parameter $t \rightsquigarrow$

$$t = \frac{d - \vec{v} \cdot \vec{n}}{(\vec{p} - \vec{v}) \cdot \vec{n}}, \quad \vec{q} = \vec{v} + \frac{d - \vec{v} \cdot \vec{n}}{(\vec{p} - \vec{v}) \cdot \vec{n}}(\vec{p} - \vec{v}) \quad (1)$$

Projection of the cube

specializing the formula (1) with $\vec{n} = (0, 0, 1)^t$, $d = 0$, $\vec{v} = (-1, 2, 3)^t \rightsquigarrow$

$$\vec{q} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \frac{3}{p_3 - 3} \begin{pmatrix} p_1 + 1 \\ p_2 - 2 \\ p_3 - 3 \end{pmatrix}$$

The points in the x_1x_2 -plane remain unchanged. Therefore, only the four vertices P with $p_3 = 1$ have to be projected:

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} - \frac{3}{1-3} \begin{pmatrix} 0+1 \\ 0-2 \\ 1-3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1 \\ 0 \end{pmatrix}$$

and, similarly,

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 1/2 \\ 0 \end{pmatrix}.$$

4.12 Intersection of Two Planes

Determine the angle φ of intersection of the two planes

$$E : x + y - 2z = 1, \quad E' : -y + z = 3$$

as well as the line of intersection g .

Resources: [Intersection of Planes](#)

Problem Variants

■ $E : -2x + 6y + 3z = 0, \quad E' : 4x + 9y + z = 7$

$\varphi = ???\pi:$

check

■ $E : 5x + 2y - 5z = -6, \quad E' : 7x + y + 2z = 6$

$\varphi = ???\pi:$

check

■ $E : -9x + 5y + 2z = 9, \quad E' : 3x + 7y - 4z = 1$

$\varphi = ??\pi:$

check

Solution

Angle of intersection

The angle φ between the two planes equals the smaller angle formed by their normal vectors $\vec{n} = (1, 1, -2)^t$ and $\vec{n}' = (0, -1, 1)^t$, i.e.

$$\cos \varphi = \frac{|\vec{n} \cdot \vec{n}'|}{|\vec{n}| |\vec{n}'|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{\sqrt{3}}{2}, \quad \varphi = \frac{\pi}{6}$$

Line of intersection

$g : \vec{p} + t\vec{u}:$

$$\bullet P \in E \cap E' \implies$$

$$p_1 + p_2 - 2p_3 = 1, \quad -p_2 + p_3 = 3$$

$$\text{choosing, e.g., } p_2 = 0 \rightsquigarrow p_3 = 3, p_1 = 7$$

$$\bullet \vec{u} \parallel \vec{n} \times \vec{n}' \implies$$

$$\vec{u} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}$$

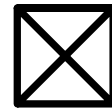
4.13 Angles between Edges and Faces of a Pyramid

Determine the angles between the edges and the faces of a symmetric pyramid with height 1 and square base area with sidelength 2.

Resources: [Scalar Product](#), [Intersection of Planes](#)

Problem Variants

- symmetric pyramid with square base area (top view), length of all edges equal to 1



angle between edges: ???, smallest angle between faces: ???:

check

- regular tetrahedron (top view), length of all edges equal to 1



angle between edges: ???, largest angle between faces: ???:

check

- tetrahedron (top view), equilateral base triangle with side length 1, height 1

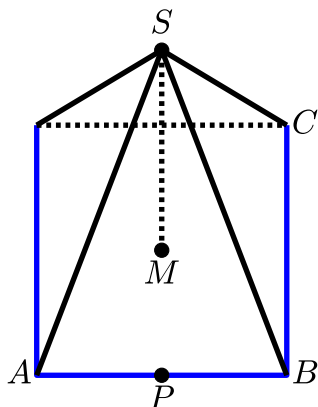


largest angle between edges: ???, smallest angle between faces: ???:

check

Solution

Choosing suitable coordinates



$$\begin{aligned} A &= (-1, -1, 0), & B &= (1, -1, 0) & C &= (1, 1, 0) \\ S &= (0, 0, 1), & M &= (0, 0, 0), & P &= (0, -1, 0) \end{aligned}$$

Angles between edges

Angle $\alpha = \sphericalangle(BAS)$ between \overline{AB} and \overline{AS} :

$$\overrightarrow{AB} = (2, 0, 0)^t, \quad \overrightarrow{AS} = (1, 1, 1)^t \quad \rightsquigarrow$$

$$\cos \alpha = \frac{\overrightarrow{AB} \cdot \overrightarrow{AS}}{|\overrightarrow{AB}| |\overrightarrow{AS}|} = \frac{2}{2\sqrt{3}} = \frac{\sqrt{3}}{3}, \quad \alpha = \arccos(\sqrt{3}/3) \approx 0.9553$$

Since the angles of a triangle sum to π , $\sphericalangle(ASB) = \pi - 2\alpha \approx 1.2310$, and, obviously, $\sphericalangle(ABC) = \pi/2 \approx 1.5708$.

Angle β between the base area and a side area

$$\tan \beta = |\overrightarrow{MS}| / |\overrightarrow{MP}| = 1/1 = 1 \quad \implies \quad \beta = \arctan(1) = \pi/4 \approx 0.7854$$

Angle γ between the triangles $D_1 = \Delta(ABS)$ and $D_2 = \Delta(BCS)$

normal vektors of the corresponding faces

$$\vec{n}_1 = \overrightarrow{AB} \times \overrightarrow{BS} = (2, 0, 0)^t \times (-1, 1, 1)^t = (0, -2, 2)^t$$

$$\vec{n}_2 = \overrightarrow{BC} \times \overrightarrow{BS} = (0, 2, 0)^t \times (-1, 1, 1)^t = (2, 0, 2)^t$$

$$\sphericalangle(D_1, D_2) \hat{=} \sphericalangle(\vec{n}_1, \vec{n}_2) \quad \rightsquigarrow$$

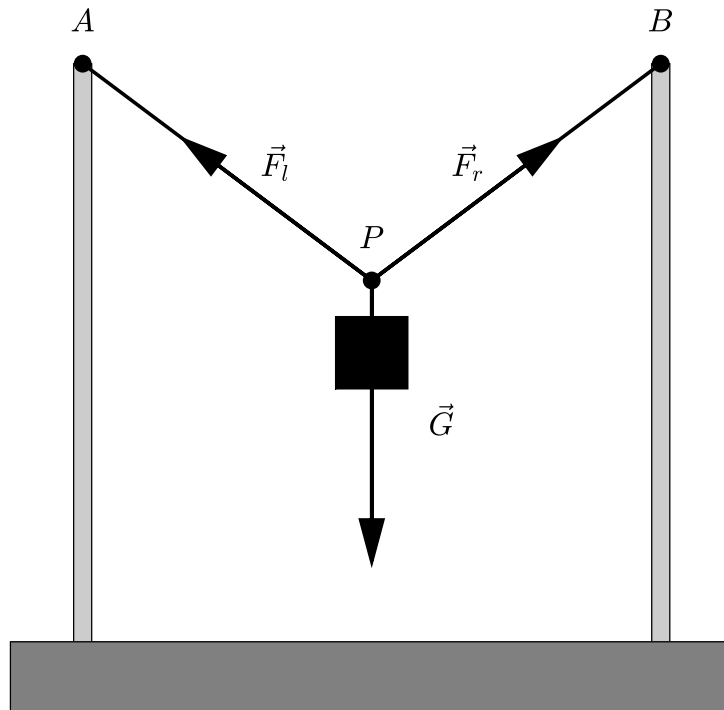
$$\cos \gamma = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|} = \frac{4}{\sqrt{(-2)^2 + 2^2} \sqrt{8}} = \frac{1}{2}, \quad \gamma = \arccos(1/2) = \pi/3 \approx 1.0472$$

Chapter 5

Calculus Highlights

5.1 Force Acting on a Rope

A force of $G = 50\text{ N}$ is exerted by a weight in a middle of a rope, which is fixed at two posts at a distance of 8 m .



- Compute the force $F = |\vec{F}_l| = |\vec{F}_r|$ in the direction of the rope for a rope with length $L = 10\text{ m}$.
- Determine the minimal length L_* of the rope, for which the force F does not exceed 100 N .

Resources: [Addition and Scalar Multiplication of Vectors](#)

Solution

Balance of forces

The forces acting at the point P sum to $\vec{0}$:

$$\vec{G} + \vec{F}_l + \vec{F}_r = (0, 0)^t \quad (1)$$

with

- $\vec{G} = (0, -50)^t$
- $\vec{F}_r = s \overrightarrow{PB} = s \left(\begin{array}{c} 8/2 \\ \sqrt{|\overrightarrow{PB}|^2 - (8/2)^2} \end{array} \right) = s \left(\begin{array}{c} 4 \\ \sqrt{(L/2)^2 - 16} \end{array} \right)$
(computation of the second component of \overrightarrow{PB} with the theorem of Pythagoras)
- $\vec{F}_l = s \left(\begin{array}{c} -4 \\ \sqrt{(L/2)^2 - 16} \end{array} \right)$ because of symmetry

substituting into (1) \rightsquigarrow

$$\left(\begin{array}{c} 0 \\ -50 \end{array} \right) + s \left(\begin{array}{c} -4 \\ \sqrt{(L/2)^2 - 16} \end{array} \right) + s \left(\begin{array}{c} 4 \\ \sqrt{(L/2)^2 - 16} \end{array} \right) = \left(\begin{array}{c} 0 \\ 0 \end{array} \right)$$

second component of the equation $\implies s = 50 / (2\sqrt{(L/2)^2 - 16})$ and

$$|\vec{F}_r| = s |\overrightarrow{PB}| = \frac{50}{2\sqrt{(L/2)^2 - 16}} \cdot \frac{L}{2} = \frac{25L}{\sqrt{L^2 - 64}} \quad (2)$$

Force for a rope with length 10 m

substituting $L = 10$ into (2) \rightsquigarrow

$$F = 25 \cdot 10 / \sqrt{100 - 64} \text{N} = 125/3 \text{N} \approx 41.6666 \text{N}$$

Minimal length

inverting (2) \rightsquigarrow

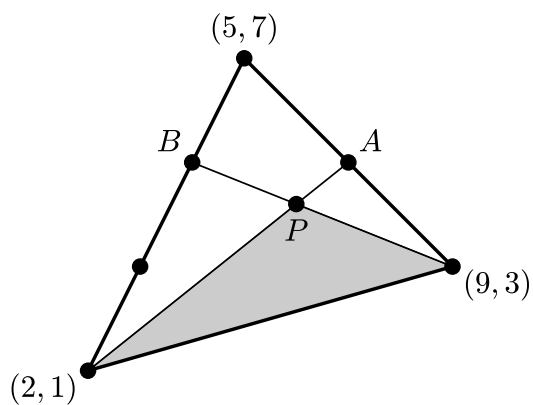
$$|\vec{F}_r|^2(L^2 - 64) = 625L^2, \quad L = \frac{8|\vec{F}_r|}{\sqrt{|\vec{F}_r|^2 - 625}}$$

substituting $F = |\vec{F}_r| = 100$ \rightsquigarrow minimal length

$$L_* = 800/\sqrt{10000 - 625} \text{ m} = 32/\sqrt{15} \text{ m} \approx 8.2624 \text{ m}$$

5.2 Subarea of a Triangle

Determine the points A and B which divide the right and left edges of the depicted triangle in the ratios $1 : 1$ and $2 : 1$, respectively. Moreover, compute the area of the shaded subtriangle.



Resources: [Addition and Scalar Multiplication of Vectors](#)

Solution

Points dividing the edges

- right edge: ratio 1 : 1 \rightsquigarrow

$$\vec{a} = \frac{1}{2} \begin{pmatrix} 9 \\ 3 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

- left edge: ratio 2 : 1 \rightsquigarrow

$$\vec{b} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \frac{2}{3} \begin{pmatrix} 5 \\ 7 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

representation of P as intersection of the line segments from $(2, 1)$ to $A = (7, 5)$ and from $(9, 3)$ to $B = (4, 5)$ \rightsquigarrow

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + r \begin{pmatrix} 7-2 \\ 5-1 \end{pmatrix} = \vec{p} = \begin{pmatrix} 9 \\ 3 \end{pmatrix} + s \begin{pmatrix} 4-9 \\ 5-3 \end{pmatrix}$$

solving the resulting linear system

$$5r + 5s = 7, \quad 4r - 2s = 2$$

$\rightsquigarrow r = 4/5, s = 3/5$, i.e. $P = (6, 21/5)$

Area

applying the formula $\text{area } \Delta(A, P, B) = |\det(\vec{p} - \vec{a}, \vec{p} - \vec{b})|/2$ with $A = (2, 1)$, $B = (9, 3)$, and $P = (6, 21/5)$ \rightsquigarrow

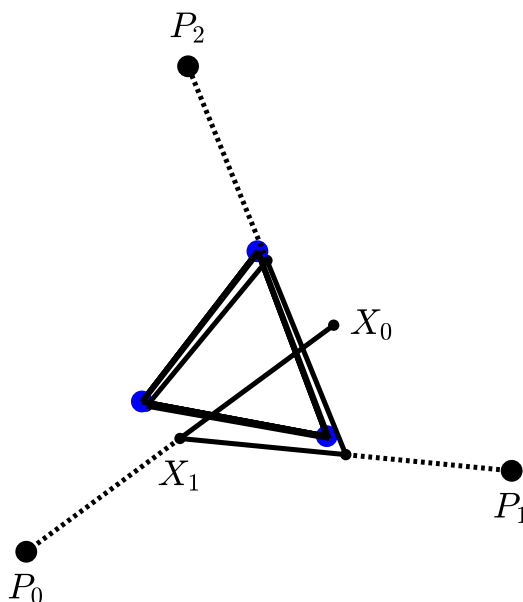
$$\begin{aligned} \text{area } \Delta(A, P, B) &= \frac{1}{2} \left| \det \begin{pmatrix} 6-2 & 6-9 \\ 21/5-1 & 21/5-3 \end{pmatrix} \right| \\ &= \frac{1}{2} \left| \det \begin{pmatrix} 4 & -3 \\ 16/5 & 6/5 \end{pmatrix} \right| = \frac{1}{2} |24/5 - (-48/5)| = 7.2 \end{aligned}$$

5.3 Jumping Flea

Starting at some point X_0 in the plane, a flea jumps half the distance towards the point P_0 . Then, half the distance towards P_1 and towards P_2 . He repeats these movements over and over again. Mathematically speaking, the position of the flea after $n + 1$ jumps is given by

$$\vec{x}_{n+1} = (\vec{p}_{n \bmod 3} + \vec{x}_n)/2, \quad n = 0, 1, \dots$$

Does there exist a limit? Will the flea eventually come to rest? No! There are **three** accumulation points P_k^* (blue dots); the sequence (X_n) tends to a 3-cycle.



Determine P_k^* , $k = 0, 1, 2$. Moreover, plot the sequence of points (X_n) with the points P_k in the recursion (1) chosen randomly. This leads to a pattern known as Sierpinski triangle¹.

Resources: [Addition and Scalar Multiplication of Vectors](#), [Vector Operations With MATLAB®](#)

¹You might also want to consider more than 3 points P_k .

Solution

Accumulation points

The figure suggests to consider the subsequence $Y_n = X_{3n}$, $n = 0, 1, \dots$, with the recursion

$$\begin{aligned}\vec{y}_0 &= \vec{x}_0 \\ \vec{x}_1 &= (\vec{p}_0 + \vec{x}_0)/2 = \vec{p}_0/2 + \vec{y}_0/2 \\ \vec{x}_2 &= (\vec{p}_1 + \vec{x}_1)/2 = \vec{p}_1/2 + \vec{p}_0/4 + \vec{y}_0/4 \\ \vec{y}_1 = \vec{x}_3 &= (\vec{p}_2 + \vec{x}_2)/2 = \underbrace{\vec{p}_2/2 + \vec{p}_1/4 + \vec{p}_0/8}_{=:\vec{q}} + \vec{y}_0/8 \\ &\dots \\ \vec{y}_2 &= \vec{q} + \vec{y}_1/8 = \vec{q} + \vec{q}/8 + \vec{y}_0/64 \\ \vec{y}_3 &= \vec{q} + \vec{y}_2/8 = \vec{q} + \vec{q}/8 + \vec{q}/64 + \vec{y}_0/512 \\ &\dots\end{aligned}$$

iteration \rightsquigarrow limit P_0^* of the subsequence (Y_n) ($\hat{=}$ accumulation point of (X_n)) in terms of a geometric series:

$$\vec{p}_0^* = \sum_{k=0}^{\infty} 8^{-k} \vec{q} = \frac{1}{1 - 1/8} \vec{q} = \frac{8}{7} \vec{q} = \frac{4}{7} \vec{p}_2 + \frac{2}{7} \vec{p}_1 + \frac{1}{7} \vec{p}_0$$

Two more accumulation points (limits of the subsequences X_1, X_4, \dots and X_2, X_5, \dots) are obtained by a cyclic shift of the indices:

$$\vec{p}_1^* = \frac{4}{7} \vec{p}_0 + \frac{2}{7} \vec{p}_2 + \frac{1}{7} \vec{p}_1, \quad \vec{p}_2^* = \frac{4}{7} \vec{p}_1 + \frac{2}{7} \vec{p}_0 + \frac{1}{7} \vec{p}_2.$$

Sierpinski triangle

MATLAB[®] script for generating the random point sequence (X_n) starting from the midpoint of an equilateral triangle with vertices P_n :

```
hold on % overlay of several subsequent plots

% vertices of an equilateral triangle
P = [0 0; 1 0; 1/2 sqrt(3)/2]';
plot(P(1,[1 2 3 1]),P(2,[1 2 3 1]),'.k','MarkerSize',20);
```



```

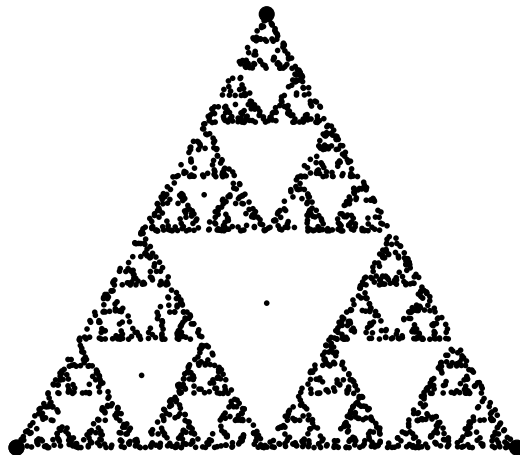
% starting point
X = [1/2; sqrt(3)/6];

% generate and plot the random sequence (X_n)
for n=1:1000
    % select a vertex of the triangle
    ind = 1+fix(3*rand); Pn = P(:,ind);
    % compute and append the next point
    X = [X (X(:,end)+Pn)/2];
end
plot(X(1,:),X(2,:),'.k','MarkerSize',10);

axis off, axis equal % no coordinate axes, equal scaling
hold off

```

resulting image:



5.4 Corners of a Cube

The three corners

$$A = (5, 1, -3), \quad B = (-6, 0, 2), \quad C = (3, 4, 3)$$

determine a cube uniquely. Compute the length of the edges and the coordinates of the remaining 5 corners.

Resources: [Norm](#), [Addition and Scalar Multiplication of Vectors](#)

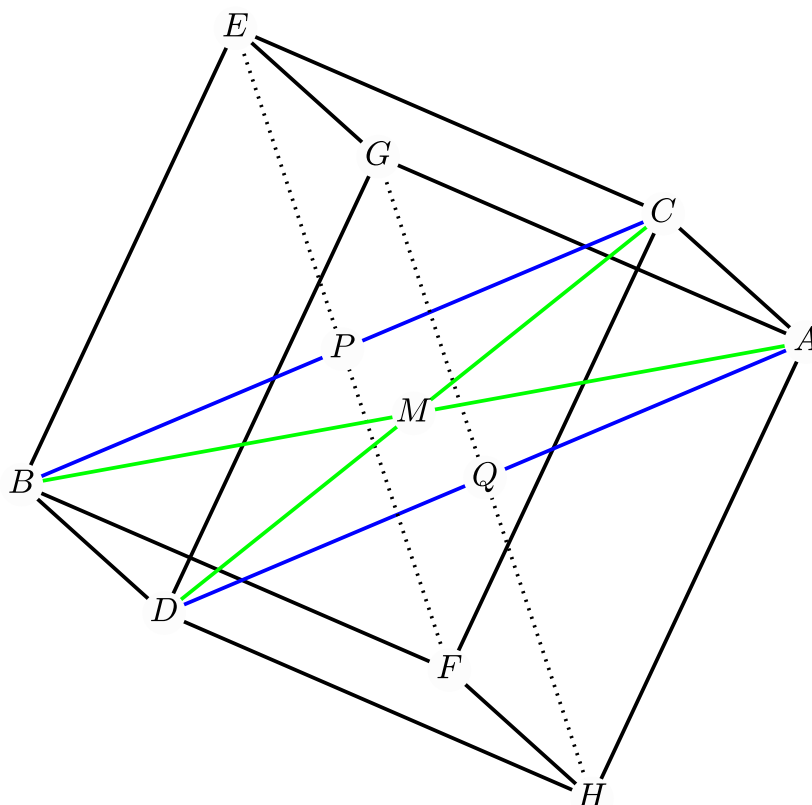
Solution

distances of the given points $A = (5, 1, -3)$, $B = (-6, 0, 2)$, $C = (3, 4, 3)$:

$$\begin{aligned} |\overline{AB}| &= \sqrt{(-6-5)^2 + (0-1)^2 + (2+3)^2} = 7\sqrt{3} \\ |\overline{AC}| &= \sqrt{(3-5)^2 + (4-1)^2 + (3+3)^2} = 7 \\ |\overline{BC}| &= \sqrt{(3+6)^2 + (4-0)^2 + (3-2)^2} = 7\sqrt{2} \end{aligned}$$

comparing the values \implies

\overline{AB} : space diagonal (green), \overline{AC} : edge with length 7, \overline{BC} : diagonal of a boundary square (blue)



midpoint M of the cube:

$$\vec{m} = (\vec{a} + \vec{b})/2 = (-1/2, 1/2, -1/2)^t$$

second space diagonal, passing through C and $M \rightsquigarrow$ another corner D with position vector

$$\vec{d} = \vec{c} + 2(\vec{m} - \vec{c}) = 2\vec{m} - \vec{c} = (-4, -3, -4)^t$$

The blue diagonals split the two boundary squares of the cube with midpoints

$$P : (\vec{b} + \vec{c})/2 = (-3/2, 2, 5/2)^t, \quad Q : (\vec{a} + \vec{d})/2 = (1/2, -1, -7/2)^t$$

into congruent right-angled triangles.

normal of the rectangle $\square(ACBD)$ (parallel to the dotted diagonals d_P and d_Q of the boundary squares with midpoints P and Q):

$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC} = (-11, -1, 5)^t \times (-2, 3, 6)^t = (-21, 56, -35)^t$$

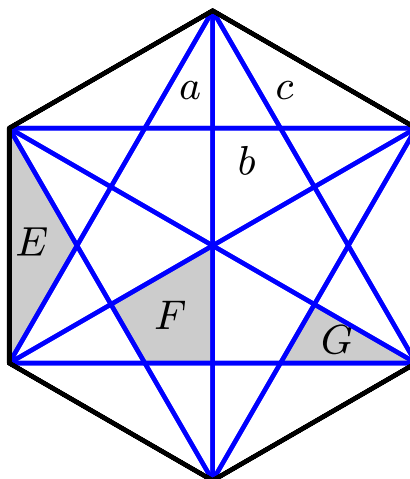
normalizing: $|\vec{n}| = 49\sqrt{2} \rightsquigarrow \vec{n}^\circ = \vec{n}/|\vec{n}| = (-3, 8, -5)^t/(7\sqrt{2})$

The remaining corners E, F, G, H of the cube lie on the diagonals d_P and d_Q with distance $|\overline{BC}|/2 = 7\sqrt{2}/2$ to the square midpoints:

$$\begin{aligned} \vec{p} \pm \frac{7\sqrt{2}}{2} \vec{n}^\circ &= (-3/2, 2, 5/2)^t \pm \frac{1}{2}(-3, 8, -5)^t \\ &\rightsquigarrow E = (-3, 6, 0), \quad F = (0, -2, 5) \\ \vec{q} \pm \frac{7\sqrt{2}}{2} \vec{n}^\circ &= (1/2, -1, -7/2)^t \pm \frac{1}{2}(-3, 8, -5)^t \\ &\rightsquigarrow G = (-1, 3, -6), \quad H = (2, -5, -1) \end{aligned}$$

5.5 Geometry of a Hexagon

The figure shows a regular hexagon with edge length 1 and with lines (blue) connecting the vertices. Determine the lengths a , b , and c of the small line segments as well as the areas of the triangles E and G and the quadrilateral F .



Resources: [Trigonometric Theorems](#)

Solution

Line Segments

The triangle $\Delta(A, C, P)$ is one half of the equilateral triangle $\Delta(A, B, C)$. Hence, the side lengths are

$$\begin{aligned} |\overline{AC}| &= 1, \\ a &= |\overline{AP}| = 1/2, \\ |\overline{CP}| &= \sqrt{1^2 - (1/2)^2} = \sqrt{3}/2. \end{aligned}$$

Since the triangle $\Delta(M, A, P)$ is similar to $\Delta(A, C, P)$ (equal angles), the ratios of corresponding side lengths are equal and

$$\begin{aligned} |\overline{CP}| : |\overline{AP}| &= \frac{\sqrt{3}}{2} : \frac{1}{2} = \sqrt{3} \implies \\ |\overline{AP}| : |\overline{MP}| &= \frac{1}{2} : b = \sqrt{3}, \quad |\overline{AC}| : |\overline{AM}| = 1 : c = \sqrt{3}, \end{aligned}$$

i.e. $b = 1/(2\sqrt{3}) = \sqrt{3}/6$, $c = 1/\sqrt{3} = \sqrt{3}/3$.

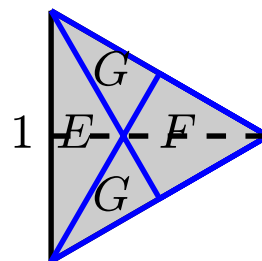
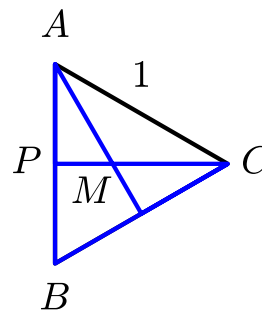
Areas of the triangles and the quadrilateral

The three bisectors divide the equilateral triangle Δ into 6 congruent small triangles with area

$$\frac{1}{6} \text{ area } \Delta = \frac{1}{6} \frac{\sqrt{3}}{4} = \frac{\sqrt{3}}{24} = \text{area } G$$

and

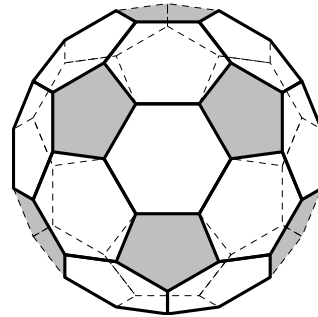
$$\text{area } E = \text{area } F = 2 \text{ area } G = \frac{\sqrt{3}}{12}.$$



5.6 Seams of a Soccer Ball

Assume, contrary to Sepp Herberger's^a statement, "The ball is round", that a soccer ball is a polyhedron. Determine the length L of an edge for a ball with radius 15 cm of the circumscribed sphere.

^aa famous former German soccer coach



Resources: [Scalar Product](#), [Norm](#)

Solution

Preliminaries

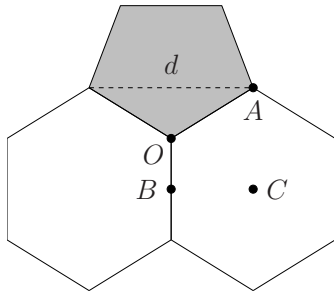
edge length $L \sim$ radius R

\rightsquigarrow choose $L = 1$ to determine the ratio $c = L/R$, i.e.

$$L = (1/R_\star)R \quad (1)$$

with R_\star the radius for a ball with edge length $L = 1$

convenient position of the soccer ball with $M = (0, -1/2, -q)$ its midpoint and \overline{MB} parallel to the z - and \overline{OB} parallel to the y -axis of the coordinate system



$$\begin{aligned} O &= (0, 0, 0) \\ A &= (d/2, 1/2, -p) \\ B &= (0, -1/2, 0) \\ C &= (d/2, -1/2, -p) \end{aligned}$$

Note that the hexagons are tilted slightly downwards with respect to the xy -plane. The coordinates $-p$ and $-q$ need to be determined which will yield a formula for $R = |\overline{OM}|$.

Main Argument

formula for the length of the diagonal of a pentagon with side length $|\overline{OA}| = L = 1$:

$$d = \frac{1 + \sqrt{5}}{2}$$

substituting this expression into $1 = |\vec{a}|^2 = (d/2)^2 + (1/2)^2 + p^2$ and solving for $p > 0$ \rightsquigarrow

$$p^2 = 1 - \frac{1 + 2\sqrt{5} + 5}{16} - \frac{1}{4} \iff p = \sqrt{6 - 2\sqrt{5}}/4$$

simplifying $\rightsquigarrow p = (\sqrt{5} - 1)/4$

$$\overrightarrow{CO} \perp \overrightarrow{CM} \implies$$

$$0 = (d/2, -1/2, -p)^t \cdot (d/2, 0, q - p)^t$$

solving for q and substituting the formulas for d and p \rightsquigarrow

$$q = \frac{d^2/4 + p^2}{p} = \left(\frac{1 + 2\sqrt{5} + 5}{16} + \frac{6 - 2\sqrt{5}}{16} \right) / \left(\frac{\sqrt{5} - 1}{4} \right)$$

$$= \frac{3}{\sqrt{5} - 1} = \frac{3 + 3\sqrt{5}}{4}$$

radius of the soccer ball with edge length $L = 1$:

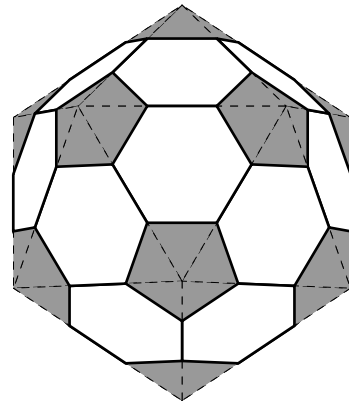
$$R_\star = |\overline{OM}| = \sqrt{(1/2)^2 + q^2} = \sqrt{\frac{1}{4} + \frac{9 + 18\sqrt{5} + 45}{16}} = \frac{\sqrt{58 + 18\sqrt{5}}}{4}$$

substituting into (1) with $R = 15$ \rightsquigarrow

$$L = (1/R_\star) R = \frac{60}{\sqrt{58 + 18\sqrt{5}}} \approx 6.0532$$

Alternative Solution

A (polyhedral) soccer ball can be constructed from an icosahedron^a by cutting off each of the vertices, resulting in the 12 pentagons and replacing the triangles by 20 hexagons. The cuts split the sides of the triangles of the icosahedron into three equal segments.



^aPlatonic solid bounded by 20 equilateral triangles, 5 of which meet at each of the 12 vertices.

volume of an icosahedron with edge length $a = 3L$ and L the edge length of the soccer ball:

$$V = \frac{5}{12} \sqrt{14 + 6\sqrt{5}} a^3$$

computation of the volume as union of 20 triangular pyramids with an equilateral triangle Δ as base and the radius r of the inscribed sphere as height \rightsquigarrow

$$V = 20 \cdot \frac{1}{3} \underbrace{\left(\frac{1}{2} \cdot a \cdot \frac{\sqrt{3}}{2} \cdot a \right)}_{\text{area } \Delta} \cdot r = \frac{5}{\sqrt{3}} a^2 r$$

equating the expressions for the volume \implies

$$L = \frac{1}{3}a = \frac{4}{\sqrt{42 + 18\sqrt{5}}} r$$

theorem of Pythagoras, applied to the triangle $\Delta(O, C, M)$ with $|\overline{OC}| = L$, $|\overline{OM}| = R$, and $|\overline{CM}| = r \implies$

$$r^2 = R^2 - L^2$$

substituting into the squared expression for $L \rightsquigarrow$

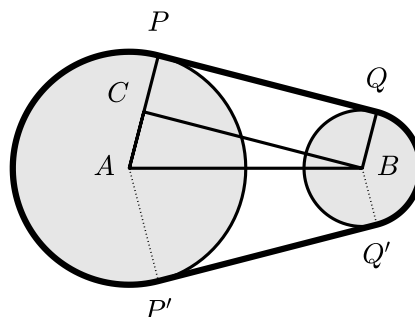
$$L^2 = \frac{16}{42 + 18\sqrt{5}} (R^2 - L^2)$$

and after solving for L

$$L = cR, \quad c = \frac{4}{\sqrt{58 + 18\sqrt{5}}} \quad \checkmark$$

5.7 Length of a V-Belt

Determine the length of the depicted V-belt for the radii $|\overline{AP}| = 20$ cm, $|\overline{BQ}| = 10$ cm, and the distance $|\overline{AB}| = 40$ cm.



Resources: [Trigonometric Theorems](#)

Solution

Lengths of the straight segments

orthogonality of the tangents to the radius vectors \overrightarrow{AP} and \overrightarrow{BQ} and the given data \rightsquigarrow

$$\begin{aligned} \overline{QP} &\perp \overline{AP}, \quad \overline{QP} \perp \overline{BQ}, \quad \overline{QP} \parallel \overline{BC} \\ |\overline{AB}| &= 40, \quad |\overline{AC}| = |\overline{CP}| = |\overline{BQ}| = 10 \end{aligned}$$

(units: cm)

Theorem of Pythagoras for the triangle $\Delta(A, B, C)$ with right angle at C
 \implies

$$|\overline{BC}| = \sqrt{|\overline{AB}|^2 - |\overline{AC}|^2} = \sqrt{40^2 - 10^2} = \sqrt{1500} \approx 38.730$$

\rightsquigarrow lengths of the straight segments of the V-belt

$$L_t = |\overline{QP}| = |\overline{Q'P'}| = |\overline{BC}| = \sqrt{1500}$$

Lengths of the circular segments

Law of cosines

$$|\overline{BC}|^2 = |\overline{AB}|^2 + |\overline{AC}|^2 - 2|\overline{AB}||\overline{AC}|\cos\alpha$$

with $|\overline{BC}| = \sqrt{1500}$, $|\overline{AB}| = 40$, $|\overline{AC}| = 10$, and $\alpha = \sphericalangle(\overrightarrow{AP}, \overrightarrow{AB}) \implies$

$$\cos\alpha = \frac{1600 + 100 - 1500}{2 \cdot 40 \cdot 10} = \frac{1}{4}, \quad \alpha = \arccos(1/4) \approx 1.3181$$

\rightsquigarrow lengths L_P and L_Q of the circular segments of the V-belt from P to P' (angle of the sector = $2\pi - 2\alpha$) and from Q' to Q (angle of the sector = 2α)

$$\begin{aligned} L_P &= \text{angle of the sector} \cdot \text{radius} \\ &= (2\pi - 2\alpha)|\overline{AP}| \approx 36.470 \cdot 2 = 72.939 \\ L_Q &= 2\alpha|\overline{BQ}| \approx 26.362 \end{aligned}$$

Total length

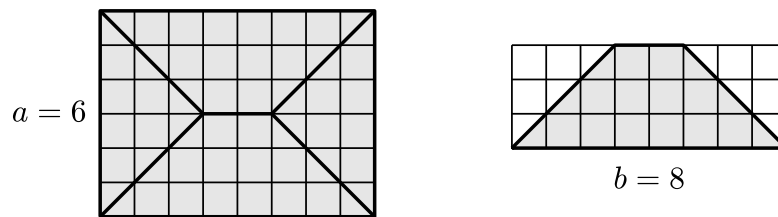
adding the lengths of the segments \rightsquigarrow length

$$2 \cdot L_t + L_P + L_Q \approx 77.460 + 72.939 + 26.362 = 176.761$$

of the V-belt

5.8 Surface Area and Enclosed Space of a Hipped Roof

The figure shows a hipped roof with a roof pitch of 45° degrees - view from above and from one side (lengths in meters).

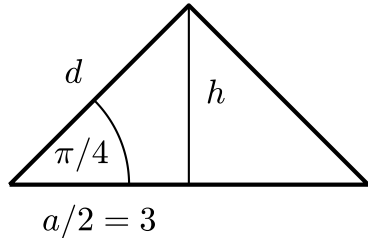


Determine the surface area of the roof and the enclosed space (volume of the attic).

Resources: [Trigonometric Theorems](#)

Solution

Cross-sectional area in the middle of the roof



height of the roof: $h = 3$ (view from the side)
isosceles right-angled cross-section \implies
distance of the roof edge and the roof ridge

$$d = \sqrt{(6/2)^2 + 3^2} = 3\sqrt{2} \quad (\text{Pythagoras})$$

Surface area of the roof

- triangles: height = d , computed as before \rightsquigarrow

$$F_1 = \frac{\text{base side} \times \text{height}}{2} = \frac{a \cdot d}{2} = \frac{6 \cdot 3\sqrt{2}}{2} = 9\sqrt{2}$$

- trapezoids:

$$\begin{aligned} F_2 &= \text{mean side length} \times \text{height} = \frac{b + (b - 2 \cdot 3)}{2} \cdot d \\ &= \frac{8 + (8 - 6)}{2} \cdot 3\sqrt{2} = 15\sqrt{2} \end{aligned}$$

\rightsquigarrow surface area of the roof: $2F_1 + 2F_2 = 18\sqrt{2} + 30\sqrt{2} = 48\sqrt{2} \approx 67.8823$

Enclosed space

- pyramids with right-angled base area:

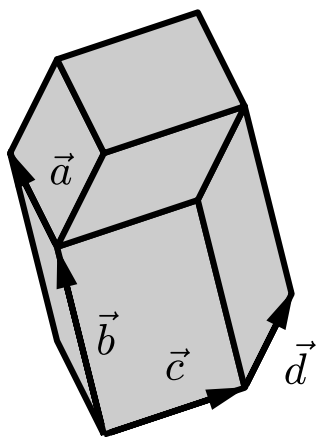
$$V_1 = \frac{\text{base area} \cdot \text{height}}{3} = \frac{(a \cdot 3) \cdot h}{3} = \frac{18 \cdot 3}{3} = 18$$

- prism with triangular base area (cross-section of the roof):

$$V_2 = \text{base area} \cdot \text{length} = (a \cdot h/2) \cdot (b - 2 \cdot 3) = (6 \cdot 3/2) \cdot 2 = 18$$

\rightsquigarrow volume: $V = 2V_1 + V_2 = 36 + 18 = 54$

5.9 Volume of a Union of Parallelepipeds



Compute the volume of the depicted solid^a

$$S : \alpha \vec{a} + \beta \vec{b} + \gamma \vec{c} + \delta \vec{d}, \quad 0 \leq \alpha, \beta, \gamma, \delta \leq 1,$$

with edges consisting of the vectors $\vec{a} = (1, 3, 0)^t$, $\vec{b} = (0, 1, 3)^t$, $\vec{c} = (3, 0, 1)^t$, and $\vec{d} = (2, 2, 2)^t$.

^aSuch solids arise as supports of multivariate B-splines (cf. C. de Boor, K. Höllig, and S. Riemenschneider: *Box-Splines*, Springer, 1993).

Resources: [Triple Product](#)

Solution

S : union of four parallelepipeds, each generated by three of the four vectors
volumes: absolute values of the corresponding triple products

- parallelepiped, generated by $\vec{a}, \vec{b}, \vec{c}$:

$$\begin{aligned} V_d &= |[\vec{a}, \vec{b}, \vec{c}]| = |\vec{a} \cdot (\vec{b} \times \vec{c})| \\ &= \left| \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \cdot \left(\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right) \right| = \left| \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 9 \\ -3 \end{pmatrix} \right| = 28 \end{aligned}$$

- parallelepiped, generated by $\vec{b}, \vec{c}, \vec{d}$:

$$V_a = \left| \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \cdot \left(\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} \right) \right| = \left| \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ -4 \\ 6 \end{pmatrix} \right| = 14$$

symmetry $\implies V_a = V_b = V_c = 14$

for example:

cyclic permutation of the coordinates of \vec{b} and \vec{c} ($1 \rightarrow 3, 2 \rightarrow 1, 3 \rightarrow 2$) \rightsquigarrow
 \vec{a} and \vec{b} , i.e. $V_a = V_c$

adding the values \rightsquigarrow

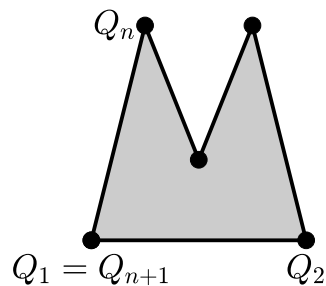
$$V = V_a + V_b + V_c + V_d = 3 \cdot 14 + 28 = 70$$

5.10 Area of a Polygonal Domain

Prove that the area of a domain D bounded by a polygon with vertices Q_k can be computed with the formula

$$\text{area } D = \frac{1}{2} \sum_{k=1}^n \vec{PQ}_k \times \vec{PQ}_{k+1},$$

where P is an arbitrary point, the vertices Q_k of the polygon are numbered counterclockwise, and the cross product of two dimensional vectors is defined as $\vec{u} \times \vec{v} = u_1v_2 - u_2v_1$.



Resources: [Cross Product](#)

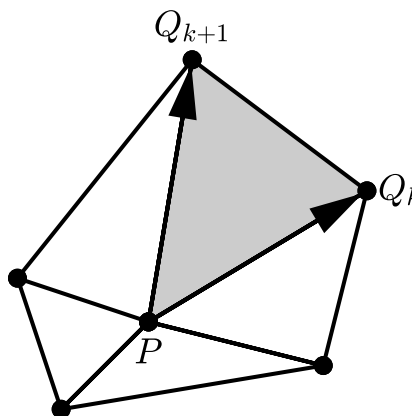
Solution

Motivation

If P is a point inside of the domain D and none of the line segments $\overline{PQ_k}$ intersects the bounding polygon, then the formula just adds the areas of the triangles since

$$\text{area } \Delta(P, Q_k, Q_{k+1}) = \frac{1}{2} \overrightarrow{PQ_k} \times \overrightarrow{PQ_{k+1}}.$$

Note that the order of the points (counter-clockwise orientation of the polygon) ensures that the cross product is positive, because the vectors $\overrightarrow{PQ_k}$, $\overrightarrow{PQ_{k+1}}$, and the normal of the xy -plane in the z -direction are oriented according to the right-hand rule.



It remains to show that the formula remains valid also when parts of the triangles $\Delta(P, Q_k, Q_{k+1})$ lie outside of the domain D .

Special case of a triangle

specializing the formula \rightsquigarrow

$$2\Delta(A, B, C) \stackrel{!}{=} \overrightarrow{PA} \times \overrightarrow{PB} + \overrightarrow{PB} \times \overrightarrow{PC} + \overrightarrow{PC} \times \overrightarrow{PA} \quad (1)$$

Using that $\overrightarrow{FH} = \overrightarrow{FG} + \overrightarrow{GH}$, $\overrightarrow{FH} = -\overrightarrow{HF}$, $\vec{u} \times \vec{u} = 0$, and $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$, the sum of cross products can be rewritten as

$$\begin{aligned} \underbrace{\overrightarrow{PB} \times (\overrightarrow{AP} + \overrightarrow{PC})}_{\text{first two cross products}} + \underbrace{(\overrightarrow{PA} + \overrightarrow{AC}) \times \overrightarrow{PA}}_{\overrightarrow{PC}} &= \overrightarrow{PB} \times \overrightarrow{AC} + \overrightarrow{AC} \times \overrightarrow{PA} \\ &= \overrightarrow{AC} \times (\overrightarrow{BP} + \overrightarrow{PA}) = \overrightarrow{AC} \times \overrightarrow{BA} = \overrightarrow{AB} \times \overrightarrow{AC}, \end{aligned}$$

i.e. the manipulations lead to the standard formula for the area of a triangle in terms of the cross product of two spanning vectors.

Union of triangles

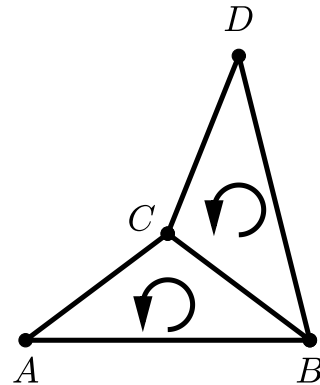
illustration for two triangles:

In view of the validity of the formula for a triangle, the area of the quadrilateral $\square(A, B, C, D)$ is the sum of the expressions

$$\frac{1}{2} \left(\vec{PA} \times \vec{PB} + \vec{PB} \times \vec{PC} + \vec{PC} \times \vec{PA} \right) \quad (1)$$

and

$$\frac{1}{2} \left(\vec{PB} \times \vec{PD} + \vec{PD} \times \vec{PC} + \vec{PC} \times \vec{PB} \right) \quad (2)$$

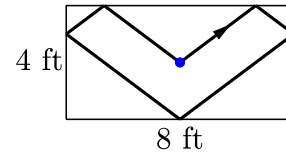


In view of the consistent orientation of the triangle, the two terms involving the points B and C have opposite signs and cancel. Hence the sum of the expressions (1) and (2) reduces to the four cross products of the asserted formula.

Generalizing the argument, we can prove the cross product formula using a triangulation of the polygonal domain, observing the cancellation of all cross products corresponding to inner edges.

5.11 Closed Paths of a Billiard Ball

The figure shows the path of a billiard ball starting from the midpoint of the pool table. Determine the points where the ball touches the cushions. How many different paths with five reflections exist?

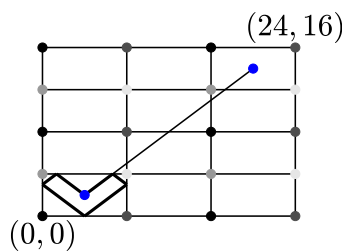


Resources: [Straight Line](#)

Solution

Surprisingly simple: Instead of reflecting the path of the billiard ball at the cushions of the pool table, the pool table is reflected at the cushion of contact and the path continues without changing its direction.

For the depicted path, the straight line segment obtained in this way leads to the midpoint $(20, 14)$ of the 5 times reflected pool table $[16, 24] \times [12, 16]$ (units in feet [ft]). The intersections with the rectangular grid correspond to the reflection points at the cushions.



point-slope form of the straight line segment, starting at the midpoint $(4, 2)$ of the pool table $[0, 8] \times [0, 4]$:

$$g : y = 2 + \frac{14 - 2}{20 - 4}(x - 4) = -1 + 3x/4$$

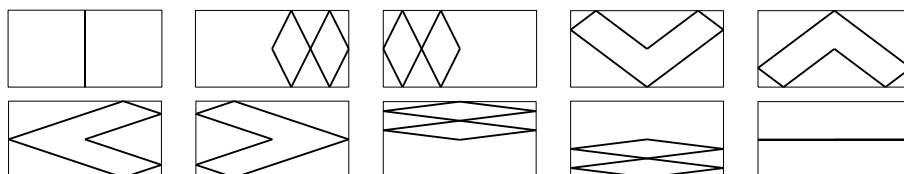
- intersection with the horizontal grid line $h : y = 4$:

$$4 = -1 + 3x/4, \quad \text{i.e. } x = 20/3$$

\rightsquigarrow reflection point $(20/3, 4)$ at the upper cushion

- intersection with the vertical grid line $v : x = 8$: $y = -1 + 3 \cdot 8/4 = 5$
reflection at $h \rightsquigarrow$ reflection point $(8, 3)$ at the right cushion
- symmetry \rightsquigarrow further reflection points at the lower, left, and upper cushions: $(4, 0)$, $(0, 3)$, $(4/3, 4)$

All possible paths with 5 reflections starting at the midpoint of the pool table are obtained by connecting the midpoints of the relevant multiple reflected pool tables, without distinguishing different path directions and not considering paths which lead to corners of the pool table.



5.12 Lines of Equal Distance

Determine the set of points P (union of 4 straight lines) which have equal distance from the straight lines

$$g_a : t \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad g_b : t \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad g_c : t \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}.$$

Resources: [Distance Point-Line](#)

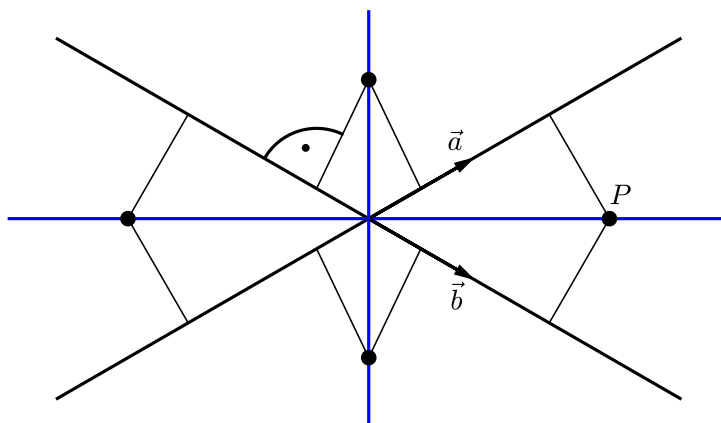
Solution

The two-dimensional problem

The points with equal distance from two lines

$$g_a : t \vec{a}, \quad g_b : t \vec{b}$$

lie on the two angle bisectors of g_a and g_b . There exist four points P with a prescribed distance.



Regarded as lines in three dimensions, points with equal distance from g_a and g_b lie on the planes which contain an angle bisector and which are perpendicular to the plane spanned by \vec{a} and \vec{b} . For a third line g_c , forming intersections with the planes corresponding to the pairs g_a, g_c and g_b, g_c yields lines of equal distance. This seem complicated. But, the analytical solution, explained below, is surprisingly simple.

Lines of equal distance

distance of a point P from the line $g_a : t \vec{a}$ with a normalized direction vector ($|\vec{a}| = 1$):

$$d_a = |\vec{p} \times \vec{a}| = |\vec{p}| \sin \angle(\vec{p}, \vec{a})$$

by definition of the cross product

using $\sin^2 + \cos^2 = 1$ and computing the cosine via a scalar product \rightsquigarrow

$$d_a = |\vec{p}| \sqrt{1 - (\vec{p} \cdot \vec{a} / |\vec{p}|)^2} = \sqrt{|\vec{p}|^2 - (\vec{p} \cdot \vec{a})^2}$$

Hence, the points P with equal distance from the three lines g_a, g_b, g_c with normalized direction vectors $\vec{a}, \vec{b}, \vec{c}$ satisfy $d_a = d_b = d_c$, i.e.

$$\sqrt{|\vec{p}|^2 - (\vec{p} \cdot \vec{a})^2} = \sqrt{|\vec{p}|^2 - (\vec{p} \cdot \vec{b})^2} = \sqrt{|\vec{p}|^2 - (\vec{p} \cdot \vec{c})^2}.$$

Since $|\vec{p} \cdot \vec{v}| \leq |\vec{p}|$, the difference $|\vec{p}|^2 - (\vec{p} \cdot \vec{v})^2$ is non-negative for $\vec{v} = \vec{a}, \vec{b}, \vec{c}$, and the equation can be squared without gaining further solutions:

$$(\vec{p} \cdot \vec{a})^2 = (\vec{p} \cdot \vec{b})^2 = (\vec{p} \cdot \vec{c})^2 \iff \vec{p} \cdot \vec{a} = \sigma_b \vec{p} \cdot \vec{b} = \sigma_c \vec{p} \cdot \vec{c}.$$

Depending on the choice of the signs $\sigma \in \{-1, +1\}$, one obtains 4 pairs of linear equations for \vec{p} .

- $\sigma_b = \sigma_c = -1$:

$$\vec{p} \cdot (\vec{a} + \vec{b}) = 0, \quad \vec{p} \cdot (\vec{a} + \vec{c}) = 0 \iff \vec{p} \parallel (\vec{a} + \vec{b}) \times (\vec{a} + \vec{c}),$$

i.e. P lies on the straight line $g_{--} : t(\vec{a} + \vec{b}) \times (\vec{a} + \vec{c})$

- $\sigma_b = -1, \sigma_c = 1 \rightsquigarrow g_{-+} : t(\vec{a} + \vec{b}) \times (\vec{a} - \vec{c})$
- $\sigma_b = 1, \sigma_c = -1 \rightsquigarrow g_{+-} : t(\vec{a} - \vec{b}) \times (\vec{a} + \vec{c})$
- $\sigma_b = \sigma_c = 1 \rightsquigarrow g_{++} : t(\vec{a} - \vec{b}) \times (\vec{a} - \vec{c})$

computing the cross product with the normalized direction vectors

$$\vec{a} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \vec{b} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{c} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

for the first choice $\sigma_b = \sigma_c = -1 \rightsquigarrow$

$$\begin{aligned} & (\vec{a} + \vec{b}) \times (\vec{a} + \vec{c}) \\ &= \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right) \times \left(\frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right) \\ &= \frac{1}{5} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -7 \\ -1 \\ 5 \end{pmatrix} \end{aligned}$$

\rightsquigarrow line of equal distance $g_{--} : t(-1.4, -0.2, 1)^t$

The other lines of equal distance are:

$$g_{-+} : t \begin{pmatrix} 1 \\ -1.4 \\ -0.2 \end{pmatrix}, \quad g_{+-} : t \begin{pmatrix} -0.2 \\ 1 \\ -1.4 \end{pmatrix}, \quad g_{++} : t \begin{pmatrix} 0.6 \\ 0.6 \\ 0.6 \end{pmatrix}.$$

5.13 Distance of Flight Corridors

An airplane takes off in north-easterly direction; a second airplane from an airport 50 km to the north in easterly direction. Both airplanes have the same velocity and an ascent rate of 10 %. Determine a lower bound for the minimal distance of the two airplanes for

- a) the same b) arbitrary

take-off times.

Resources: [Straight Line](#), [Distance of Straight Lines](#)

Solution

Coordinates of the airports and flight directions

y -axis pointing north \rightsquigarrow

$$P = (0, 0, 0), \vec{u} = (\sqrt{2}/2, \sqrt{2}/2, 1/10)^t, \quad Q = (0, 50, 0), \vec{v} = (1, 0, 1/10)^t$$

For an ascent rate of 10 %, moving a unit distance horizontally, the z -coordinate changes by 1/10.

flight paths (point-direction form of straight lines):

$$\vec{x}(t) = \vec{p} + t\vec{u}, \quad \vec{y}(s) = \vec{q} + s\vec{v}$$

The common velocity of the airplanes is irrelevant for the distance computation, hence chosen as $|\vec{u}| = |\vec{v}|$ for convenience.

Equal take-off times

distance of the airplanes:

$$d(t)^2 = |\vec{p} - \vec{q} + t(\vec{u} - \vec{v})|^2 = (t\sqrt{2}/2 - t)^2 + (t\sqrt{2}/2 - 50)^2$$

quadratic function of t which is minimal if

$$0 = \frac{d}{dt}d(t)^2 = 2t(\sqrt{2}/2 - 1)^2 + t - 50\sqrt{2} \iff t_{\min} = \frac{50\sqrt{2}}{4 - 2\sqrt{2}} = 25(\sqrt{2} + 1)$$

substituting t_{\min} into the expression for $d(t)^2$ \rightsquigarrow

$$d_{\min}^2 = \left(25(\sqrt{2} + 1)(\sqrt{2}/2 - 1)\right)^2 + \left(25(\sqrt{2} + 1)\sqrt{2}/2 - 50\right)^2$$

and after simplification

$$d_{\min} = 25\sqrt{2 - \sqrt{2}} \approx 19.1342$$

Arbitrary take-off times

computation of the minimal distance of the straight lines d and h using the triple product \rightsquigarrow

$$d = \frac{|[\vec{p} - \vec{q}, \vec{u}, \vec{v}]|}{|\vec{u} \times \vec{v}|} = \frac{|(\vec{p} - \vec{q}) \cdot \vec{n}|}{|\vec{n}|}, \quad \vec{n} = \vec{u} \times \vec{v}$$

substituting the given values \rightsquigarrow

$$\vec{p} - \vec{q} = (0, -50, 0)^t$$

$$\vec{n} = (\sqrt{2}/2, \sqrt{2}/2, 1/10)^t \times (1, 0, 1/10)^t = \frac{\sqrt{2}}{20}(1, \sqrt{2} - 1, -10)^t$$

and

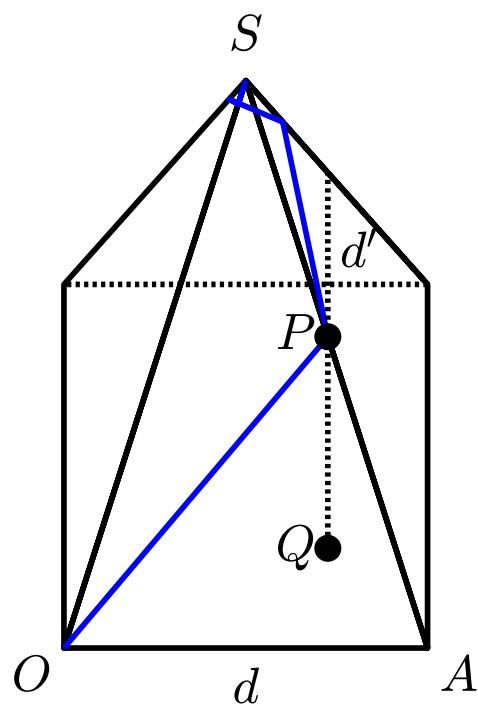
$$d = \frac{50(\sqrt{2} - 1)}{\sqrt{1 + (\sqrt{2} - 1)^2 + 100}} \stackrel{(\star)}{=} \frac{50}{\sqrt{304 + 202\sqrt{2}}} \approx 2.0590,$$

where, for (\star) , the fraction was expanded with $\sqrt{2} + 1$, noting that $(\sqrt{2} - 1)(\sqrt{2} + 1) = 1$

5.14 Path to the Tip of a Pyramid

Determine the length of a piecewise straight path (blue line segments) with a constant slope of 50% from a corner of a quadratic pyramid to its tip as a function of the length d of the edges (black line segments).

Note that the slope of the path is defined as $|\overline{QP}|/|\overline{OQ}|$ with Q the projection of P onto the base square of the pyramid.



Resources: [Straight Line](#), [Trigonometric Theorems](#)

Solution

Height H of the pyramid

$$A = (d, 0, 0), S = (d/2, d/2, H), |\overline{AS}| = d \implies$$

$$\underbrace{(d/2 - d)^2 + (d/2)^2 + H^2}_{|\overline{AS}|^2} = d^2 \iff d^2/2 + H^2 = d^2,$$

$$\text{i.e. } H = d/\sqrt{2}$$

First segment \overline{OP} of the path

$$P \in \overline{AS} \implies$$

$$\vec{p} = \vec{a} + t(\vec{s} - \vec{a}) = \begin{pmatrix} d \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -d/2 \\ d/2 \\ d/\sqrt{2} \end{pmatrix} = \begin{pmatrix} d - td/2 \\ td/2 \\ td/\sqrt{2} \end{pmatrix}$$

$$\text{slope} = 50 \% \iff (\text{height reached}):(\text{horizontal distance}) = 50 : 100, \text{ i.e.}$$

$$|\overline{QP}| : |\overline{OQ}| = 1 : 2 \quad (1)$$

with $Q = (d - td/2, td/2, 0)$ the projection of P onto the xy -plane
resulting equation:

$$\frac{1}{2} = \frac{|\overline{QP}|}{|\overline{OQ}|} = \frac{td/\sqrt{2}}{\sqrt{(d - td/2)^2 + (td/2)^2}} = \frac{t}{\sqrt{2(1 - t/2)^2 + 2(t/2)^2}}$$

squaring and simplifying \rightsquigarrow

$$3t^2 + 2t - 2 = 0$$

with the relevant positive solution $t = (\sqrt{7} - 1)/3$
height reached at the end of the first path segment:

$$h_1 = |\overline{QP}| = \frac{t}{\sqrt{2}} d = \frac{\sqrt{7} - 1}{3\sqrt{2}} d$$

(1) $\implies |\overline{OQ}| = 2h_1$ and applying the theorem of Pythagoras to the right-angled triangle $\Delta(O, Q, P)$ \rightsquigarrow length of the first path segment

$$L_1 = |\overline{OP}| = \sqrt{(2h_1)^2 + h_1^2} = \sqrt{5} h_1 = \frac{\sqrt{5}(\sqrt{7} - 1)}{3\sqrt{2}} d$$

Total length

The length of the second segment can be computed in the same way, applying the arguments to a pyramid with height

$$H' = H - h_1 = \frac{1}{\sqrt{2}} d - \frac{\sqrt{7} - 1}{3\sqrt{2}} d = \underbrace{\frac{4 - \sqrt{7}}{3}}_{=:q} \underbrace{\frac{1}{\sqrt{2}}}_{H} d .$$

By similarity, all lengths are reduced by the same factor q . In particular, $d' = qd$, and $L_2 = qL_1$ is the length of the second path segment. Hence, by the formula for a geometric series, the total length is

$$\begin{aligned} L &= L_1 + qL_1 + q^2L_1 + \cdots = L_1 \sum_{k=0}^{\infty} q^k = \frac{L_1}{1 - q} \\ &= \frac{\sqrt{5}(\sqrt{7} - 1)}{3\sqrt{2}} d / \underbrace{\left(1 - \frac{4 - \sqrt{7}}{3}\right)}_{\frac{\sqrt{7}-1}{3}} = \frac{\sqrt{5}}{\sqrt{2}} d \end{aligned}$$

5.15 Designing a Problem with MATLAB[®]

Write a MATLAB[®] program which generates randomly vectors for the following problem:

Determine the coefficients $c = (c_u, c_v, c_w)$ of the vector \vec{x} with respect to the basis $\{\vec{u}, \vec{v}, \vec{w}\}$.

The objective is that all vectors involved should have integer coordinates less than 10 to avoid tedious calculations². Moreover, other additional esthetic requirements could be imposed.

Resources: [Vector Operations with MATLAB[®]](#)

²Can I bring a pocket calculator to the exam? The answer was: Not necessary; All numbers involved will be reasonably SMALL!

Solution

```
function problem_data(n_samples)

% open a data file for writing
fid = fopen('problem_data.txt','w');
for k=1:n_samples
    % 12 random integers in {-9,-8,...,9}
    R = fix(19*rand(12,1))-9;
    % basis
    u = R(1:3); v = R(4:6); w = R(7:9);
    % coefficients and corresponding vector x
    c = R(10:12); x = [u v w]*c;
    % determinants appearing in Cramer's rule
    D = [det([u,v,w]);det([x,u,v]);det([u,x,w]);det([u,v,x])];
    % avoid zeros which would lead to simplification
    % enforce the bound 9 on all determinants and coordinates
    if prod([D;u;v;w;c;x])~=0 & norm([D;x],inf) <= 9
        [u v w c x]
        % write admissible data to the data file
        fprintf(fid,'u=[%d;%d;%d], v=[%d;%d;%d], w=[%d;%d;%d]\n',u,v,w);
        fprintf(fid,'c=[%d;%d;%d], x=[%d;%d;%d]\n\n',c,x);
    end
end
% close the data file
fclose(fid);

end
```

With `n_samples = 100000`, the program finds the following data sets.

```
u=[3;2;-1], v=[-4;-7;2], w=[-9;6;1]
c=[-6;-3;-1], x=[3;3;-1]
```

```
u=[8;2;-5], v=[-8;-7;7], w=[-6;5;1]
c=[-2;-1;-1], x=[-2;-2;2]
```

```
u=[-4;3;7], v=[-9;2;6], w=[1;2;4]
c=[-4;2;3], x=[1;-2;-4]
```



```
u=[2;4;-2] , v=[6;-8;-2] , w=[-8;9;3]
c=[1;3;3] , x=[-4;7;1]
```

```
u=[-9;-4;-2] , v=[-8;4;1] , w=[-7;-7;-3]
c=[-8;3;6] , x=[6;2;1]
```

```
u=[5;2;-1] , v=[2;-5;-5] , w=[-2;-4;-2]
c=[-1;2;-2] , x=[3;-4;-5]
```

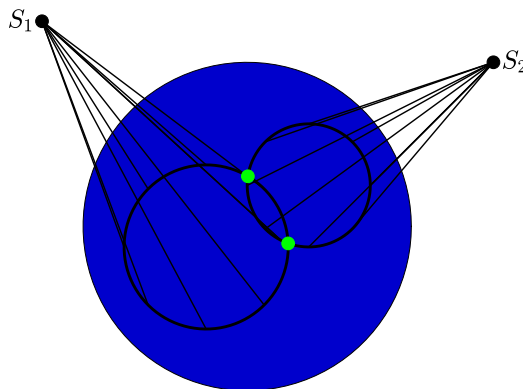
Clearly, enlarging the bound 9 and allowing one or several zeros would substantially increase the possible choices. On the other hand, one can impose more esthetic restrictions. For example, one can require that no repetitions occur. This is accomplished by using the MATLAB[®] function `R = randperm(n)` which returns a random permutation of $\{1, \dots, n\}$. Hence, the commands

```
R = randperm(19)'; R = -9+R(1:12),
```

yield 12 different integers with absolute value less than 10 to define the vectors \vec{u} , \vec{v} , \vec{w} , and \vec{c} .

5.16 Intersection of Spheres with MATLAB[®]

Write a MATLAB[®] program which determines the location P on the surface of the earth from the distances d_j to two satellites S_j ³. You may exclude degenerate configurations, i.e. you may assume that two distinct solutions P_1, P_2 (green dots) exist.



Test your program for two satellites, circulating at a height of 20000 km, and for distances $d_1 = 21633$ km, $d_2 = 20659$ km corresponding to the positions $S_1 = (40.72, -74.02)$ (New York), $S_2 = (48.86, 2.34)$ (Paris) in decimal GPS coordinates. Use $R = 6367$ km as average radius of the earth.

Resources: [Vector Operations with MATLAB[®]](#)

³This problem illustrates the basic idea of GPS: determination of positions on the earth from distance measurements to satellites. GPS algorithms are considerably more complicated than the simple example of the problem, in particular, because of inaccuracies in time measurements, which are needed for the distance computations. Data from at least four satellites are needed.

Solution

Outline of the computations

- Conversion to cartesian coordinates:
GSP coordinates

$$(\Theta, \Phi) \in [-90, 90] \times [-180, 180]$$

\rightsquigarrow spherical coordinates

$$\vartheta = \Theta \cdot (\pi/2)/90, \quad \varphi = \Phi \cdot \pi/180$$

and cartesian coordinates⁴

$$x = r \cos \vartheta \cos \varphi, \quad y = r \cos \vartheta \sin \varphi, \quad z = r \sin \vartheta,$$

where r is the distance to the center of the earth ($r = 20000 + R$, with $R = 6367$ the radius of the earth, for the two satellites S_j of the problem)

cartesian coordinates of S_j :

$$S = \begin{pmatrix} S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} 5502 & -19212 & 17201 \\ 17332 & 708 & 19857 \end{pmatrix}$$

- Intersection of the spheres corresponding to the satellites with the surface of the earth:
The points $P = (p_1, p_2, p_3)$ with distance d_j from the satellite S_j satisfy

$$\sum_{k=1}^3 (p_k - s_{j,k})^2 = d_j^2, \quad j = 1, 2$$

(equations of spheres with radii d_j and centers $(s_{j,1}, s_{j,2}, s_{j,3})$)
subtracting the equation for the surface of the earth,

$$\sum_{k=1}^3 p_k^2 = R^2, \tag{1}$$

⁴Note that, in accordance with MATLAB[®] syntax, ϑ is the elevation angle, i.e. the angle between the position vector and the xy -plane, and not the angle formed with the z -axis.

and rearranging terms \rightsquigarrow

$$\sum_{k=1}^3 s_{j,k} p_k = \underbrace{\left(\sum_{k=1}^3 s_{j,k}^2 + R^2 - d_j^2 \right)}_{b_j} / 2, \quad j = 1, 2$$

The solution set of these two linear equations for the three unknowns p_k , which can be interpreted as the intersection of two planes, is a straight line:

$$p_k = q_k + t u_k \quad (2)$$

with q a particular solution and u perpendicular to $(s_{1,1}, s_{1,2}, s_{1,3})$, $(s_{2,1}, s_{2,2}, s_{2,3})$, e.g. u equal to the cross product of these vectors.

substituting (2) into the equation (1) \rightsquigarrow quadratic equation for the line parameter t :

$$\left(\sum_k u_k^2 \right) t^2 + \left(2 \sum_k q_k u_k \right) t + \left(\sum_k q_k^2 - R^2 \right) = 0$$

solutions t_1, t_2 \rightsquigarrow cartesian coordinates of two intersections P_1, P_2

MATLAB[®] program

```
% satellite positions
GPS1 = [40.72, -74.02]; % New York
GPS2 = [48.86, 2.34]; % Paris
r = 20000+6367; % sum of orbit height and earth radius

% distances
d = [21633; 20659];

% conversion to spherical coordinates
S_spherical = [GPS1([2 1])*pi/180, r; GPS2([2 1])*pi/180, r];

% conversion to cartesian coordinates
[x,y,z] = sph2cart(S_spherical(:,1),S_spherical(:,2),S_spherical(:,3));
S = [x(:),y(:),z(:)];

% intersection of the 3 spheres
% positions determined by the satellite distances
```

```

P = intersect_spheres(S,d);

% conversion to spherical and then to GPS coordinates
[phi,theta,r] = cart2sph(P(1,:),P(2,:),P(3,:));
GPS_P = [theta*180/pi; phi*180/pi]'

% ----

function [P] = intersect_spheres(S,d)
% solves the equations
% |P| = R, |P-S(j,:)| = d(j), j=1,2

    R = 6367;    % earth radius

    % right side of the difference of the sphere equations
    b = ((sum((S.^2)'))'+R^2-d.^2)/2;

    % particular solution of S*p=b and solution of S*p=[0;0]
    q = S\b; u = cross(S(1,:)',S(2,:)' );

    % coefficients of the equation c(1)*t^2+c(2)*t+c(3)=0
    % resulting from |p|^2 - R = 0 with p = q+t*u
    c = [u'*u, 2*q'*u, q'*q-R^2];

    % zeros of the quadratic equation
    t = roots(c);

    % points of intersection
    P = [q+t(1)*u, q+t(2)*u];
end

```

GPS coordinates of the positions determined by the distance measurements (rows of the array `GPS_P` computed with the MATLAB[®] script):

$$P_1 : (37.73, -25.67), \quad P_2 : (66.96, -25.71).$$

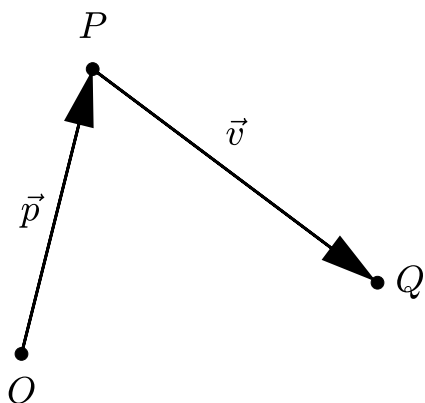
Given a choice, you will prefer *Ponta Delgada* on the Azores over a hostile location in the midst of the *North Atlantic*.

Chapter 6

Lexicon

6.1 Vectors

Points and Vectors



$$P = (p_1, p_2, p_3)$$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \overrightarrow{PQ} = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$$

position vector of P : $\vec{p} = \overrightarrow{OP}$

transposition of vectors^a:

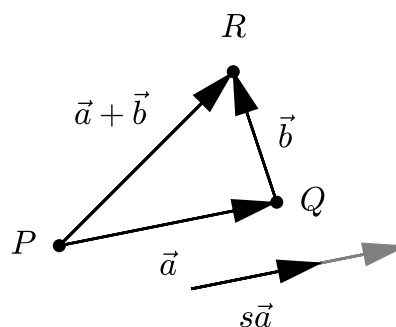
$$\vec{v}^t = (v_1, v_2, v_3), \quad \vec{v} = (v_1, v_2, v_3)^t$$

^aconvenient in text passages to avoid excessive height of lines and essential for matrix/vector operations in linear algebra

Addition and Scalar Multiplication of Vectors

$$\vec{a} + \vec{b} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}, \quad \overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR}$$

$$s\vec{a} = \begin{pmatrix} sa_1 \\ sa_2 \\ sa_3 \end{pmatrix}$$



Norm

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{\vec{a} \cdot \vec{a}}$$

Triangle Inequalities: $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|, \quad ||\vec{a}| - |\vec{b}|| \leq |\vec{a} - \vec{b}|$

Unit Vector: $\vec{a}^\circ = \vec{a}/|\vec{a}|$

Vector Operations with Matlab

```
v = [1; 2; 3] % definition
% transposition
v_t = v', v_t = [1, 2, 3]
addition, subtraction, and scalar multiplication
v_p = v1+v2, v_m = v1-v2, v_s = s*v
% norm
v_norm = norm(v)

% special vectors
[1:3] % -> [1, 2, 3]
ones(3,1), zeros(3,1) % -> [1; 1; 1], [0; 0; 0]
v([3, 3, 1]) % -> [3; 3; 1]
```

Vector Operations with Maple™

```
with(LinearAlgebra) # loading relevant commands
# various vector definitions
v := <1,2,3>; v := Vector([1,2,3]); v := Vector(3,k->k);
# vector with symbolic entries v_1,v_2,v_3
v_s := Vector(3,symbol=v)
# transposition (several alternatives)
v_t := Transpose(v); v_t := <1|2|3>; v_t := Vector[row]([1,2,3]);
# addition an scalar multiplication
v_p := v1+v2; v_m := v1-v2; v_s := s*v
# norm
v_norm := norm(v,2)
```

Coordinate Transformation

$(x_1, x_2) \rightarrow (x'_1, x'_2)$:

- **Translation** by (p_1, p_2) : $x'_1 = x_1 - p_1, x'_2 = x_2 - p_2$
- **Rotation** by φ : $x'_1 = \cos \varphi x_1 + \sin \varphi x_2, x'_2 = -\sin \varphi x_1 + \cos \varphi x_2$

Coordinate Systems

- **Cylindrical coordinates:** $(x, y, z) \sim (\varrho, \varphi, z)$, $\varrho \geq 0$

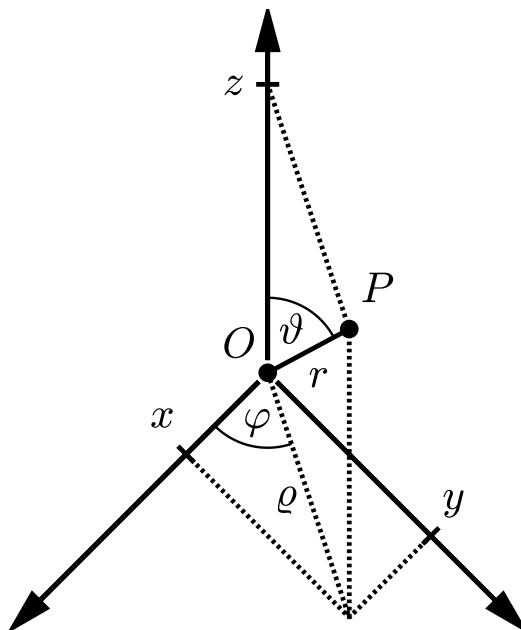
$$x = \varrho \cos \varphi, \quad y = \varrho \sin \varphi, \quad \varrho = \sqrt{x^2 + y^2}, \quad \varphi = \arctan(x/y) + \sigma\pi$$

- **Spherical coordinates:** $(x, y, z) \sim (r, \vartheta, \varphi)$, $r \geq 0, 0 \leq \vartheta \leq \pi$

$$x = r \sin \vartheta \cos \varphi, \quad y = r \sin \vartheta \sin \varphi, \quad z = r \cos \vartheta$$

$$r = \sqrt{x^2 + y^2 + z^2}, \quad \vartheta = \arccos(z/r), \quad \varphi = \arctan(x/y) + \sigma\pi$$

$\arctan : [-\infty, \infty] \rightarrow [-\pi/2, \pi/2] \rightsquigarrow$ necessity of adding $\sigma\pi$:
 $\sigma = 0$ for $x \geq 0$, $\sigma = 1$ for $x < 0 \wedge y \geq 0$ (second quadrant), and $\sigma = -1$ for $x < 0 \wedge y < 0$ (third quadrant) $\rightsquigarrow \varphi \in (-\pi, \pi]$ (standard interval)
 addition of 2π for $\varphi \in (-\pi, 0)$ if the alternative standard interval $[0, 2\pi)$ is used



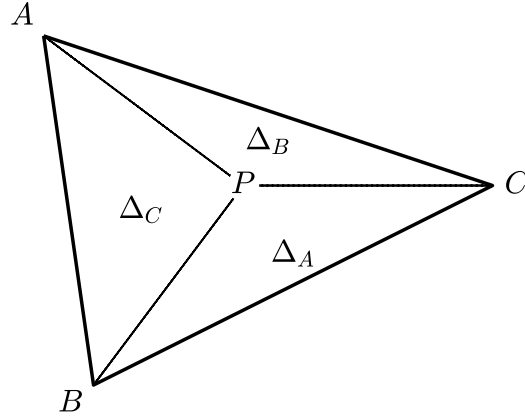
Polar coordinates: $(x, y) \sim (r, \varphi)$, $r \geq 0 \hat{=} \text{cylindrical or spherical coordinates with } z = 0 \hat{=} \vartheta = \pi/2 \text{ and } \varrho = r$:

$$x = r \cos \varphi, \quad y = r \sin \varphi$$

Barycentric Coordinates

$$P = \frac{\Delta_A A + \Delta_B B + \Delta_C C}{\Delta_A + \Delta_B + \Delta_C}$$

where Δ_* are the signed areas ($\Delta_A < 0$ if A and P lie on opposite sides of the edge \overline{BC} with the analogous convention for Δ_B and Δ_C) of the triangles determined by P



Coordinate Transformations with MATLAB[®]

- **Polar coordinates:**

$$\begin{aligned} [\text{phi}, \text{r}] &= \text{cart2pol}(x, y) \\ [x, y] &= \text{pol2cart}(\text{phi}, \text{r}) \end{aligned}$$

With the z -coordinate left unchanged, the commands can also be used for the conversion to cylindrical coordinates.

- **Spherical coordinates:**

$$\begin{aligned} [\text{phi}, \text{theta}, \text{r}] &= \text{cart2sph}(x, y, z) \\ [x, y, z] &= \text{cart2sph}(\text{phi}, \text{theta}, \text{r}) \end{aligned}$$

Note that MATLAB[®] defines ϑ as the elevation angle from the xy -plane, i.e. the north (south) pole corresponds to $\vartheta = \pi/2$ ($\vartheta = -\pi/2$).

6.2 Scalar Product

Scalar Product

$$\vec{a} \cdot \vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}| |\vec{b}| \underbrace{\cos \angle(\vec{a}, \vec{b})}_{\varphi}$$

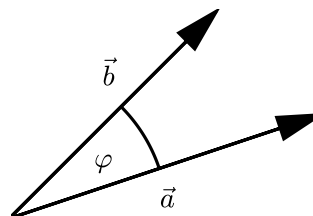
linear: $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$, $\vec{a} \cdot (s\vec{b}) = s\vec{a} \cdot \vec{b}$

symmetric: $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$,

$\vec{a} \cdot \vec{b} = 0 \iff \vec{a} \perp \vec{b}$

special values of $\cos \varphi$

φ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
$\cos \varphi$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	1/2	0



$$\cos \varphi = \cos(-\varphi) = -\cos(\pi - \varphi)$$

MATLAB[®] :

$$a_dot_b = a' * b$$

$$phi = \text{acos}(a_dot_b / (\text{norm}(a) * \text{norm}(b)))$$

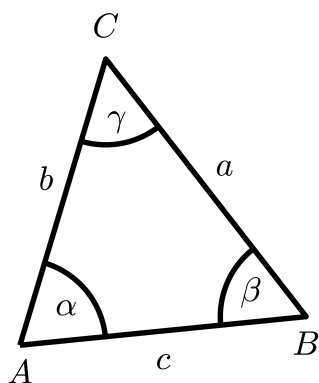
Maple[™] :

`with(LinearAlgebra)`

`a_dot_b := DotProduct(a,b)`

`phi := arccos(a_dot_b / (norm(a,2)*norm(b,2)))`

Trigonometric Theorems



- **Theorem of Pythagoras:**

$$\gamma = \pi/2 \implies c^2 = a^2 + b^2$$

- **Law of Sines:**

$$\sin \alpha : \sin \beta = a : b$$

In particular, for similar triangles, i.e. triangles with the same angles, the ratios of corresponding side lengths are equal.

- **Law of Cosines:**

$$c^2 = a^2 + b^2 - 2ab \cos \gamma$$

Orthogonal Basis

$$\vec{u} \perp \vec{v} \perp \vec{w} \perp \vec{u}$$

$$\begin{aligned}\vec{a} &= \frac{\vec{a} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} + \frac{\vec{a} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \vec{v} + \frac{\vec{a} \cdot \vec{w}}{\vec{w} \cdot \vec{w}} \vec{w} \\ |\vec{a}|^2 &= \frac{|\vec{a} \cdot \vec{u}|^2}{|\vec{u}|^2} + \frac{|\vec{a} \cdot \vec{v}|^2}{|\vec{v}|^2} + \frac{|\vec{a} \cdot \vec{w}|^2}{|\vec{w}|^2}\end{aligned}$$

Orthonormal Basis: $|\vec{u}| = |\vec{v}| = |\vec{w}| = 1$

\rightsquigarrow formulas without denominators

6.3 Cross Product and Triple Product

Cross Product

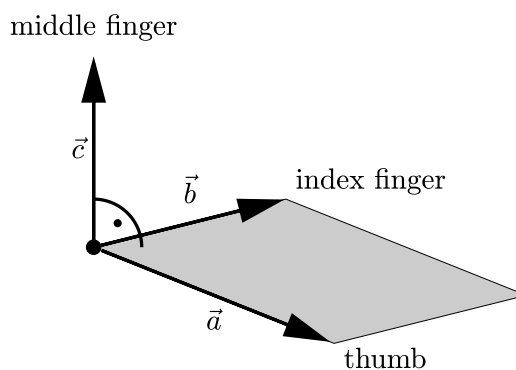
$$\vec{c} = \vec{a} \times \vec{b} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}, \quad |\vec{c}| = |\vec{a}||\vec{b}| \sin \angle(\vec{a}, \vec{b})$$

Geometric Interpretation:

$$\vec{c} \perp \vec{a}, \vec{b}$$

oriented according to the right-hand rule

$|\vec{c}|$: area of the parallelogram spanned by \vec{a} and \vec{b}



linear with respect to \vec{a} and \vec{b} , antisymmetric: $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$ and, in particular, $\vec{a} \times \vec{a} = \vec{o} = (0, 0, 0)^t$, $\vec{a} \times \vec{b} = \vec{o} \iff \vec{a} \parallel \vec{b}$

- **Grassmann's identity:** $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$
- **Lagrange's identity:** $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{a} \cdot \vec{c})(\vec{b} \cdot \vec{d}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

For two-dimensional vectors, it is convenient to define $\vec{u} \times \vec{v} = u_1v_2 - u_2v_1$.

MATLAB[®] :

```
a_cross_b = cross(a,b)
```

Maple[™] :

```
a_cross_b := CrossProduct(a,b)
```

Epsilon Tensor

$$\varepsilon_{1,2,3} = \varepsilon_{2,3,1} = \varepsilon_{3,1,2} = 1, \quad \varepsilon_{1,3,2} = \varepsilon_{2,1,3} = \varepsilon_{3,2,1} = -1, \quad \text{otherwise} = 0$$

representation of the cross product $\vec{c} = \vec{a} \times \vec{b}$: $c_j = \sum_{k,\ell} \varepsilon_{j,k,\ell} a_k b_\ell$

Triple Product

$$\begin{aligned} [\vec{a}, \vec{b}, \vec{c}] &= \vec{a} \cdot (\vec{b} \times \vec{c}) = \underbrace{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}_{\text{determinant}} = \sum_{j,k,\ell} \varepsilon_{j,k,\ell} a_j b_k c_\ell \\ &= a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - a_1 c_2 b_3 - b_1 a_2 c_3 - c_1 b_2 a_3 \end{aligned}$$

- multilinear (linear with respect to \vec{a} , \vec{b} , and \vec{c})
- cyclic: $\vec{a} \rightarrow \vec{b} \rightarrow \vec{c} \rightarrow \vec{a}$,
- sign change after permutation
- $[\vec{a}, \vec{b}, \vec{c}] = 0 \iff \vec{a}, \vec{b}, \vec{c}$ linearly dependent
- absolute value: volume of a parallelepiped (= 6 · volume of a tetrahedron) spanned by \vec{a} , \vec{b} , and \vec{c}

MATLAB[®] :

```
s = a'*cross(b,c)
s = det([a, b, c]) % alternative
```

Maple[™] :

```
with(LinearAlgebra)
s := DotProduct(a,CrossProduct(b,c))
s := Determinant(<a|b|c>) # alternative
```

Basis Coefficients

For a basis $\{\vec{u}, \vec{v}, \vec{w}\}$, any vector \vec{a} can be represented as linear combination $\vec{a} = r\vec{u} + s\vec{v} + t\vec{w}$ with

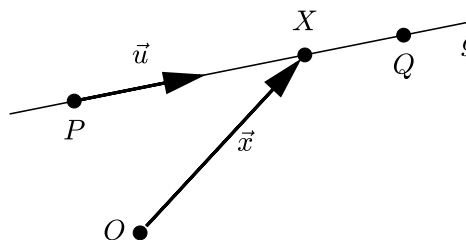
$$r = \frac{[\vec{a}, \vec{v}, \vec{w}]}{[\vec{u}, \vec{v}, \vec{w}]}, \quad s = \frac{[\vec{u}, \vec{a}, \vec{w}]}{[\vec{u}, \vec{v}, \vec{w}]}, \quad t = \frac{[\vec{u}, \vec{v}, \vec{a}]}{[\vec{u}, \vec{v}, \vec{w}]}.$$

6.4 Lines and Planes

Straight Line

different representations of points X on a straight line g

- **Point-Direction Form:** $\vec{x} = \vec{p} + t\vec{u}$
- **Two-Point Form:** $\vec{x} = \vec{p} + t(\vec{q} - \vec{p})$
- **Moment Form:** $\vec{x} \times \vec{u} = \vec{p} \times \vec{u}$

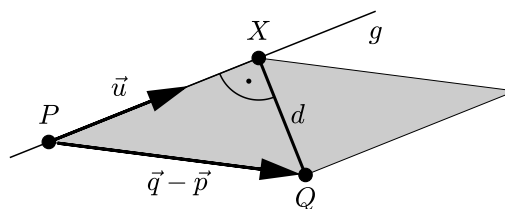


Distance Point-Line

$g : \vec{p} + t\vec{u}$, X : projection of a point Q on g , i.e. $(\vec{x} - \vec{q}) \perp \vec{u}$

$$\underbrace{\text{dist}(Q, g)}_d = |\overrightarrow{QX}| = \frac{|\overbrace{(\vec{q} - \vec{p})}^{\vec{v}} \times \vec{u}|}{|\vec{u}|} = |\vec{v}| \sin \angle(\vec{v}, \vec{u}) = \sqrt{|\vec{v}|^2 - (\vec{v} \cdot \vec{u}^\circ)^2}$$

$$\vec{x} = \vec{p} + \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|^2} \vec{u}$$



shaded area: norm of the cross product

Distance of Straight Lines

$g : \vec{x} = \vec{p} + s\vec{u}$, $h : \vec{y} = \vec{q} + t\vec{v}$

$$\text{dist}(g, h) = \frac{|[\vec{q} - \vec{p}, \vec{u}, \vec{v}]|}{|\vec{u} \times \vec{v}|}, \quad = |(\vec{q} - \vec{p}) \times \vec{u}| / |\vec{u}| \quad \text{for parallel straight lines}$$

points X, Y of minimal distance: solutions of the linear system

$$(\vec{y} - \vec{x}) \cdot \vec{u} = 0, \quad (\vec{y} - \vec{x}) \cdot \vec{v} = 0$$

for s and t , $\vec{y} - \vec{x} \parallel \vec{u} \times \vec{v}$

Plane

different representations of points X on a plane E

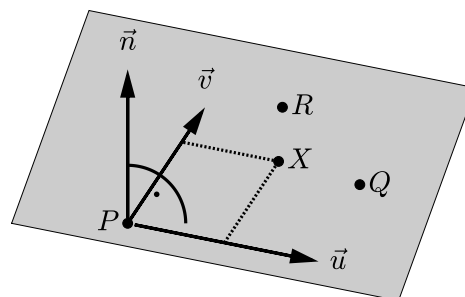
- **Parametric Form:**

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

- **Implicit Form:**

$$\vec{x} \cdot \vec{n} = d, \quad d = \vec{p} \cdot \vec{n}$$

normal vector $\vec{n} \perp E$
 choosing $|\vec{n}| = 1$, $d \geq 0$ (distance to the origin) \rightsquigarrow **Hesse Form**



- **Three-Point Form:**

$$[\vec{q} - \vec{p}, \vec{r} - \vec{p}, \vec{x} - \vec{p}] = 0 \iff \begin{vmatrix} p_1 & q_1 & r_1 & x_1 \\ p_2 & q_2 & r_2 & x_2 \\ p_3 & q_3 & r_3 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

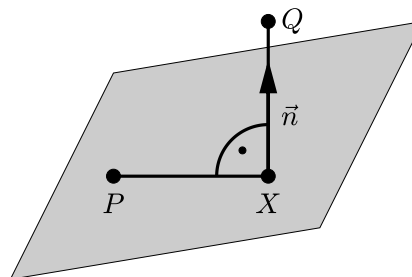
Distance Point-Plane

X : projection of a point Q onto the plane

E : $\vec{x} \cdot \vec{n} = d$, i.e. $(\vec{q} - \vec{x}) \parallel \vec{n}$

$$\text{dist}(Q, E) = |\overrightarrow{QX}| = \frac{|\vec{q} \cdot \vec{n} - d|}{|\vec{n}|}$$

$$\vec{x} = \vec{q} - \frac{\vec{q} \cdot \vec{n} - d}{|\vec{n}|^2} \vec{n}$$



$d = \vec{p} \cdot \vec{n}$ with P any point in E

Intersection of Planes

E_k : $\vec{x} \cdot \vec{n}_k = d_k$

- angle of the intersection: $\varphi = \arccos(|\vec{n}_1 \cdot \vec{n}_2| / (|\vec{n}_1| |\vec{n}_2|)) \in [0, \pi/2]$
- direction of the straight line g of intersection: $\vec{u} = \vec{n}_1 \times \vec{n}_2$
- point P on g : solution (not unique) of the equations of the planes $\vec{p} \cdot \vec{n}_k = d_k$, $k = 1, 2$