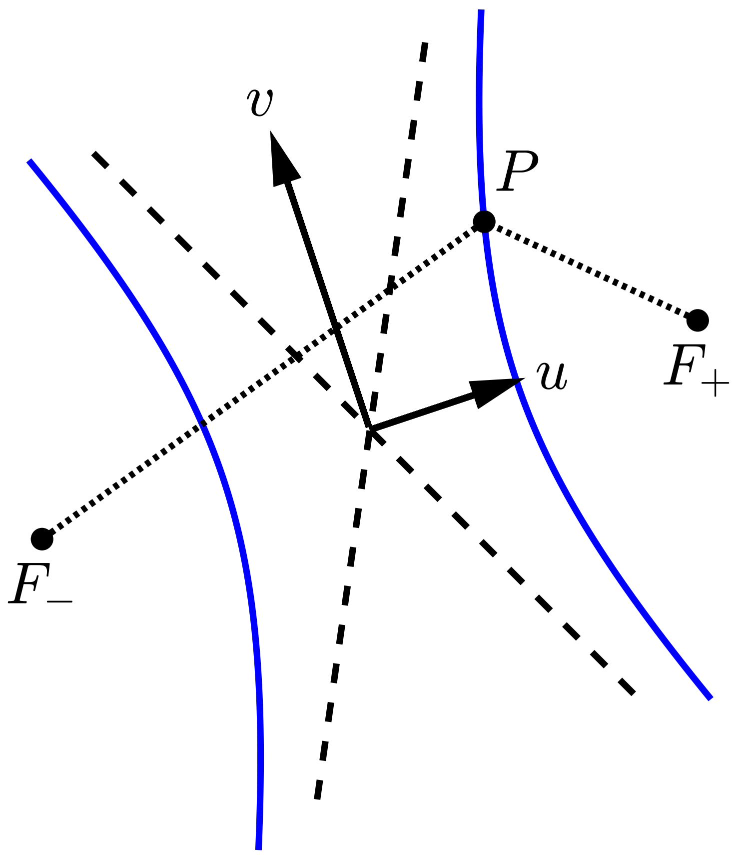
MathTraining

Problems and Solutions for Linear Algebra



Klaus Höllig

Partially translated from
Aufgaben und Lösungen zur Höheren Mathematik 2, 4. Auflage
by Klaus Höllig and Jörg Hörner

Preface

The problem set of the book covers all basic topics of a course on Linear Algebra. It can be used to practice for exams, to facilitate the completion of homework assignments, and to review course material. Interactive variants to model problems with detailed solutions permit the student reader to test his comprehension of the relevant techniques. In addition to the collection of problems, a small mathematics lexicon contains brief descriptions of the relevant theorems, methods, and definitions.

There exists also a sportive aspect of mathematics - challenging problems requiring ideas beyond the standard techniques. The problems in the chapter *Calculus Highlights* are perhaps too difficult for undergraduates. They are included to initiate or strengthen fascination for mathematics. It is definitely not a mistake to practice substantially harder than necessary ...!

The book is partially translated from

Aufgaben und Lösungen zur Höheren Mathematik 2

by Jörg Hörner and the author. It supplements this textbook by providing detailed solutions to tests for the chapters on *Linear Algebra*. Moreover, the book includes additional problems, in particular problem variants for the topics of the tests.

The author wishes the readers success in their studies **and** hopes that mathematics will become one of their favorite subjects!

Klaus Höllig

Contents

Introduction

The book contains problems with detailed solutions, problem variants with interactive result verification, and a mathematics lexicon for the principal topics which are usually subject of a course on *Linear Algebra*:

- Groups and Fields
- Vector Spaces, Scalar Products, and Bases
- Linear Maps and Matrices
- Determinants
- Linear Systems
- Eigenvalues and Normal Forms
- Least Squares and Singular Value Decomposition
- Reflections and Rotations
- Conic Sections and Quadrics
- Calculus Highlights

The problem set can be used to practice for exams, to facilitate the completion of homework assignments, and to deepen the comprehension of course material. How is this accomplished most effectively? Remembering his own student days, the author makes the following recommendations to a student reader.

Consider, as an example, a problem from the chapter on *Linear Systems*:

5.2 Inverse of a Symmetric 3×3 Matrix

Determine the inverse $C = A^{-1}$ of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 2 \end{pmatrix}$$

with Cramer's rule.

Resources: Cramer's Rule

Before looking at the solution of the problem, review the relevant theory or methods (resources). Clicking on the link leads to the following brief description of the relevant formulas from the *Lexicon* in chapter 11.

Cramer's Rule

- solution of $Ax = b$, $A = (a_1, \ldots, a_n)$:

$$x_j = \det \underbrace{(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n)}_{\text{column } a_j \text{ of } A \text{ replaced by } b} \,/\, \det A$$

- inverse matrix $C = A^{-1}$:

$$c_{j,k} = \underbrace{(-1)^{j+k} \det \tilde{A}_{k,j}}_{\text{cofactor}} \,/\, \det A$$

with $\tilde{A}_{k,j}$ obtained from A by deleting row k and column j

2×2 matrix:

$$x_1 = \frac{b_1 a_{2,2} - b_2 a_{1,2}}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}}, \quad x_2 = \frac{b_2 a_{1,1} - b_1 a_{2,1}}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}}$$

and

$$\begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} = \frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}$$

Try to solve the problem with these instructions. Then compare your computations with the solution given in the book:

Solution

Cramer's rule $\rightsquigarrow$ formula for the elements of $C = A^{-1}$:

$$c_{j,k} = \underbrace{(-1)^{j+k} \det \tilde{A}_{k,j}}_{\text{cofactor}} / \det A \qquad (1)$$

with $\tilde{A}_{k,j}$ obtained from A by deleting row k and column j

⚠ Note the permutation of indices: $c_{j,k} \leftrightarrow \tilde{A}_{k,j}$.

- determinant of A: expanding with respect to the first row $\rightsquigarrow$

$$\det A = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = -1 \cdot (1 \cdot 2 - 0 \cdot 3) = -2$$

- application of formula (1):

$$c_{1,1} = (-1)^{1+1} |\tilde{A}_{1,1}|/(-2) = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} /(-2) = 5/2$$

($\tilde{A}_{1,1} = A$ without row 1 and column 1)

$$c_{1,2} = (-1)^{1+2} |\tilde{A}_{2,1}|/(-2) = -\begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} /(-2) = 1$$

($\tilde{A}_{2,1} = A$ without row 2 and column 1)
analogously

$$c_{1,3} = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} /(-2) = -\frac{3}{2},\; c_{2,2} = \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} /(-2) = 0,\; c_{2,3} = 0,\; c_{3,3} = \frac{1}{2}$$

symmetry of A $\implies$ $C^{\mathrm{t}} = (A^{-1})^{\mathrm{t}} = (A^{\mathrm{t}})^{-1} = A^{-1} = C$ and, consequently, $c_{1,2} = c_{2,1}$, $c_{1,3} = c_{3,1}$, $c_{2,3} = c_{3,2}$, i.e.,

$$C = \begin{pmatrix} 5/2 & 1 & -3/2 \\ 1 & 0 & 0 \\ -3/2 & 0 & 1/2 \end{pmatrix}$$

Verification with MATLAB®

```
A = [0 1 0; 1 2 3; 0 3 2], C = inv(A)
```

The solutions are written in a keyword-like style, as you would employ when you comment your solutions in an exam or for homework problems. For example, the phrase

"Cramer's rule $\rightsquigarrow$ formula for the elements of $C = A^{-1}$"

stands for "By applying Cramer's rule we obtain a formula for the elements of the inverse matrix C of A". Other examples of typical phrases are "simplifying $\rightsquigarrow$ ..." or "alternative argument: ...". There is just as much detail included as is necessary for the mathematical arguments.

To gain more practice with the solution technique, it is highly recommended to solve some (preferably all ...) of the problem variants following the principal model problem for each topic. For *Inverse of a Symmetric* 3×3-*Matrix* the variants are:

Problem Variants

■ $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$\max_{j,k} c_{j,k} =?$:

[ ] check

■ $A = \begin{pmatrix} 1 & -3 & -1 \\ -3 & 2 & 2 \\ -1 & 2 & 1 \end{pmatrix}$

$\max_{j,k} c_{j,k} =?$:

[ ] check

■ $A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$

$\max_{j,k} c_{j,k} =?.??$:

[ ] check

You can check your solution by typing your answer in the field adjacent to the check - box, replacing every question mark by a character (digit or letter). Convert your result to a decimal, truncated to the number of digits indicated. For example,

$$2/3 \to 0.6666\ldots \xrightarrow{?.??} 0.66, \qquad \text{answer}: \boxed{066}\,.$$

⚠ Note that the period is omitted; only the characters corresponding to the question marks are typed.

The solutions for the three problem variants are

$$(1)\;\; C = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

$$(2)\;\; C = \begin{pmatrix} 2 & -1 & 4 \\ -1 & 0 & -1 \\ 4 & -1 & 7 \end{pmatrix}$$

$$(3)\;\; C = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & 2 \end{pmatrix}$$

Hence, the correct input is

$$(1) \qquad \max_{j,k} c_{j,k} = ? \quad \to \quad \boxed{1}$$

$$(2) \qquad \max_{j,k} c_{j,k} = ? \quad \to \quad \boxed{7}$$

$$(3) \qquad \max_{j,k} c_{j,k} = ?.?? \quad \to \quad \boxed{066}$$

As mentioned in the beginning, the problem set can also assist you in completing homework assignments. Just look for a similar problem and study its solution. Similarly, for methods and examples presented in class, practice with the relevant problems.

The above remarks pertain to the first nine chapters which exclusively discuss the solution of standard problems. Usually, such problems constitute the major portion of an exam or homework assignment. Hence, to review the basic techniques involved is of primary importance. Applying these techniques to more advanced problems is a natural next step. The chapter *Calculus Highlights* contains examples of rather challenging applications. You do not have to be disappointed if you cannot solve any of these problems; they are definitely very difficult. It is legitimate to immediately look at the solutions and learn how the methods from the previous chapters are applied in an advanced setting. Also, as mentioned in the Preface, it is not a mistake to practice substantially harder than necessary ...!

You have solved some of the problems in chapter 10 without resorting to the solutions. Then ...

... **you can take pride in your mastery of the principal techniques for solving problems in Linear Algebra!**

With the previous explanations aimed at student readers, instructors could (obviously) also benefit from the interactive problem collection. The solutions of the model problems can be used as examples in class, and some of the variants assigned as homework problems. Students will welcome the possibility of checking results before submitting or presenting their solutions in the exercise sections.

Disclaimer: Although the solutions and answers to the variants have been thoroughly checked, mistakes can always occur[1]. Please, write to the author (`Klaus.Hoellig@gmail.com`) if you find any errors.

[1]A statement by a teaching assistant to encourage students, which the author will always remember: "This year, the final exam is not too difficult - your professor could check the results without committing any errors!".

Chapter 1

Groups and Fields

1.1 Verification of Group Axioms

Which of the maps,

$$f_r : (x, y) \mapsto (x, y + rx) \quad \text{or} \quad h_r : (x, y) \mapsto (x, y + ry)\,,$$

$r \in \mathbb{R}$, form a group with respect to composition of maps as group operation?

Resources: Group

Problem Variants

■ $f_r : (x, y) \mapsto (rx + y, x)$, $h_r : (x, y) \mapsto (x + ry, y)$, $\quad r \in \mathbb{R}$

---

f/h ?:

[ ] check

■ $f_{r,s} : (x, y) \mapsto (rx, sx + y)$, $h_{r,s} : (x, y) \mapsto (rx, x + sy)$, $\quad r, s \in \mathbb{R}, r \neq 0$

---

f/h ?:

[ ] check

■ $f_{r,s} : (x, y) \mapsto (rx+sy, -sx+ry)$, $h_{r,s} : (x, y) \mapsto (rx+sy, sx+ry)$, $\quad r, s \in \mathbb{R}$, $|r| + |s| \neq 0$

---

f/h: ?:

[ ] check

Solution

$G = \{f_r : (x,y) \mapsto (x, y+rx) | \, r \in \mathbb{R}\}$

First, the admissibility of the group operation has to be checked, i.e. that

$$f_r \circ f_s \in G \quad \forall r, s \in \mathbb{R}\,.$$

definition of the maps in $G \quad \Longrightarrow$

$$\begin{aligned} f_s : \quad & (x,y) \mapsto (x', y') = (x, y+sx) \\ f_r \circ f_s \quad & (x,y) \mapsto (x', y'+rx') = (x, y+sx+rx)\,, \end{aligned}$$

i.e.

$$f_r \circ f_s = f_t, \quad t = s + r \quad \checkmark \tag{1}$$

Verification of the group axioms:

- associativity: valid, in general, for composition of maps, and easily confirmed in the special case considered

  $$\begin{aligned} (f_r \circ f_s) \circ f_t : \quad & (x,y) \mapsto (f_r \circ f_s)(x, y+tx) \underset{(1)}{=} (x, y+tx+(s+r)x), \\ f_r \circ (f_s \circ f_t) : \quad & (x,y) \mapsto f_r(x, y+(t+s)x) = (x, y+(t+s)x+rx)\,, \end{aligned}$$

  i.e. $((f_r \circ f_s) \circ f_t)\,(x,y) = (f_r \circ (f_s \circ f_t))\,(x,y) = (x, y+(r+s+t)x)$ ✓

- neutral map: $f_0 : (x,y) \mapsto (x,y) \quad \Longrightarrow \quad f_r \circ f_0 = f_0 \circ f_r = f_r$

- inverse map: $(f_r)^{-1} = f_{-r}$ since

  $$f_r \circ f_{-r} : (x,y) \mapsto f_r(x, y-rx) = (x + y - rx + rx) = (x,y) = f_0(x,y)$$

All group axioms are satisfied.

$G = \{h_r : (x,y) \mapsto (x, y+ry) | \, r \in \mathbb{R}\}$

The maps in G do not form a group, since, e.g., $h_{-1} : (x,y) \mapsto (x,0)$ does not possess an inverse map $h_r \in G$.

1.2 Generators of a Group

Specify as few permutations as possible which generate the permutation group of $\{1, 2, 3\}$ by iterated composition.

Resources: Group

Problem Variants

■ $G = \{1, \ldots 6\}$, $\diamond$: multiplication modulo 7

---

minimal number of generators ?:

[ ] check

■ G congruence maps of the square $\square(A, B, C, D)$

---

minimal number of generators ?:

[ ] check

■ $G = \{0, 1, 2\} \times \{0, 1, 2\}$, $\diamond$: addition of vectors modulo 3

---

minimal number of generators ?:

[ ] check

Solution

The transposition

$$p = (1\,2)(3): \quad 1 \mapsto 1,\, 2 \mapsto 1,\, 3 \mapsto 3$$

and the 3-cycle

$$q = (1\,2\,3): \quad 1 \mapsto 2,\, 2 \mapsto 3,\, 3 \mapsto 1$$

generate subgroups G_p and G_q of the permutation group G with 2 and 3 permutations, respectively:

$$\begin{aligned} G_p &= \{p,\, p \circ p\} = \{(1\,2)(3),\, (1)(2)(3)\}, \\ G_q &= \{q,\, q \circ q,\, q \circ q \circ q\} = \{(1\,2\,3),\, (1\,3\,2),\, (1)(2)(3)\}\,. \end{aligned}$$

The union $G_p \cup G_q$, consisting of 4 permutations, generates a subgroup $G_{p,q}$ with $n = |G_{p,q}| \geq 4$ permutations. Since n divides $|G| = 6$, n equals 6, and, consequently, p and q generate the entire permutation group with 6 elements[1].
In particular,

$$(1\,2)(3) \circ (1\,2\,3) = (2\,3)(1), \quad (1\,2)(3) \circ (1\,3\,2) = (1\,3)(2)\,.$$

[1] It is true, in general, that the permutation group of $\{1, \ldots, n\}$ is generated by the transposition $(1\,2)(3) \cdots (n)$ and the n-cycle $(1\,2\,\ldots\,n)$.

1.3 Subgroups

Determine all subgroups of the group $G = \{1, 2, 3, 4, 5, 6\}$ with respect to multiplication modulo 7.

Resources: Group

Problem Variants

■ $G = \{1, 2, \ldots, 10\}$, $\diamond$: multiplication modulo 11

---

number of nontrivial subgroups ?:

[ ] check

■ $G = \{0, 2, \ldots, 10, 11\}$, $\diamond$: addition modulo 12

---

number of nontrivial subgroups ?:

[ ] check

■ $G = \{0, 1, 2\} \times \{0, 1, 2\}$, $\diamond$: addition of vectors modulo 3

---

number of nontrivial subgroups ?:

[ ] check

Solution

cyclic subgroups U of $G = \{1, 2, \ldots, 7\}$ (generated by a single element $g \in G$):

- $g = 2$:
  $2 \cdot 2 \bmod 7 = 4$, $4 \cdot 2 \bmod 7 = 1$
  $\rightsquigarrow \quad U_3 = \{1, 2, 4\}$

- $g = 3$:
  $3 \cdot 3 \bmod 7 = 2$, $2 \cdot 3 \bmod 7 = 6$, $6 \cdot 3 \bmod 7 = 4$, $4 \cdot 3 \bmod 7 = 5$, $5 \cdot 3 \bmod 7 = 1$
  $\rightsquigarrow \quad U = G$ (trivial subgroup; $g = 3$ is a generator of G)

- $g = 4 \in U_3$ (different generator of U_3)

- $g = 5$: $\ldots \quad \rightsquigarrow \quad G$

- $g = 6$:
  $6 \cdot 6 \bmod 7 = 1$
  $\rightsquigarrow \quad U_2 = \{1, 6\}$

$\Longrightarrow \quad$ 2 nontrivial subgroups: U_2 und U_3

More nontrivial subgroups do not exist, since a subgroup U of G which is not cyclic must have at least 4 elements. This is not possible, since $|U|$ divides $|G| = 6$.

1.4 Cycles, Inverse, and Sign of a Permutation

Determine the cycles, the inverse, and the sign of the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 8 & 1 & 9 & 2 & 4 & 6 & 5 & 7 \end{pmatrix}.$$

Resources: Permutation

Problem Variants

■

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 9 & 8 & 7 & 4 & 6 & 5 & 2 \end{pmatrix}$$

---

lengths of the cycles in increasing order ??:

[ ] check

■

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 9 & 8 & 7 & 3 & 6 & 5 & 4 & 2 & 1 \end{pmatrix}$$

---

lengths of the cycles in increasing order ????:

[ ] check

■

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 9 & 2 & 7 & 8 & 5 & 1 & 6 & 3 \end{pmatrix}$$

---

lengths of the cycles in increasing order ???:

[ ] check

Solution

Cycles

compute the images $\pi(1)$, $\pi(\pi(1))$, ...

$$1 \mapsto 3,\ 3 \mapsto 1 \quad \rightsquigarrow \quad 2-\text{cycle}\,(1\,3)$$

continuing with the smallest numbers, not part of the previously computed cycles $\rightsquigarrow$

$$\begin{array}{l} 2 \mapsto 8,\ 8 \mapsto 5,\ 5 \mapsto 2 \quad \rightsquigarrow \quad 3-\text{cycle}\,(2\,8\,5) \\ 4 \mapsto 9,\ 9 \mapsto 7,\ 7 \mapsto 6,\ 6 \mapsto 4 \quad \rightsquigarrow \quad 4-\text{cycle}\,(4\,9\,7\,6) \end{array}$$

Inverse Permutation

inverting the cycles $\rightsquigarrow$

$$\pi^{-1} = (1\,3)(2\,5\,8)(4\,6\,7\,9)\,,$$

i.e.

$$\pi^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 5 & 1 & 6 & 8 & 7 & 9 & 2 & 4 \end{pmatrix}$$

Sign

applying the formula

$$\sigma(\pi) = (-1)^m, \quad m = \sum_k (m_k - 1)$$

with m_k the lengths of the cycles of π $\rightsquigarrow$

$$\sigma(\pi) = (-1)^{(2-1)+(3-1)+(4-1)} = (-1)^6 = 1$$

1.5 Computing with Permutations Using Cycles

Represent the permutations

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 5 & 4 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 5 & 6 & 1 & 2 \end{pmatrix}$$

by their cycles and determine

$$p^{-1},\ q^{-1},\ p \circ q,\ q \circ p\,.$$

Resources: Permutation

Problem Variants

■

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 1 & 3 & 4 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 6 & 4 & 3 & 1 & 2 \end{pmatrix}$$

$(p \circ q \circ p^{-1})(2)$ =?:

[ ] check

■

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{pmatrix}$$

$(q \circ p \circ q^{-1})(3)$ =?:

[ ] check

■

$$p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 4 & 1 & 6 & 2 & 5 & 3 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 7 & 1 & 2 & 5 & 6 \end{pmatrix}$$

$(p \circ q \circ p^{-1})(4)$ =?:

[ ] check

Solution

<u>Cycles</u>

map corresponding to $p = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 2 & 6 & 5 & 4 \end{pmatrix}$:

$$1 \mapsto 3,\ 2 \mapsto 1,\ 3 \mapsto 2,\ 4 \mapsto 6,\ 5 \mapsto 5,\ 6 \mapsto 4\,,$$

i.e. $p:\ 1 \mapsto 3 \mapsto 2 \mapsto 1, \quad 4 \mapsto 6 \mapsto 4, \quad 5 \mapsto 5$
$\rightsquigarrow$ representation by three cycles (including also the trivial 1-cycle which is usually omitted)

$$p = (132)(46)(5)$$

similarly: $q = (146235)$, i.e.

$$q:\ 1 \mapsto 4 \mapsto 6 \mapsto 2 \mapsto 3 \mapsto 5 \mapsto 1$$

<u>Inverse</u>

inverting the map, corresponding to p (separately for each cycle)

$$\begin{array}{rcl} 1 \mapsto 3 \mapsto 2 \mapsto 1 & \rightsquigarrow & 1 \mapsto 2 \mapsto 3 \mapsto 1 \\ 4 \mapsto 6 \mapsto 4 & \rightsquigarrow & 4 \mapsto 6 \mapsto 4 \\ 5 \mapsto 5 & \rightsquigarrow & 5 \mapsto 5 \end{array}$$

$\Longrightarrow \quad p^{-1} = (123)(46)(5)$

similarly: $q^{-1} = (532641) = (153264)$
The representations are invariant under cyclic translation. Therefore, cycles usually begin with the smallest integer.

<u>Composition</u>

$$\begin{array}{rcl} q(1) = 4,\ p(4) = 6 & \Longrightarrow & (p \circ q)(1) = 6 \\ q(6) = 2,\ p(2) = 1 & \Longrightarrow & (p \circ q)(6) = 1 \\ q(2) = 3,\ p(3) = 2 & \Longrightarrow & (p \circ q)(2) = 2 \\ q(3) = 5,\ p(5) = 5 & \Longrightarrow & (p \circ q)(3) = 5 \\ & \ldots & \end{array}$$

i.e. $p \circ q$: $1 \mapsto 6 \mapsto 1, \quad 2 \mapsto 2, \quad 3 \mapsto 5 \mapsto 3, \quad 4 \mapsto 4$

$$\rightsquigarrow \quad p \circ q = (16)(2)(35)(4)$$

similarly: $q \circ p = (15)(24)(3)(6)$

1.6 Group Table

Complete the depicted table of a commutative group.

	a	b	c	d	e
a	a				
b		c		e	
c		d		a	
d				b	
e					

Resources: Group

Problem Variants

■

	a	b	c
a	a		
b			
c			

missing entries, rowwise ????????:

[ ] check

■

	a	b	c	d
a	a			
b		a		
c			a	
c				

missing entries, rowwise ?????????????:

[ ] check

■

	a	b	c	d
a	a			
b		d		
c				
c				

missing entries, rowwise ??????????????:

[ ] check

Solution

Since $a \diamond a = a$ (entry $(1,1)$ of the table), the first row and the first column equal $(a\,b\,c\,d\,e)$ because $x \diamond a = a \diamond x = x$.

Now it is used that
(1) each row and each column of the group table contains every element of the group
and
(2) the group table of a commutative group is symmetric (entry (j,k) equals entry (k,j)).

With the aid of these properties the table can be completed as is indicated below:

	a	b	c	d	e
a	a	b	c	d	e
b	b	c		e	
c	c	d		a	
d	d			b	
e	e				

$\xrightarrow[(2)]{}$

	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	
c	c	d		a	
d	d	e	a	b	
e	e				

$\xrightarrow[(1),(2)]{}$

	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d		a	
d	d	e	a	b	c
e	e	a		c	

In row 3, the elements b and e are missing. Since e appears in column 5, row 3 equals $(c\,d\,e\,a\,b)$. Then, property (1) yields the last two missing entries in row 5, and the resulting group table is

	a	b	c	d	e
a	a	b	c	d	e
b	b	c	d	e	a
c	c	d	e	a	b
d	d	e	a	b	c
e	e	a	b	c	d

.

This table corresponds, e.g., to the group $\{0,1,2,3,4\}$ with addition modulo 5 as group operation.

1.7 Quadratic Equation in a Prime Field

Determine the solutions x_k of the quadratic equation

$$x^2 + 8x = 7 \bmod 11$$

in the prime field $\mathbb{Z}_{11}$.

Resources: Field

Problem Variants

■ $2x^2 + 2x = 11 \bmod 13$

$x_1 =? > x_2 =?$:

[ ] check

■ $3x^2 + 5x = 1 \bmod 7$

$x_1 =? > x_2 =?$:

[ ] check

■ $4x^2 - 5x = -9 \bmod 17$

$x =?$:

[ ] check

Solution

completing the square $\quad\rightsquigarrow$

$$(x+4)^2 = 7 + 16 \bmod 11 = 1 \bmod 11$$

There exist two square roots of 1 in $\mathbb{Z}_{11}$:

$$1^2 = 1 \bmod 11, \quad 10^2 = 100 = 1 \bmod 11\,.$$

$\rightsquigarrow$ solutions

$$x_1 = 1 - 4 \bmod 11 = 8, \quad x_2 = 10 - 4 \bmod 11 = 6$$

1.8 Greatest Common Divisor of Two Integers

Determine the gcd of 4641 and 4389 with the Euclidean algorithm.

Resources: Euclid's Algorithm

Problem Variants

■ 14014, 10659

??:

[ ] check

■ 26013, 51051

??:

[ ] check

■ 131670, 215878

??:

[ ] check

Solution

Starting with $n_1 = 4641$, $n_2 = 4389$, the Euclidean algorithm generates a decreasing sequence of positive integers by successive division:

$$n_{k-1} = q_k n_k + \underbrace{n_{k+1}}_{<n_k},$$

i.e. n_{k+1} is the remainder when n_{k-1} is divided by n_k. The algorithm terminates if $n_{K+1} = 0$; n_K being the greatest common divisor of n_1 and n_2.

resulting sequence for the given integers:

$$\begin{array}{rcrcrl}
4641 & : & 4389 & = & 1 & \text{remainder}\, 252 \\
4389 & : & 252 & = & 17 & \text{remainder}\, 105 \\
252 & : & 105 & = & 2 & \text{remainder}\, 42 \\
105 & : & 42 & = & 2 & \text{remainder} \quad \underbrace{\mathbf{21}}_{\gcd(4641,4389)} \\
42 & : & \mathbf{21} & = & 2 & \text{remainder}\, 0
\end{array}$$

1.9 Greatest Common Divisor of Two Polynomials

Determine the gcd of the polynomials

$$p_1(x) = x^3 - 8x - 3, \quad p_2(x) = x^3 - 2x^2 - 9\,.$$

Resources: Euclid's Algorithm

Problem Variants

■ $p_1(x) = x^4 - 1$, $p_2(x) = x^3 + 2x^2 - 5x - 6$

---

gcd: $?{\cdot}? + 1$:

[ ] check

■ $p_1(x) = 2x^3 + x^2 - 5x + 2$, $p_2(x) = 2x^3 - 3x^2 - 3x + 2$

---

gcd: $?? - 1$:

[ ] check

■ $p_1(x) = 4x^4 - 10x^3 + 2x^2 + 5x - 2$, $p_2(x) = -2x^4 + 5x^3 - 6x^2 + 10x - 4$

---

gcd: $-?x^2 + ?x - 2$:

[ ] check

Solution

application of Euclid's algorithm to the polynomials

$$p_1(x) = x^3 - 8x - 3, \quad p_2(x) = x^3 - 2x^2 - 9\,,$$

i.e. computing successively a sequence of polynomials $p_3, p_4 \ldots$ with polynomial division:

$$p_{k-1} = q_k p_k + p_{k+1} \iff p_{k-1}/p_k = q_k \,\text{remainder}\, p_{k+1},\ k = 2, 3, \ldots \qquad (1)$$

The degrees of $p_2, p_3, \ldots$ are strictly decreasing. Hence, the sequence terminates with $p_K = 0$ yielding p_{K-1} as the gcd.

- $k = 2$:

$$p_1(x)/p_2(x) = q_2(x) = 1 \quad \text{remainder}\, p_3(x) = p_1(x) - p_2(x) = 2x^2 - 8x + 6$$

- $k = 3$:

$$\begin{array}{rrrrl}
(\; x^3 & -2x^2 & +0x & -9 &) : (\; 2x^2 - 8x + 6 \;) = \; q_3(x) = \frac{1}{2}x + 1 \\
x^3 & -4x^2 & +3x & & \\
\hline
 & 2x^2 & -3x & -9 & \\
 & 2x^2 & -8x & +6 & \\
\hline
 & & 5x & -15 & = \quad p_4(x)
\end{array}$$

- $k = 4$:

$$\begin{array}{rrrl}
(\; 2x^2 & -8x & +6 &) : (\; 5x - 15 \;) = \; q_4(x) = \frac{2}{5}x - \frac{2}{5} \\
2x^2 & -6x & & \\
\hline
 & -2x & +6 & \\
 & -2x & +6 & \\
\hline
 & & 0 & = \quad p_5(x)
\end{array}$$

$\implies \qquad K = 5$ and gcd $= p_4(x) = 5x - 15$

backward substitution, according to (1) $\rightsquigarrow$

$$p_3 = q_4 p_4,\ p_2 = q_3 \underbrace{(q_4 p_4)}_{p_3} + p_4,\ p_1 = q_2 \underbrace{(q_3 q_4 + 1) p_4}_{p_2} + p_3 = [\ldots] p_4\,,$$

confirming that p_4 divides both, p_1 and p_2.

As another test, p_1 and p_2 are evaluated at the zero $x_* = 15/5 = 3$ of the gcd p_4:

$$p_1(3) = 3^3 - 8 \cdot 3 - 3 = 0, \quad p_2(3) = 27 - 18 - 9 = 0\,,$$

i.e. x_* is a common zero of p_1 and p_2 and $(x - x_*)$ is a common linear factor.
✓

1.10 Application of the Chinese Remainder Theorem

Determine the smallest positive solution $x \in \mathbb{N}_0$ of the congruences

$$x = 1 \bmod 11, \quad x = 9 \bmod 13\,.$$

Resources: Chinese Remainder Theorem

Problem Variants

■ $x = 5 \bmod 7, \quad x = 7 \bmod 9$

---

x = ??:

[ ] check

■ $x = 13 \bmod 17, \quad x = 5 \bmod 19$

---

x = ??:

[ ] check

■ $x = 3 \bmod 5, \quad x = 7 \bmod 9$

---

x = ??:

[ ] check

Solution

Chinese Remainder Theorem $\implies$

$$x = \Big(1 \cdot (13^{-1} \bmod 11) \cdot 13 + 9 \cdot (11^{-1} \bmod 13) \cdot 11\Big) \bmod(11 \cdot 13) \qquad (1)$$

with $r = q^{-1} \bmod p \iff r \cdot q = 1 \bmod p$

testing the products $13 \cdot 2, 13 \cdot 3, \ldots \quad \rightsquigarrow \quad 13 \cdot 6 = 78 = 1 \bmod 11$, i.e. $13^{-1} \bmod 11 = 6$

analogously: $11 \cdot 6 = 1 \bmod 13 \quad \implies \quad 11^{-1} \bmod 13 = 6$

substituting into (1) $\quad \rightsquigarrow$

$$x = \Big(1 \cdot 6 \cdot 13 + 9 \cdot 6 \cdot 11\Big) \bmod 143 = 672 \bmod 143 = 100$$

<u>Alternative Solution</u>

direct solution of the equations $x = 1 + 11m = 9 + 13n$

solving for $m \quad \rightsquigarrow$

$$m = \frac{8 + 13n}{11} = \left[\frac{8 + 13p}{11}\right] + \left\{\frac{13(n-p)}{11}\right\}$$

choose p so that $[\ldots] \in \mathbb{N}$, i.e. $p = 7$ and $[\ldots] = 9$

$\{\ldots\} \in \mathbb{N} \quad \implies \quad \frac{n-p}{11} = k \in \mathbb{N}$ and

$$x = 1 + 11 \cdot (\,[9]\; + \;\{13k\}\,) = 100 + 143k$$

Chapter 2

Vector Spaces, Scalar Products, and Bases

2.1 Criteria for Subspaces

Which of the following conditions define subspaces of the vector space V of continuous functions $f : [0,1] \to \mathbb{R}$?

a) $f(1) = 0$ b) $f(0) = 1$ c) $f(0) = f(1)$

Resources: Vector Space

Problem Variants

■ V: polynomials p
a) $q(x) = x^2 + 1$ divides p b) degree$(p) = 10$ c) $p(-x) = -p(x)$

??:
[ ] check

■ V: rational functions p/q
a) no zeros b) no poles c) $\lim_{x\to\pm\infty} p(x) = 0$

??:
[ ] check

■ V: complex functions f
a) $\operatorname{Re} f(0) \cdot \operatorname{Im} f(0) = 0$ b) $\lim_{|z|\to 0} f(z) = 0$ c) $\lim_{|z|\to\infty} f(z) = \infty$

?:
[ ] check

Solution

A subset U of a real vector space V is a subspace if

$$f + g \in U,\, rf \in U \quad \forall f, g \in U,\, r \in \mathbb{R}\,, \tag{1}$$

i.e. if U is closed under addition and scalar multiplication.
This characterization is applied to subsets U of the vector space V of continuous functions $f : [0,1] \to \mathbb{R}$.

$\underline{U = \{f \in V : f(1) = 0\}}$

$$f(1) = 0,\, g(1) = 0 \quad \Longrightarrow \quad (f+g)(1) = 0,\, rf(1) = 0\,,$$

i.e. the subspace condition (1) is satisfied

$\underline{U = \{f \in V : f(0) = 1\}}$

Condition (1) is violated since, e.g.,

$$f(0) = 1,\, r = 2 \quad \Longrightarrow \quad (rf)(0) = 2\,,$$

i.e., $rf \notin U$.

$\underline{U = \{f \in V : f(0) = f(1)\}}$

$f(0) = f(1),\, g(0) = g(1)$ i.e. (1) is valid, U is a subspace.

2.2 Basis for the Intersection of two Subspaces

Construct a basis for the intersection U of the two subspaces of $\mathbb{R}^4$, spanned by the vectors

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right\},$$

respectively.

Resources: Vector Space

Problem Variants

■ $\left\{ \begin{pmatrix} 0 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} -3 \\ -1 \\ -5 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ -0 \end{pmatrix} \right\}$

$\dim U =?$:

[ ] check

■ $\left\{ \begin{pmatrix} 1 \\ 4 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} -3 \\ 2 \\ 1 \\ 5 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} -2 \\ 1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \\ -5 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ 5 \\ -4 \end{pmatrix} \right\}$

$\dim U =?$:

[ ] check

■ $\left\{ \begin{pmatrix} 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \\ 1 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 2 \\ 0 \end{pmatrix} \right\}$ and $\left\{ \begin{pmatrix} 2 \\ 1 \\ 3 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ 2 \\ 1 \end{pmatrix} \right\}$

$\dim U =?$:

[ ] check

Solution

A vector u belongs to the intersection U of the subspaces spanned by the vectors v_k and w_k if

$$u = \sum_k x_k v_k = \sum_k y_k w_k\,,$$

i.e., if

$$z = \begin{pmatrix} x \\ y \end{pmatrix} \in \ker \underbrace{\begin{pmatrix} v_1 & v_2 & \ldots & -w_1 & -w_2 & \ldots \end{pmatrix}}_{A}\,.$$

For the given vectors, $\ker A = \{z : Az = (0,0,\ldots)^{\mathrm{t}}\}$ is determined via transformation of A to row echelon form:

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix} \to$$
$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1 & 0 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

Prescribing z_k for the columns $k = 5$ and $k = 6$, which do not contain pivots, yields basis vectors z for $\ker A$ and corresponding basis vectors for the intersection U.

- $z_5 = 1,\ z_6 = 0$

$$Az = (0,0,0,0)^{\mathrm{t}} \quad \rightsquigarrow \quad z^{(1)} = (\underbrace{1,1,1}_{x},1,1,0)^{\mathrm{t}}$$

- $z_6 = 1,\ z_5 = 0$

$$Az = (0,0,0,0)^{\mathrm{t}} \quad \rightsquigarrow \quad z^{(2)} = (\underbrace{0,-1,1}_{x},-1,0,1)^{\mathrm{t}}$$

corresponding basis vectors $u = \sum_k x_k v_k$ for the intersection of the subspaces:

$$\begin{aligned} u^{(1)} &= v_1 + v_2 + v_3 = (1,2,2,1)^{\mathrm{t}} \\ u^{(2)} &= -v_2 + v_3 = (0,-1,0,1)^{\mathrm{t}} \end{aligned}$$

2.3 Distance from a Complex Straight Line

Determine the distance d of $p = (1+\mathrm{i}, 0)^{\mathrm{t}} \in \mathbb{C}^2$ from

$$g: \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} + t \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}.$$

Resources: Norm

Problem Variants

■ $p = \begin{pmatrix} 4+5\mathrm{i} \\ -5+3\mathrm{i} \end{pmatrix}$, $g: \begin{pmatrix} 1-4\mathrm{i} \\ -5-3\mathrm{i} \end{pmatrix} + t \begin{pmatrix} 2\mathrm{i} \\ -2-\mathrm{i} \end{pmatrix}$

---

d^2 =???:

[ ] check

■ $p = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, $g: \begin{pmatrix} 2\mathrm{i} \\ -3\mathrm{i} \\ -2\mathrm{i} \end{pmatrix} + t \begin{pmatrix} 1-\mathrm{i} \\ -1+2\mathrm{i} \\ -3 \end{pmatrix}$

---

d^2 =??:

[ ] check

■ $p = \begin{pmatrix} 2 \\ 1 \\ -2-2\mathrm{i} \end{pmatrix}$, $g: t \begin{pmatrix} -1+2\mathrm{i} \\ -\mathrm{i} \\ -2+\mathrm{i} \end{pmatrix}$

---

d^2 =??:

[ ] check

Solution

square of the distance of $(1+\mathrm{i},0)^{\mathrm{t}}$ from a point $(1,\mathrm{i})^{\mathrm{t}}+t(\mathrm{i},1)^{\mathrm{t}}$ on g:

$$\begin{aligned} d(t)^2 &= |(1+t\mathrm{i}-1-\mathrm{i},\mathrm{i}+t)^{\mathrm{t}}|^2 = |(0+(t-1)\mathrm{i},t+\mathrm{i})^{\mathrm{t}}|^2 \\ &= 0+(t-1)^2+t^2+1 = 2t^2-2t+2 \end{aligned}$$

in view of the definition of the norm of a complex vector ($|(z_1,z_2)|^2 = |z_1|^2 + |z_2|^2$, $|x+\mathrm{i}y|^2 = x^2+y^2$)

minimization:

$$\frac{\mathrm{d}}{\mathrm{d}t}d(t)^2 = 4t-2 = 0 \quad \Longrightarrow \quad t_{\min} = 1/2$$

and

$$d(t_{\min}) = \sqrt{2(1/2)^2 - 2(1/2) + 2} = \sqrt{3/2}$$

2.4 Angle Between Functions

Determine $\sphericalangle(f, g)$ for the functions

$$f(x) = x^2, \quad g(x) = x^3$$

with respect to the scalar product $\langle f, g\rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$.

Resources: Scalar Product

Problem Variants

■ $f(x) = \cos(\pi x)$, $g(x) = \sin(\pi x/2)$

?.??:

[ ] check

■ $f(x) = \mathrm{e}^x$, $g(x) = \mathrm{e}^{-x}$

?.??:

[ ] check

■ $f(x) = \dfrac{1}{1+x}$, $g(x) = \dfrac{1}{2+x}$

?.??:

[ ] check

Solution

substituting

$$\langle f,g\rangle = \int_0^1 x^2x^3\,\mathrm{d}x = \frac{1}{6},$$
$$\|f\|^2 = \int_0^1 (x^2)^2\mathrm{d}x = \frac{1}{5},\ \|g\|^2 = \int_0^1 (x^3)^2\mathrm{d}x = \frac{1}{7}$$

into the formula $\cos\sphericalangle(f,g) = \langle f,g\rangle/(\|f\|\,\|g\|) \quad \rightsquigarrow$

$$\sphericalangle(f,g) = \arccos\left(\frac{1/6}{\sqrt{1/5}\,\sqrt{1/7}}\right) = \arccos(\sqrt{35}/6) \approx 0.1674$$

2.5 Properties of Real Scalar Products

Which of the three defining properties (positivity, symmetry, linearity) of a real scalar product for vectors $x, y \in \mathbb{R}^2$ are satisfied by the following functions p?

a) $p(x, y) = (2|x_1| + x_2)(y_1 + 2|y_2|)$ b) $p(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$

Resources: Scalar Product

Problem Variants

Indicate the validity of the properties *positivity, symmetry, linearity* with a 0/1-sequence, e.g. 001 means that only linearity is satisfied.

■ $p(x, y) = (x_1 + x_2)(y_1 + y_2)$

???:
[ ] check

■ $p(x, y) = x_1y_1 + 2x_1y_2 + 2x_2y_1 + x_2y_2$

???:
[ ] check

■ $p(x, y) = 2x_1y_1 - x_1y_2 - x_2y_1 + x_2y_2$

???:
[ ] check

Solution

a) $p(x,y) = (2|x_1| + x_2)(y_1 + 2|y_2|)$

- positivity - no:

$$x = (0,-1)^{\mathrm{t}} \quad \Longrightarrow \quad p(x,x) = (2|0| - 1)(0 + 2|-1|) = -2 < 0$$

- symmetry - no:
  $x = (1,0)^{\mathrm{t}}$, $y = (0,1)^{\mathrm{t}}$ $\Longrightarrow$

$$\begin{aligned} p(x,y) &= (2|1| + 0)(0 + 2|1|) = 4 \\ &\neq 1 = (2|0| + 1)(1 + 2|0|) = p(y,x) \end{aligned}$$

- linearity - no:
  $s = -1$, $x = (1,0)^{\mathrm{t}} = y$ $\Longrightarrow$

$$\begin{aligned} p(sx,y) &= (2|-1| + 0)(1 + 2|0|) = 2 \\ &\neq -2 = (-1)(2|1| + 0)(1 + 2|0|) = sp(x,y) \end{aligned}$$

b) $p(x,y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2$

- positivity - yes:
  $2ab \le a^2 + b^2$ $\Longrightarrow$

$$\begin{aligned} p(x,x) &\ge 2x_1^2 - (x_1^2 + x_2^2) + 2x_2^2 \\ &= x_1^2 + x_2^2 \ge 0 \end{aligned}$$

  equal to zero only for $x_1 = x_2 = 0$

- symmetry - yes:

$$\begin{aligned} p(x,y) &= 2x_1y_1 + x_1y_2 + x_2y_1 + 2x_2y_2 \\ &= 2y_1x_1 + y_1x_2 + y_2x_1 + 2y_2x_2 = p(y,x) \end{aligned}$$

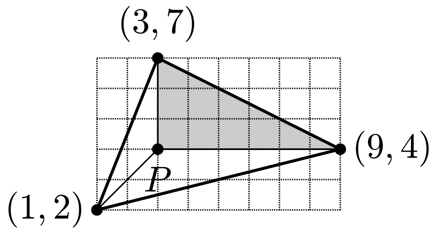
- linearity - yes:

$$\begin{aligned} & p((sx + t\tilde{x}), y) \\ =\ & 2(sx_1 + t\tilde{x}_1)y_1 + (sx_1 + t\tilde{x}_1)y_2 + (sx_2 + t\tilde{x}_2)y_1 + 2(sx_2 + t\tilde{x}_2)y_2 \\ =\ & 2sx_1y_1 + sx_1y_2 + sx_2y_1 + 2sx_2y_2 + 2t\tilde{x}_1y_1 + t\tilde{x}_1y_2 + t\tilde{x}_2y_1 + 2t\tilde{x}_2y_2 \\ =\ & sp(x,y) + tp(\tilde{x}, y) \end{aligned}$$

$\rightsquigarrow$ all properties of a scalar product are valid

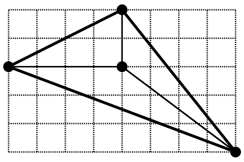
2.6 Barycentric Coordinates of a Point in a Triangle

Represent the point P as convex combination of the vertices of the depicted triangle.



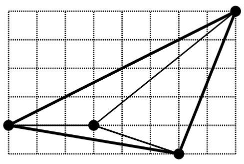
Resources: Convex Combination, Cramer's Rule

Problem Variants



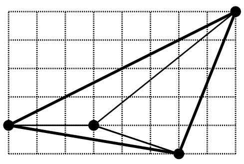
coefficients: ?/7 <?/7 <?/7:

check



coefficients: ?/32 <??/32 <??/32:

check



coefficients: ?/37 <??/37 <??/37:

check

Solution

The convex combination P of the vertices $A=(1,2)^{\mathrm{t}}$, $B=(9,4)^{\mathrm{t}}$, $C=(3,7)^{\mathrm{t}}$ is a linear combination with non-negative coefficients s_k which sum to 1:

$$P = s_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + s_2 \begin{pmatrix} 9 \\ 4 \end{pmatrix} + s_2 \begin{pmatrix} 3 \\ 7 \end{pmatrix}, \quad \sum_k s_k = 1\,.$$

$\rightsquigarrow$ linear system

$$\begin{pmatrix} 1 & 9 & 3 \\ 2 & 4 & 7 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$$

application of Cramer's rule $\rightsquigarrow$ computation of s_k as quotient of determinants:

$$s_1 = \begin{vmatrix} 3 & 9 & 3 \\ 4 & 4 & 7 \\ 1 & 1 & 1 \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 9 & 3 \\ 2 & 4 & 7 \\ 1 & 1 & 1 \end{vmatrix} \underset{(1)}{=} \underbrace{\begin{vmatrix} 3 & 6 & 0 \\ 4 & 0 & 3 \\ 1 & 0 & 0 \end{vmatrix}}_{\det(B-P,C-P)} \Bigg/ \underbrace{\begin{vmatrix} 1 & 8 & 2 \\ 2 & 2 & 5 \\ 1 & 0 & 0 \end{vmatrix}}_{\det(B-A,C-A)} \underset{(2)}{=} \frac{18}{36} = \frac{1}{2}\,.$$

(1) simplifying the determinants by subtracting the first column from columns two and three
(2) expanding with respect to the last row

geometric interpretation
Since $|\det(B-A,C-A)|/2$ is equal to the area of the triangle $\Delta(A,B,C)$,

$$s_1 = \text{area}\,\underbrace{\Delta(P,A,B)}_{\text{grey}}\ /\ \text{area}\,\Delta(A,B,C) = 9/18 = 1/2\,.$$

The areas of the two other sub-triangles sharing the point P can also be easily read off from the sketch (subtracting the areas of right-angled triangles from the area of the enclosing rectangle):

$$\begin{aligned} s_2 &= \frac{\text{area}\,\Delta(P,A,C)}{\text{area}\,\Delta(A,B,C)} = \frac{2\cdot 5 - 5 - 2}{18} = \frac{3}{18} = \frac{1}{6} \\ s_3 &= \frac{\text{area}\,\Delta(P,A,B)}{\text{area}\,\Delta(A,B,C)} = \frac{2\cdot 8 - 8 - 2}{18} = \frac{6}{18} = \frac{1}{3} \end{aligned}$$

2.7 Linear Dependence of Vectors in $\mathbb{R}^3$

Are the vectors

$$\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

linearly independent?

Resources: Linear Independence, Determinant

Problem Variants

■

$$\begin{pmatrix} 1 \\ 4 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 9 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 5 \\ 7 \\ 8 \end{pmatrix}$$

---

y/n: ?:

[ ] check

■

$$\begin{pmatrix} 8 \\ 6 \\ 7 \end{pmatrix}, \quad \begin{pmatrix} 9 \\ 1 \\ 5 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 2 \\ 3 \end{pmatrix}$$

---

y/n: ?:

[ ] check

■

$$\begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 9 \\ 8 \\ 7 \end{pmatrix}$$

---

y/n: ?:

[ ] check

Solution

linear independence of three vectors $v_k \in \mathbb{R}^3 \quad \Longleftrightarrow \quad \det(v_1, v_2, v_3) \neq 0$

computation of the determinant for the given vectors

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$$

with the Sarrus scheme $\quad \rightsquigarrow$

$$\begin{aligned} \begin{vmatrix} 0 & 1 & 3 \\ 1 & 0 & 2 \\ 2 & -2 & -2 \end{vmatrix} &= 0 + 1 \cdot 2 \cdot 2 + 3 \cdot 1 \cdot (-2) - 0 - 1 \cdot 1 \cdot (-2) - 0 \\ &= 4 - 6 + 2 = 0 \end{aligned}$$

$\Longrightarrow$ linear dependence, i.e. there exists a nontrivial linear combination equation to $(0,0,0)^{\mathrm{t}}$.
For example,

$$2v_1 + 3v_2 - v_3 = (0,0,0)^{\mathrm{t}} \,.$$

2.8 Linear Independence of Vectors in $\mathbb{R}^4$

Decide if the following vectors are linearly independent:

a) $u = (5, 2, 0, 4)^t$, $v = (5, 4, 5, 3)^t$, $w = (4, 2, 1, 3)^t$
b) $u = (2, 4, 0, -1)^t$, $v = (2, 3, -3, 1)^t$, $w = (4, 5, -5, 0)^t$

Resources: Linear Combination, Linear Independence

Problem Variants

■
a) $u = (-2, 1, -3, -3)^t$, $v = (3, 2, -2, -1)^t$, $w = (-1, 3, 1, 2)^t$
b) $u = (-1, -3, -3, -2)^t$, $v = (3, 2, 2, 3)^t$, $w = (-2, 1, 1, -1)^t$

---

linearly independent/dependent (i/d): a) ?, b) ?:

check

■
a) $u = (1, -1, -2, -1)^t$, $v = (3, 2, 1, -3)^t$, $w = (-3, 2, -2, 3)^t$
b) $u = (-3, 2, -2, 3)^t$, $v = (1, 1, -1, -1)^t$, $w = (3, 2, -2, 3)^t$

---

linearly independent/dependent (i/d): a) ?, b) ?:

check

■
a) $u = (-3, 3, -3, 3)^t$, $v = (-2, 2, 1, -1)^t$, $w = (-1, 1, -2, 2)^t$
b) $u = (3, 1, 2, -2)^t$, $v = (-1, -3, -3, -1)^t$, $w = (2, -2, 3, 1)^t$

---

linearly independent/dependent (i/d): a) ?, b) ?:

check

Solution

criteria for linear independence

- The vector $(0,0,0,0)^{\mathrm{t}}$ admits only the trivial representation as linear combination of u, v, w (all coefficients equal to 0):

$$ru + sv + tw = (0,0,0,0)^{\mathrm{t}} \quad \Longrightarrow \quad r = s = t = 0\,.$$

- The rank of the matrix $A = (u, v, w)$ is equal to the number of vectors/columns ($= 3$). Equivalently, the echelon form of A has 3 pivots.

a) $u = (5,2,0,4)^{\mathrm{t}}$, $v = (5,4,5,3)^{\mathrm{t}}$, $w = (4,2,1,3)^{\mathrm{t}}$

applying the first criterion:
$\rightsquigarrow$ linear system

$$\begin{array}{rcrcrcl} 5r & + & 5s & + & 4t & = & 0 \\ 2r & + & 4s & + & 2t & = & 0 \\ & & 5s & + & t & = & 0 \\ 4r & + & 3s & + & 3t & = & 0 \end{array}$$

equation 3 $\quad \Longrightarrow \quad t = -5s$
substitution into equation 4 $\quad \Longrightarrow \quad 4r + 3s - 15s = 0$, i.e. $r = 3s$
consistent with the first two equations:

$$15s + 5s - 20s = 0\,\checkmark, \quad 6s + 4s - 10s = 0\,\checkmark$$

$\Longrightarrow$ linear dependence of u, v, w, since $r = s = t = 0$ cannot be concluded
for example: $s = 1$, $r = 3$, $t = -5$ $\quad \rightsquigarrow$

$$3u + v - 5w = 3(5,2,0,4)^{\mathrm{t}} + (5,4,5,3)^{\mathrm{t}} - 5(4,2,1,3)^{\mathrm{t}} = (0,0,0,0)^{\mathrm{t}}$$

b) $u = (2,4,0,-1)^{\mathrm{t}}$, $v = (2,3,-3,1)^{\mathrm{t}}$, $w = (4,5,-5,0)^{\mathrm{t}}$

application of the second criterion:
transforming

$$A = (u, v, w) = \begin{pmatrix} 2 & 2 & 4 \\ 4 & 3 & 5 \\ 0 & -3 & -5 \\ -1 & 1 & 0 \end{pmatrix}$$

to echelon form $\quad \rightsquigarrow$

$$\underset{(1)}{\to} \begin{pmatrix} 2 & 2 & 4 \\ 0 & -1 & -3 \\ 0 & -3 & -5 \\ 0 & 2 & 2 \end{pmatrix} \underset{(2)}{\to} \begin{pmatrix} 2 & 2 & 4 \\ 0 & -1 & -3 \\ 0 & 0 & 4 \\ 0 & 0 & -4 \end{pmatrix} \underset{(3)}{\to} \begin{pmatrix} \boxed{2} & 2 & 4 \\ 0 & \boxed{-1} & -3 \\ 0 & 0 & \boxed{4} \\ 0 & 0 & 0 \end{pmatrix} =: E$$

(1): $\quad$ row 2 $\leftarrow$ row 2 $-$ 2 $\cdot$ row 1, $\quad$ row 4 $\leftarrow$ row 4 + (1/2) $\cdot$ row 1

(2): $\quad$ row 3 $\leftarrow$ row 3 $-$ 3 $\cdot$ row 2, $\quad$ row 4 $\leftarrow$ row 4 + 2 $\cdot$ row 2

(3): $\quad$ row 4 $\leftarrow$ row 4 + row 3

echelon form E with 3 pivots (boxed) $\quad \Longrightarrow \quad$ linear independence

2.9 Coefficients of a Vector with Respect to a Basis

Represent $b = (2, -2, 1)^{\mathrm{t}}$ as linear combination of the basis vectors

$$u = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}, \quad v = \begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}.$$

Resources: Basis, Linear System, Cramer's Rule, Gauß Elimination

Problem Variants

■

$$u = \begin{pmatrix} -1 \\ -3 \\ 4 \end{pmatrix}, \quad v = \begin{pmatrix} 4 \\ -2 \\ -3 \end{pmatrix}, \quad w = \begin{pmatrix} -4 \\ 4 \\ 2 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$x = (?, ?, ?)$:

[ ] check

■

$$u = \begin{pmatrix} 4 \\ -3 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}, \quad w = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}, \quad b = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$$

$x = (-?, ?, ?)$:

[ ] check

■

$$u = \begin{pmatrix} -1 \\ -1 \\ 4 \end{pmatrix}, \quad v = \begin{pmatrix} -3 \\ -4 \\ 2 \end{pmatrix}, \quad w = \begin{pmatrix} -3 \\ -2 \\ 3 \end{pmatrix}, \quad b = \begin{pmatrix} 4 \\ 4 \\ 3 \end{pmatrix}$$

$x = (?, -?, -?)$:

[ ] check

Solution

The coefficients c_u, c_v, c_w of the linear combination

$$c_u \underbrace{\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}}_{u} + c_v \underbrace{\begin{pmatrix} -2 \\ -4 \\ 3 \end{pmatrix}}_{v} + c_w \underbrace{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}}_{w} = \underbrace{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}_{b} \iff \underbrace{(u, v, w)}_{A}\, c = b$$

can be determined by solving the linear system $Ac = b$ with Gauß elimination.

- transformation of the tableau

$$(A|b) = \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 2 & -4 & -1 & -2 \\ -3 & 3 & -1 & 1 \end{array}\right)$$

  to upper triangular form:

$$(A|b) \underset{(1)}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & 2 & 7 \end{array}\right) \underset{(2)}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -2 & 1 & 2 \\ 0 & -3 & 2 & 7 \\ 0 & 0 & -3 & -6 \end{array}\right)$$

  (1): row 2 ← row 2 − 2 · row 1, row 3 ← row 3 + 3 · row 1

  (2): interchange row 2 and row 3

- backward substitution:
  equation 3: $-3c_w = -6 \implies c_w = 2$
  equation 2: $-3c_v + 2c_w = -3c_v + 4 = 7 \implies c_v = -1$
  equation 1: $c_u - 2c_v + c_w = c_u + 2 + 2 = 2 \implies c_u = -2$

verification with MATLAB® :

```
u=[1;2,-3]; v=[-2;-4;3]; w=[1;-1;-1]; b=[2;-2;1];
c = [u,v,w]\b
```

Alternative solution

solution of the linear system $(u, v, w)c = b$ with Cramer's rule:

$$c_u = \frac{\det(b, v, w)}{\det(u, v, w)}, \quad c_v = \frac{\det(u, b, w)}{\det(u, v, w)}, \quad c_w = \frac{\det(u, v, b)}{\det(u, v, w)}$$

2.10 Coefficients of a Vector with Respect to an Orthogonal Basis

Represent $v = (-3, -4, 6, 2)^{\mathrm{t}}$ as linear combination of the basis vectors

$$u_1 = \begin{pmatrix} 4 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 1 \\ 4 \\ -2 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -2 \\ 4 \\ 1 \\ 2 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 1 \\ -2 \\ 2 \\ 4 \end{pmatrix}.$$

Resources: Orthogonal Basis

Problem Variants

■ $b = (1, 8)^{\mathrm{t}}$

$$u_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$$

---

coefficients: $?.?, -?.?$:

[ ] check

■ $b = (-3, 3, 7, -9)^{\mathrm{t}}$

$$u_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

---

coefficients: $-?, ?, ?, -?$:

[ ] check

■ $b = (2, 1, -7)^{\mathrm{t}}$

$$u_1 = \begin{pmatrix} 1 \\ -4 \\ -8 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -4 \\ 7 \\ -4 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -8 \\ -4 \\ 1 \end{pmatrix}$$

---

coefficients: $?.??, ?.??, -?.??$:

[ ] check

Solution

application of the formula

$$v = \sum_k \underbrace{\frac{u_k^{\mathrm{t}} v}{u_k^{\mathrm{t}} u_k}}_{c_k} u_k$$

for the coefficients c_k of a vector v with respect to an orthogonal basis $\{u_1, u_2, \ldots\}$

substituting $v = (-3, -4, 6, 2)^{\mathrm{t}}$ and $u_1 = (4, 2, -2, 1)^{\mathrm{t}}$, $u_2 = (2, 1, 4, -2)^{\mathrm{t}}$, $u_3 = (-2, 4, 1, 2)^{\mathrm{t}}$, $u_4 = (1, -2, 2, 4)^{\mathrm{t}}$ $\rightsquigarrow$

$$\begin{aligned} c_1 &= \frac{(4, 2, -2, 1)(-3, -4, 6, 2)^{\mathrm{t}}}{(4, 2, -2, 1)(4, 2, -2, 1)^{\mathrm{t}}} = \frac{-12 - 8 - 12 + 2}{16 + 4 + 4 + 1} = -\frac{30}{25} = -\frac{6}{5} \\ c_2 &= \frac{(2, 1, 4, -2)(-3, -4, 6, 2)^{\mathrm{t}}}{(2, 1, 4, -2)(2, 1, 4, -2)^{\mathrm{t}}} = \frac{10}{25} = \frac{2}{5} \end{aligned}$$

similarly: $c_3 = 0/25 = 0$, $c_4 = 25/25 = 1$

2.11 Orthogonal Basis in $\mathbb{C}^2$

Determine a vector $v \in \mathbb{C}^2$ which is orthogonal to $u = (5+\mathrm{i}, 1-3\mathrm{i})^{\mathrm{t}}$, normalize the two vectors, and compute the coordinates x_u and x_v of $x = (4-\mathrm{i}, 2)^{\mathrm{t}}$ with respect to the basis $B = \{u^\circ, v^\circ\}$.

Resources: Scalar Product, Orthogonal Basis

Problem Variants

■ $u = (1+\mathrm{i},\ 1-\mathrm{i})^{\mathrm{t}}$, $x = (2,\ -1-\mathrm{i})^{\mathrm{t}}$

$|x_v|^2 =$?:

[ ] check

■ $u = (2+\mathrm{i},\ 2\mathrm{i})^{\mathrm{t}}$, $x = (6+5\mathrm{i},\ -7-4\mathrm{i})^{\mathrm{t}}$

$|x_v|^2 =$??:

[ ] check

■ $u = (6,\ 3-2\mathrm{i})^{\mathrm{t}}$, $x = (3-8\mathrm{i},\ 7-5\mathrm{i})^{\mathrm{t}}$

$|x_v|^2 =$??:

[ ] check

Solution

Orthogonal vector v to $u = (5+\mathrm{i}, 1-3\mathrm{i})^{\mathrm{t}}$

$0 \stackrel{!}{=} \langle u, v\rangle = \bar{u}_1 v_1 + \bar{u}_2 v_2 \quad \rightsquigarrow \quad$ convenient choice

$$v_1 = -\bar{u}_2 = -1-3\mathrm{i}, \quad v_2 = \bar{u}_1 = 5-\mathrm{i} \tag{1}$$

⚠ The complex conjugation of u is essential. Proceeding as for a real scalar product, there would exist nontrivial (complex) vectors w with $|w|^2 = \langle w, w\rangle = 0$.

Normalization

$|u|^2 = |u_1|^2 + |u_2|^2 = |5+\mathrm{i}|^2 + |1-3\mathrm{i}|^2 = (5^2+1^2) + (1^2+3^2) = 36 \quad \rightsquigarrow$

$$u^\circ = u/|u| = (5+\mathrm{i}, 1-3\mathrm{i})^{\mathrm{t}}/6$$

special choice (1) of v with $|v_1| = |-\bar{u}_2| = |u_2|$, $|v_2| = |\bar{u}_1| = |u_1| \quad \Longrightarrow$

$$|v| = |u|, \quad v^\circ = (-1-3\mathrm{i}, 5-\mathrm{i})^{\mathrm{t}}/6$$

Coordinates of $x = (4-\mathrm{i}, 2)^{\mathrm{t}}$

$$\begin{aligned} x_u &= \langle u^\circ, x\rangle = (\bar{u}^\circ)^{\mathrm{t}} x = \frac{1}{6}\begin{pmatrix} 5-\mathrm{i}, & 1+3\mathrm{i}\end{pmatrix}\begin{pmatrix} 4-\mathrm{i} \\ 2\end{pmatrix} \\ &= \frac{20-5\mathrm{i}-4\mathrm{i}-1}{6} + \frac{2+6\mathrm{i}}{6} = \frac{7}{2} - \frac{1}{2}\mathrm{i} \\ x_v &= \underbrace{\frac{1}{6}\begin{pmatrix} -1+3\mathrm{i}, & 5+\mathrm{i}\end{pmatrix}}_{(\bar{v}^\circ)^{\mathrm{t}}}\underbrace{\begin{pmatrix} 4-\mathrm{i} \\ 2\end{pmatrix}}_{x} = \cdots = \frac{3}{2} + \frac{5}{2}\mathrm{i} \end{aligned}$$

⚠ The order of the arguments of the scalar products is important. Interchanging the arguments, the coordinates would not be linear in x.

coordinates $\quad \rightsquigarrow \quad$ representation

$$\begin{pmatrix} 4-\mathrm{i} \\ 2\end{pmatrix} = x_u u^\circ + x_v v^\circ = \frac{7-\mathrm{i}}{2}\begin{pmatrix} 5+\mathrm{i} \\ 1-3\mathrm{i}\end{pmatrix}\Big/6 + \frac{3+5\mathrm{i}}{2}\begin{pmatrix} -1-3\mathrm{i} \\ 5-\mathrm{i}\end{pmatrix}\Big/6$$

Verification with MATLAB®

```
u = [5+i; 1-3*i], v = [-1-3*i; 5-i], x = [4-i; 2]
u0 = u/norm(u), v0 = v/norm(v)
xu = u0'*x, xv = v0'*x
```

2.12 Projection onto a Plane

Project $x = (0, 1, 0)^t$ onto the plane spanned by the vectors $u = (1, 1, 0)^t$ and $v = (0, 1, 1)^t$.

Resources: Orthogonal Projection, Gram-Schmidt Algorithm

Problem Variants

■ $x = (-3, 0, -4)^t$, $u = (3, -2, 2)^t$, $v = (4, 1, -1)^t$

$(-?, ?, -?)^t$:

[ ] check

■ $x = (4, 0, -2)^t$, $u = (3, -1, 2)^t$, $v = (1, -4, -3)^t$

$(?, -?, ?)^t$:

[ ] check

■ $x = (-3, 4, 3)^t$, $u = (1, 2, -4)^t$, $v = (-1, -2, 0)^t$

$(?, ?, ?)^t$:

[ ] check

Note that in each variant the vector elements are permutations of $\{-4, -3, \ldots, 4\}$ and the projections are integer vectors.

Solution

Orthogonal basis $\{u, w\}$ for the plane

applying the algorithm of Gram-Schmidt to the vectors $u = (1, 1, 0)^{\mathrm{t}}$, $v = (0, 1, 1)^{\mathrm{t}}$, spanning the plane $\rightsquigarrow$

$$w = v - \frac{v^{\mathrm{t}}u}{u^{\mathrm{t}}u}u = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

Projection y of $x = (0, 1, 0)^{\mathrm{t}}$

adding the projections of x onto the orthogonal basis vectors $\rightsquigarrow$

$$\begin{aligned} y &= \frac{(0,1,0)u}{u^{\mathrm{t}}u}u + \frac{(0,1,0)w}{w^{\mathrm{t}}w}w = \frac{1}{2}u + \frac{1/2}{3/2}w \\ &= \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \end{aligned}$$

Checking the result

$(x - y) \overset{!}{\perp} u, v$:

$$(x-y)^{\mathrm{t}}u = \begin{pmatrix} -1/3, & 1/3, & -1/3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = 0 \quad \checkmark$$

similarly: $(x - y)^{\mathrm{t}}v = 0$

2.13 Orthogonal Polynomials

Construct the first three Laguerre polynomials

$$p_k(x) = x^k + \sum_{j=0}^{k-1} c_{k,j} x^j, \quad k = 0, \ldots, n \qquad (n = 2)$$

which are orthogonal with respect to the scalar product

$$\langle f, g \rangle = \int_0^\infty f(x) g(x) \mathrm{e}^{-x} \, \mathrm{d}x \, .$$

Resources: Scalar Product, Gram-Schmidt Algorithm

Problem Variants

■ $\langle f, g \rangle = \int_0^1 f(x) g(x) \, x \, \mathrm{d}x$, $n = 3$

$p_3(1)$ =?.???:

[ ] check

■ $\langle f, g \rangle = \int_{-1}^1 f(x) g(x) \, x^2 \, \mathrm{d}x$, $n = 4$

$p_4(1)$ =?.??:

[ ] check

■ $\langle f, g \rangle = \int_0^1 f(x) g(x) \, x(1-x) \, \mathrm{d}x$, $n = 2$

$p_2(1)$ =?.?:

[ ] check

Solution

application of the Gram-Schmidt recursion

$$p_k = q_k - \sum_{j=0}^{k-1} \frac{\langle p_j, q_k\rangle}{\langle p_j, p_j\rangle} p_j, \quad k = 0, 1, \ldots$$

to the monomials $q_k : x \mapsto x^k \quad \rightsquigarrow \quad p_0(x) = q_0(x) = 1$ (the sum $\sum_{j=0}^{k-1} \cdots$ is empty for $k = 0$) and

$$\begin{aligned}
p_1 &= q_1 - \frac{\langle p_0, q_1\rangle}{\langle p_0, p_0\rangle} p_0 = q_1 - \frac{\int_0^\infty x\, \mathrm{e}^{-x}\, \mathrm{d}x}{\int_0^\infty \mathrm{e}^{-x}\, \mathrm{d}x} p_0 = q_1 - \frac{1}{1} p_0 \\
&\rightsquigarrow \ p_1(x) = x - 1 \\
p_2 &= q_2 - \frac{\langle p_0, q_2\rangle}{\langle p_0, p_0\rangle} p_0 - \frac{\langle p_1, q_2\rangle}{\langle p_1, p_1\rangle} p_1 \\
&= q_2 - \frac{\int_0^\infty x^2\, \mathrm{e}^{-x}\, \mathrm{d}x}{\int_0^\infty \mathrm{e}^{-x}\, \mathrm{d}x} p_0 - \frac{\int_0^\infty (x-1) x^2\, \mathrm{e}^{-x}\, \mathrm{d}x}{\int_0^\infty (x-1)^2\, \mathrm{e}^{-x}\, \mathrm{d}x} p_1 \\
&= q_2 - \frac{2}{1} p_0 - \frac{6-2}{2-2+1} p_1 \\
&\rightsquigarrow \ p_2(x) = x^2 - 2 - 4(x-1) = x^2 - 4x + 2
\end{aligned}$$

used: $\int_0^\infty x^k \mathrm{e}^{-x}\, \mathrm{d}x = \Gamma(k+1) = k!$

<u>Alternative solution</u>

direct construction of $p_m(x) = x^m + \sum_{j=0}^{m-1} c_j\, x^j$, using that

$$\langle p_m, q_j\rangle = 0, \quad j = 0, \ldots, m-1$$

$\rightsquigarrow$ linear system
e.g., for $m = 2$:

$$\int_0^\infty (x^2 + c_1 x + c_0)\, \mathrm{e}^{-x} \mathrm{d}x = 0, \quad \int_0^\infty (x^2 + c_1 x + c_0) x\, \mathrm{e}^{-x} \mathrm{d}x = 0$$

substituting $\int_0^\infty x^3\, \mathrm{e}^{-x}\, \mathrm{d}x = 6$, $\int_0^\infty x^2\, \mathrm{e}^{-x}\, \mathrm{d}x = 2$, $\int_0^\infty x\, \mathrm{e}^{-x}\, \mathrm{d}x = 1$, $\int_0^\infty \mathrm{e}^{-x}\, \mathrm{d}x = 1 \quad \rightsquigarrow$

$$2 + c_1 + c_0 = 0, \quad 6 + 2c_1 + c_0 = 0$$

with the solution $c_1 = -4$, $c_0 = 2$

2.14 Orthogonal Basis for the Orthogonal Complement of a Subspace

Construct an orthogonal basis $\{u_3, u_4\}$ for the orthogonal complement of the subspace, spanned by the vectors

$$v_1 = (1,1,1,0)^{\mathrm{t}},\ v_2 = (0,1,1,1)^{\mathrm{t}}\,,$$

by applying the Gram-Schmidt algorithm to the sequence $\{v_1, v_2, e_1, e_2, \ldots\}$ with e_k the unit vectors of $\mathbb{R}^4$ ($(e_k)_j = \delta_{j,k}$).

Resources: Gram-Schmidt Algorithm

Problem Variants

■ $v_1 = (0,1,-2,1)^{\mathrm{t}}$, $v_2 = (-2,-3,0,-3)^{\mathrm{t}}$

$u_4 = (?,?,?,-?)^{\mathrm{t}}/2$:

[ ] check

■ $v_1 = (3,1,-1,-2,1)^{\mathrm{t}}$, $v_2 = (-3,-1,0,-2,-2)^{\mathrm{t}}$ $v_3 = (0,0,-3,-2,-3)^{\mathrm{t}}$

$u_5 = (?,?,?,?,-?)^{\mathrm{t}}/3$:

[ ] check

■ $v_1 = (-2,-1,1,3,-1)^{\mathrm{t}}$, $v_2 = (0,0,0,-2,2)^{\mathrm{t}}$

$u_5 = (?,?,?,-?,-?)^{\mathrm{t}}/3$:

[ ] check

For these problem variants, all denominators of the rational entries of the vectors u_k are less than 5.

Solution

Orthogonal basis $\{u_1, u_2\}$ for $V = \operatorname{span}\{v_1, v_2\}$

applying one step the Gram-Schmidt recursion

$$u_k = v_k - \sum_{j=1}^{k-1} \frac{u_j^{\mathrm{t}} v_k}{u_j^{\mathrm{t}} u_j} u_j, \quad k = 1, 2, \ldots,$$

to the vectors

$$v_1 = (1,1,1,0)^{\mathrm{t}},\ v_2 = (0,1,1,1)^{\mathrm{t}},$$

spanning V $\rightsquigarrow$

$$\begin{aligned} u_1 &= v_1 = (1,1,1,0)^{\mathrm{t}} \\ u_2 &= v_2 - \frac{u_1^{\mathrm{t}} v_2}{u_1^{\mathrm{t}} u_1} u_1 = (0,1,1,1)^{\mathrm{t}} - \frac{2}{3}(1,1,1,0)^{\mathrm{t}} = (-2,1,1,3)^{\mathrm{t}}/3 \end{aligned}$$

Orthogonal basis for $V^\perp$

$$\dim V^\perp = \dim \mathbb{R}^4 - \dim V = 4 - 2 = 2$$

Two basis vectors u_3, u_3, orthogonal to V (i.e. spanning $V^\perp$), can be constructed with two more steps ($k = 3, 4$) of the Gram-Schmidt recursion. To compute u_k, any vector v_k, which does not lie in the span U_{k-1} of the previously generated vectors $u_1, \ldots, u_{k-1}$, can be used, The unit vectors are a convenient choice. If, accidentally, the unit vector used lies in U_{k-1}, then the Gram-Schmidt recursion yields $u_k = (0, \ldots, 0)^{\mathrm{t}}$. In this exceptional case, simply the next unit vector is taken and the computation is repeated. This explains the dots ($\{v_1, v_2, e_1, e_2, \ldots\}$) in the statement of the problem. Proceeding in this fashion $\rightsquigarrow$

$$\begin{aligned} u_3 &= e_1 - \frac{u_1^{\mathrm{t}} e_1}{u_1^{\mathrm{t}} u_1} u_1 - \frac{u_2^{\mathrm{t}} e_1}{u_2^{\mathrm{t}} u_2} u_2 \\ &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{-2/3}{15/9} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \Big/ 3 = \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \Big/ 5 \end{aligned}$$

$$
\begin{aligned}
u_4 &= e_2 - \frac{u_1^{\mathrm{t}} e_2}{u_1^{\mathrm{t}} u_1} u_1 - \frac{u_2^{\mathrm{t}} e_2}{u_2^{\mathrm{t}} u_2} u_2 - \frac{u_3^{\mathrm{t}} e_2}{u_3^{\mathrm{t}} u_3} u_3 \\
&= \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{1/3}{15/9} \begin{pmatrix} -2 \\ 1 \\ 1 \\ 3 \end{pmatrix} \Big/ 3 - \frac{-1/5}{10/25} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \Big/ 5 \\
&= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \Big/ 2
\end{aligned}
$$

Chapter 3

Linear Maps and Matrices

3.1 Matrix of a Linear Map between Polynomials

Determine the matrix A with respect to the monomial basis, which represents the map assigning to a polynomial p with degree ≤ 2 the secant (a polynomial q with degree $\leq n = 1$), passing through the points $(2, p(2))$, $(3, p(3))$.

Resources: Matrix of a Linear Map

Problem Variants

■ $n = 2$, $q(x) = p(0)x^2 + 2p(1)$

$\sum_{j,k} a_{j,k} = ?$:

[ ] check

■ $n = 4$, $q(x) = (x^2 + 1)p(x)$

$\sum_{j,k} a_{j,k} = ?$:

[ ] check

■ $n = 1$, $q(x) = p''(x) - 2p'(x)$

$\sum_{j,k} a_{j,k} = -?$:

[ ] check

Solution

Lagrange representation of the secant q

formula for the interpolation at two points a, b:

$$q(x) = p(a)\frac{b-x}{b-a} + p(b)\frac{x-a}{b-a}$$

application with $a = 2$, $b = 3$ $\quad\rightsquigarrow$

$$\begin{aligned} q(x) &= p(2)(3-x) + p(3)(x-2) \\ &= (3p(2) - 2p(3))x^0 + (p(3) - p(2))x^1 \end{aligned}$$

Matrix A with respect to the monomial basis

$a_{j,k}$: coefficient of x^{j-1} of the secant $q(x)$ for $p(x) = x^{k-1}$, e.g.

$$\begin{aligned} a_{1,3} &\underset{p(x)=x^2}{=} \underbrace{3p(2) - 2p(3)}_{\text{coefficient of } x^0} = 3 \cdot 2^2 - 2 \cdot 3^2 = -6 \\ a_{2,2} &\underset{p(x)=x^1}{=} \underbrace{p(3) - p(2)}_{\text{coefficient of } x^1} = 3 - 2 = 1 \end{aligned}$$

analogous computation of the other matrix elements $\quad\rightsquigarrow$

$$A = \begin{pmatrix} 1 & 0 & -6 \\ 0 & 1 & 5 \end{pmatrix}$$

3.2 Matrix of a Parallel Projection

Determine the 2×3 matrix A of the linear map which projects $(x_1, x_2, x_3)^{\mathrm{t}}$ in the direction of the vector $v = (1, 2, 3)^{\mathrm{t}}$ onto the y_1y_2-plane.

Resources: Matrix of a Linear Map

Problem Variants

- A: 2×3, $v = (0, 3, -1)^{\mathrm{t}}$, projection onto the y_1y_3-plane

---

$\sum_{j,k} a_{j,k}$ =?.??:

[ ] check

- A: 2×2, $v = (1, 2)^{\mathrm{t}}$, projection onto the straight line $g : y = s\,(2, 1)^{\mathrm{t}}$

$\sum_{j,k} a_{j,k}$ =?:

[ ] check

- A: 3×3, $v = (1, 2, 0)^{\mathrm{t}}$, projection onto the plane $E : (0, 3, 4)\,y = 0$

$\sum_{j,k} a_{j,k} = -?.?$:

[ ] check

Solution

translation in the direction of the vector $(1,2,3)^{\mathrm{t}}$:

$$(x_1,x_2,x_3)^{\mathrm{t}} \mapsto y = (x_1 - t, x_2 - 2t, x_3 - 3t)^{\mathrm{t}}$$

$y_3 = x_3 - 3t = 0$ (projection onto the x_1x_2-plane) $\implies$ $t = x_3/3$, i.e. the projection has the form

$$x \mapsto \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 - x_3/3 \\ x_2 - 2x_3/3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -2/3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

3.3 Matrix, Describing a Change of Basis

Determine the matrix A of the linear map $x \mapsto y = Ax$ describing the coordinate transformation $v = \sum_k x_k e_k \to v = \sum_k y_k f_k$ corresponding to the bases

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ e_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \quad \text{and} \quad f_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix},\ f_2 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}.$$

Resources: Change of Basis

Problem Variants

■ $e_1 = (4,2)^t$, $e_2 = (1,3)^t$
$f_1 = (2,1)^t$, $f_2 = (3,4)^t$

$A = [?, -?; ?, ?]$ (MATLAB® notation):

[ ] check

■ $e_1 = (0,1,1)^t$, $e_2 = (1,0,1)^t$, $e_3 = (1,1,0)^t$
$f_1 = (0,1,-1)^t$, $f_2 = (1,-1,1)^t$, $f_3 = (-1,1,0)^t$

$A = [?, ?, ?; ?, ?, ?; ?, ?, ?]$ (MATLAB® notation):

[ ] check

$e_1 = (1,1,-1)^t$, $e_2 = (0,1,1)^t$, $e_3 = (0,0,1)^t$
$f_1 = (1,0,0)^t$, $f_2 = (-1,1,0)^t$, $f_3 = (1,-1,1)^t$

$A = [?, ?, -?; ?, ?, ?; ?, ?, ?]$ (MATLAB® notation):

[ ] check

Solution

General formula

$e_k = \sum_j a_{j,k} f_j$ (representation of e_k with respect to the basis $\{f_1, f_2\}$) $\Longrightarrow$

$$\sum_k x_k e_k = \sum_j \underbrace{\left(\sum_k a_{j,k} x_k \right)}_{y_j} f_j \,,$$

i.e. $y = Ax$ describes the coordinate transformation

Application to the given bases

$$\begin{aligned} e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \underbrace{(1/2)}_{a_{1,1}} \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{f_1} + \underbrace{(-1/6)}_{a_{2,1}} \underbrace{\begin{pmatrix} 0 \\ 3 \end{pmatrix}}_{f_2} \\ e_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} &= \underbrace{(1)}_{a_{1,2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \underbrace{(2/3)}_{a_{2,2}} \begin{pmatrix} 0 \\ 3 \end{pmatrix} \end{aligned}$$

$\rightsquigarrow$ matrix

$$A = \begin{pmatrix} 1/2 & 1 \\ -1/6 & 2/3 \end{pmatrix}$$

Alternative solution

With E (F) the matrix with the basis vectors e_k (f_j) as columns, the ℓ-th component of the identity $e_k = \sum_j a_{j,k} f_j$ equals

$$e_{\ell,k} = \sum_j a_{j,k} f_{\ell,j} \quad \Longleftrightarrow \quad E = FA$$

multiplying by F^{-1} and applying the formula for the inverse of a 2×2 matrix

$$\begin{aligned} A &= F^{-1}E = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \\ &= \frac{1}{2 \cdot 3 - 1 \cdot 0} \begin{pmatrix} 3 & 0 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 3 & 6 \\ -1 & 4 \end{pmatrix} \end{aligned}$$

3.4 Matrix, Determined by Images of Vectors

Determine the matrix which maps $(1,1)^t$ and $(2,3)^t$ to $(1,2)^t$.

Resources: Matrix of a Linear Map

Problem Variants

■ $(0,1)^t$, $(1,2)^t \mapsto (0,2)^t$

$[?, ?; -?, ?]$ (MATLAB® notation):

check

■ $(2,1)^t \mapsto (6,5)^t$, $(4,3)^t \mapsto (8,7)^t$

$[?, -?; ?, -?]$ (MATLAB® notation):

check

■ $(9,7,2)^t$, $(5,4,1)^t$, $(3,8,6)^t \mapsto (2,1,3)^t$

$[-?, ?, -?; -?, ?, -?; -?, ?, -?]$ (MATLAB® notation):

check

Solution

expressing the assertions of the problem,

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto y = Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad x = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \mapsto y = Ax = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

in matrix form $\quad\rightsquigarrow$

$$A \underbrace{\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}}_{X} = \underbrace{\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}_{Y}$$

applying the formula for the inverse of a 2×2-matrix $\quad\rightsquigarrow$

$$A = YX^{-1} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \frac{1}{1 \cdot 3 - 1 \cdot 2} \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$$

<u>Alternative solution</u>

Since $x = (1,1)^{\mathrm{t}}$ and $y = (2,3)^{\mathrm{t}}$ are linearly independent, the matrix A maps all vectors to multiples of $v = (1,2)^{\mathrm{t}}$. Hence, the columns of A (images of the unit vectors) are parallel to v, i.e.

$$A = (rv, sv) = v\,(r,s) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (r,\ s)\,.$$

$v = Ax = v(rx_1 + sx_2) \quad \Longrightarrow \quad (rx_1 + sx_2) = 1$ and, with $x = (1,1)^{\mathrm{t}}$ and $x = (2,3)^{\mathrm{t}}$

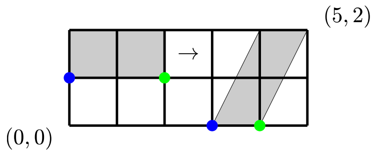
$$r + s = 1,\ 2r + 3s = 1, \quad \text{i.e.}\ r = 2, s = -1$$

consequently,

$$A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} (2,\ -1) = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix}$$

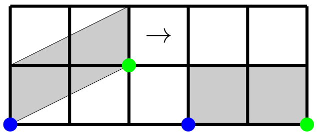
3.5 Affine Transformation of Rectangles and Parallelograms

Determine the affine transformation $x \mapsto Ax - b$, which, as depicted in the figure, maps the rectangle to the parallelogram.



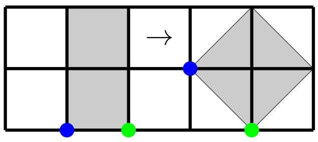
Resources: Affine Map

Problem Variants

■ 

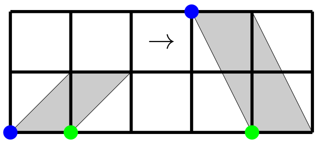
$[A, b] = [?, ?, ?; -?.?, ?, ?]$ (MATLAB® notation):

check

■ 

$[A, b] = [?, ?.?, ?; -?, ?.?, ?]$ (MATLAB® notation):

check

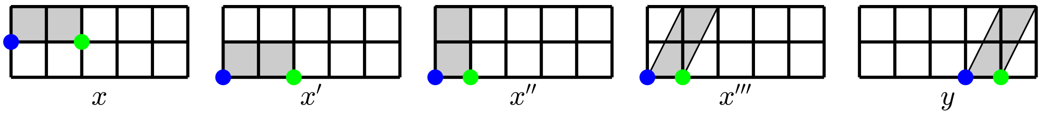
■ 

$[A, b] = [?, -?, ?; ?, ?, ?]$ (MATLAB® notation):

check

Solution

representation of the transformation $y = Ax + b$ in terms of elementary maps



- translation:

$$x \mapsto x' = \begin{pmatrix} x_1 \\ x_2 - 1 \end{pmatrix}$$

- scaling:

$$x' \mapsto x'' = \begin{pmatrix} x_1'/2 \\ 2x_2' \end{pmatrix} = \begin{pmatrix} x_1/2 \\ 2x_2 - 2 \end{pmatrix}$$

- shearing (determined by the images of the unit vectors): $(1,0)^{\mathrm{t}} \mapsto (1,0)^{\mathrm{t}}$, $(0,2)^{\mathrm{t}} \mapsto (1,2)^{\mathrm{t}} \iff (0,1)^{\mathrm{t}} \mapsto (1/2,1)^{\mathrm{t}} \implies$

$$x'' \mapsto x''' = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1'' \\ x_2'' \end{pmatrix} = \begin{pmatrix} x_1'' + x_2''/2 \\ x_2'' \end{pmatrix} = \begin{pmatrix} x_1/2 + x_2 - 1 \\ 2x_2 - 2 \end{pmatrix}$$

- translation:

$$x''' \mapsto y = \begin{pmatrix} x_1''' + 3 \\ x_2''' \end{pmatrix} = \begin{pmatrix} x_1/2 + x_2 + 2 \\ 2x_2 - 2 \end{pmatrix}$$

affine map

$$y = \underbrace{\begin{pmatrix} 1/2 & 1 \\ 0 & 2 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 2 \\ -2 \end{pmatrix}}_{b}$$

<u>Alternative solution</u>

Choosing three vertices x, x', x'', the matrix A can be determined from the equation

$$\left(\ y - y',\ y - y''\ \right) = A \left(\ x - x',\ x - x''\ \right)\ ,$$

and $b = y - Ax$ is the translation.

Matlab® -script

```
x = [0;1]; xp = [1;1]; xpp = [0;2];
y = [3;0]; yp = [4;0]; ypp = [4;2];
A = [y-yp, y-ypp]*inv([x-xp, x-xpp]), b = y-A*x
```

3.6 Products of Vectors and Matrices

For

$$A = \begin{pmatrix} i & 1 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & i \\ -i & 1 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ i \end{pmatrix}, \quad w = \begin{pmatrix} i \\ 2 \end{pmatrix},$$

compute A^*vw^*B.

Resources: Matrix Multiplication

Problem Variants

■ v^*A^*Bw

??:

[ ] check

■ AB^*vw^*

$[-?, -??; ??, -?]$ (MATLAB® notation):

[ ] check

■ $A(v^*w)B^*$

$[-?, ?; ?, ?]$ (MATLAB® notation):

[ ] check

Solution

transposition and complex conjugation ($C \to C^* = \bar{C}^{\mathrm{t}}$) $\quad \rightsquigarrow$

$$A^* = \begin{pmatrix} \mathrm{i} & 1 \\ 2 & 0 \end{pmatrix}^* = \begin{pmatrix} -\mathrm{i} & 2 \\ 1 & 0 \end{pmatrix}, \quad w^* = \begin{pmatrix} \mathrm{i} \\ 2 \end{pmatrix}^* = \begin{pmatrix} -\mathrm{i} & 2 \end{pmatrix}$$

multiplying with $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} \quad \rightsquigarrow$

$$A^*v = \begin{pmatrix} -\mathrm{i} & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix}, \quad w^*B = \begin{pmatrix} -\mathrm{i} & 2 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix} = \begin{pmatrix} -2\mathrm{i} & 3 \end{pmatrix}$$

product of the two results

$$A^*vw^*B = \begin{pmatrix} \mathrm{i} \\ 1 \end{pmatrix} \begin{pmatrix} -2\mathrm{i} & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3\mathrm{i} \\ -2\mathrm{i} & 3 \end{pmatrix}$$

3.7 Products of Matrices and Vectors with MATLAB®

Define

$$A = \begin{pmatrix} 0 & 3 \\ 2 & 1 \end{pmatrix} =: \begin{pmatrix} a & b \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} =: \begin{pmatrix} c \\ d \end{pmatrix},$$

and compute the following products:

a) Aa, $b^{\mathrm{t}}A$, Cc^{t}, dC^{t} b) A^2, AC, CC^{t}, $C^{\mathrm{t}}C$ c) $a^{\mathrm{t}}b$, $c^{\mathrm{t}}d$

Resources: Matrix Operations with MATLAB®

Problem Variants

In the following variants, compute all products (with 2 factors) of the matrices A, B, and the vectors c, d.

■ $A = \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \end{pmatrix} =: \underbrace{[[B;c],d]}_{\text{MATLAB® notation}}$

---

sum of all entries of the products: ?????:

check

■ $A = \begin{pmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 31 & 32 & 33 \\ 41 & 42 & 43 \end{pmatrix} =: \underbrace{[[B;c],[d;0]]}_{\text{MATLAB® notation}}$

---

sum of all entries of the products: ?????:

check

■ $A = \begin{pmatrix} 11 & 12 & 13 & 14 \\ 21 & 22 & 23 & 24 \\ 31 & 32 & 33 & 34 \end{pmatrix} =: \underbrace{[[B,c];d]}_{\text{MATLAB® notation}}$

---

sum of all entries of the products: ?????:

check

Solution

Definition of the matrices and vectors

```
% 2x2 matrix with columns a and b
A = [0 3; 2 1], a = A(:,1), b = A(:,2)
A =              a =          b =
    0   3            0            3
    2   1            2            1

% 2x3 matrix with rows c and d
c = [0, 1, 2], d = [2, 1, 0], C = [c; d]
c =             d =              C =
      0   1   2       2   1   0       0   1   2
                                      2   1   0
```

Matrix/vector products

matrix $*$ column vector $\to$ column vector
row vector $*$ matrix $\to$ row vector

```
A*a          b'*A          C*c'          d*C'
6            2   10        5             1   5
2                          1
```

Matrix/matrix products

$\ell \times m$ matrix $*$ $m \times n$ matrix $\to$ $\ell \times n$ matrix

```
A^2          A*C           C*C'          C'*C
6   3        6   3   0     5   1         4   2   0
2   7        2   3   4     1   5         2   2   2
                                         0   2   4
```

Vector/vector products

column vector $*$ row vector $\to$ matrix
row vector $*$ column vector of the same dimension $\to$ number

```
a*b'   % matrix with rank 1             c*d'   % scalar product
0   0                                   1
6   2
```

3.8 Vector/Matrix Operations with Maple™

For the vectors and matrices

$$\begin{array}{llll} p: & 3\times 1,\ p_j = j, & q: & 1\times 2,\ q_k = v_k, \\ A: & 2\times 2,\ a_{j,k} = j^k, & B: & 2\times 3,\ b_{j,k} = j-k, \end{array}$$

compute the products qA, Bp, AB, as well as the first row of A^2, and the last two columns of $B^{\mathrm{t}}B$.

Resources: Matrix Operations with Maple™

Problem Variants

■ For $a_{j,k} = j + k^2$, $p = (1,2)^{\mathrm{t}}$, $q = (q_1, q_2, q_3)^{\mathrm{t}}$ compute

$$p^{\mathrm{t}}Aq, \quad Aqp^{\mathrm{t}}, \quad qp^{\mathrm{t}}A\,,$$

choosing the dimensions of A appropriately.

---

largest entry of all results with $q_k = 1$: ??:

[ ] check

■ For $a_{j,k} = \mathrm{i}^{jk}$ $(\mathrm{i} = \sqrt{-1})$, $p = (0,1,2,3)^{\mathrm{t}}$ compute

$$App^{\mathrm{t}}A, \quad p^{\mathrm{t}}AAp\,.$$

---

smallest absolute value of all results: ?:

[ ] check

■ For $a_{j,k} = (j-k)^{j+k}$ compute

$$A(:,1)^{\mathrm{t}}AA(2:3,:)^{\mathrm{t}}\,.$$

---

largest entry of the result: ????:

[ ] check

Solution

Definition of the vectors and matrices

```
with(LinearAlgebra):   # loading relevant functions
p := Vector([1,2,3]); q := Vector[row](2,symbol=v);
```

$$p := \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad q := \begin{bmatrix} v_1 & v_2 \end{bmatrix}$$

```
A := Matrix([[1,1],[1,4]]); B := Matrix(2,3,(j,k)->j-k);
```

$$A := \begin{bmatrix} 1 & 1 \\ 1 & 4 \end{bmatrix}, \quad B := \begin{bmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \end{bmatrix}$$

Matrix/vector products

```
qA := VectorMatrixMultiply(q,A);
Bp := MatrixVectorMultiply(B,p);
```

$$qA := \begin{bmatrix} v_1 + v_2 & v_1 + 4v_2 \end{bmatrix}, \quad Bp := \begin{bmatrix} -8 \\ -2 \end{bmatrix}$$

Matrix/matrix products

```
AB := MatrixMatrixMultiply(A,B);
```

$$\begin{bmatrix} 1 & -1 & -3 \\ 4 & -1 & -6 \end{bmatrix}$$

```
r := MatrixMatrixMultiply(A[1,1..2],A);
C := MatrixMatrixMultiply(Transpose(B),B[1..2,2..3]);
```

$$r := \begin{bmatrix} 2 & 5 \end{bmatrix}, \quad C := \begin{bmatrix} 0 & -1 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}$$

```
# alternative
AA := MatrixMatrixMultiply(A,A): r := AA[1,1..2];
BtB := MatrixMatrixMultiply(Transpose(B),B):
C := BtB[1..3,2..3];
```

3.9 Multiplication of Sparse Matrices

The list[1]

j	1	2	3	3	5	6	7	8
k	4	8	1	7	5	2	4	6
$a_{j,k}$	4	-4	2	-3	3	-1	1	-2

contains the nonzero elements of an 8×8 matrix A. Determine the list for A^2 without assembling the matrix A explicitly.

Resources: Matrix Multiplication

Problem Variants

■

j	1	5	5	6	7	7	8	8
k	4	2	3	1	6	8	3	7
$a_{j,k}$	-3	1	-4	-1	3	2	-2	4

---

sum of the entries of A^2: ??:
[ ] check

■

j	1	2	3	3	4	5	6	7
k	1	5	5	7	8	8	4	1
$a_{j,k}$	-4	-1	-2	4	3	2	1	-3

---

sum of the entries of A^2: ??:
[ ] check

■

j	1	2	5	5	6	6	7	8
k	3	3	1	3	1	8	1	7
$a_{j,k}$	-1	-4	3	-3	2	1	-2	4

---

sum of the entries of A^2: $-$?:
[ ] check

[1] a suitable way to store *sparse matrices*

Solution

definition of the matrix product $C = A\,A$:

$$c_{j,\ell} = \sum_k a_{j,k} a_{k,\ell}$$

common middle index k of the two factors of the summands $\rightsquigarrow$ relevant combinations of index pairs (j, k) from the list

j	1	2	3	3	5	6	7	8
k	4	8	1	7	5	2	4	6
$a_{j,k}$	4	-4	2	-3	3	-1	1	-2

for example,

- $(j, k) = (1, 4)$: $a_{1,4}$ not possible as first factor since $a_{4,\ell} = 0\ \forall \ell$ (no index pair $(4, \ell)$ in the list)
- $(j, k) = (2, 8)$: $a_{2,8}$ possible as first factor since $\exists$ index pair $(k, \ell) = (8, 6)$ with first index $k = 8$ (common middle index) and

  $$a_{2,8} a_{8,6} = (-4) \cdot (-2) = 8$$

  contributes as a summand to $c_{2,6}$

analogously:

$$\begin{array}{lcl}
a_{3,1}a_{1,4} = 2 \cdot 4 = 8 & \underset{\text{contributing summand}}{\rightsquigarrow} & c_{3,4} \\
a_{3,7}a_{7,4} = (-3) \cdot 1 = -3 & \rightsquigarrow & c_{3,4} \\
a_{5,5}a_{5,5} = 3 \cdot 3 = 9 & \rightsquigarrow & c_{5,5} \\
a_{6,2}a_{2,8} = (-1) \cdot (-4) = 4 & \rightsquigarrow & c_{6,8} \\
a_{7,4}a_{4,\ell} \text{ not possible} & & \\
a_{8,6}a_{6,2} = (-2) \cdot (-1) = 2 & \rightsquigarrow & c_{8,2}
\end{array}$$

adding the contributions to equal matrix elements (in this problem only the summands contributing to $c_{3,4}$: $a_{3,1}a_{1,4} + a_{3,7}a_{7,4} = 8 - 3 = 5$) $\rightsquigarrow$ list for $C = A\,A$

j	2	3	5	6	8
k	6	4	5	8	2
$c_{j,k}$	8	5	9	4	2

Verification with MATLAB®

```
j = [1; 2; 3; 3; 5; 6; 7; 8];
k = [4; 8; 1; 7; 5; 2; 4; 6];
ajk = [4; -4; 2; -3; 3; -1; 1; -2];
A = sparse(j,k,ajk,8,8);
A^2
```

3.10 Zero Patterns of Matrices

Assume that the entries of the matrices A and B satisfy

$$a_{j,k} = 0,\ k < j + p \ \wedge\ b_{j,k} = 0,\ k < j + q\,.$$

Which entries $c_{j,k}$ of the product $C = AB$ are guaranteed to be zero? Marking possibly nonzero entries with x,

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{C} = \underbrace{\begin{pmatrix} 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{B},$$

illustrates the case $p = 1$, $q = 2$.

Resources: Matrix Multiplication

Problem Variants

■ $a_{j,k} = 0$ for $k > j - p$, $b_{j,k} = 0$ for $k > j - q$

$c_{j,k} = 0$ for $k > j???q$:

[ ] check

■ $a_{j,k} = 0$ for $|j - k| > p$, $b_{j,k} = 0$ for $|j - k| > q$

$c_{j,k} = 0$ for $|j??| > p??$:

[ ] check

■ $a_{j,k} = 0$ for $k \neq j + p$, $b_{j,k} = 0$ for $k \neq j + q$

$c_{j,k} = 0$ for $k \neq j?p??$:

[ ] check

Solution

definition of matrix multiplication:

$$C = AB \quad \Longleftrightarrow \quad c_{j,k} = \sum_{\ell} a_{j,\ell} b_{\ell,k}$$

omitting the summands with $a_{j,\ell} = 0$ $(\ell < j + p)$ or $b_{\ell,k} = 0$ $(k < \ell + q)$ $\rightsquigarrow$ restriction of the summation range:

$$\ell \geq j + p \quad \wedge \quad k \geq \ell + q \quad \Longleftrightarrow \quad j + p \leq \ell \leq k - q$$

The range of admissible indices ℓ is empty, implying $c_{j,k} = 0$, if no index ℓ satisfies both inequalities, i.e. if $k < j + p + q$.

3.11 LU-Factorization of a Matrix

Determine the unknown matrix elements of the factorization

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix}}_{L} \underbrace{\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}}_{U}$$

of A as a product of a lower and an upper triangular matrix.

Resources: Matrix Multiplication

Problem Variants

■ $A = \begin{pmatrix} 4 & -5 & -1 \\ 4 & -8 & 5 \\ -8 & 4 & 6 \end{pmatrix}$

$(x, y, z) = (?, -?, ?)$:

[ ] check

■ $A = \begin{pmatrix} 3 & 4 & -7 \\ -3 & 4 & -1 \\ -6 & 0 & 2 \end{pmatrix}$

$(x, y, z) = (-?, -?, ?)$:

[ ] check

■ $A = \begin{pmatrix} -1 & -2 & -7 \\ -1 & 0 & -1 \\ -5 & 4 & 3 \end{pmatrix}$

$(x, y, z) = (?, ?, ?)$:

[ ] check

Solution

successive comparison of the matrix elements $\quad \rightsquigarrow$

- $1 = a_{1,1} = 1 \cdot a \quad \Longrightarrow \quad a = 1$
- $0 = a_{1,2} = 1 \cdot b \quad \Longrightarrow \quad b = 0$
- $1 = a_{1,3} = 1 \cdot c \quad \Longrightarrow \quad c = 1$
- $1 = a_{2,1} = x \cdot a \quad \Longrightarrow \quad x = 1$
- $1 = a_{2,2} = x \cdot b + 1 \cdot d \quad \Longrightarrow \quad d = 1$
- $1 = a_{2,3} = x \cdot c + 1 \cdot e \quad \Longrightarrow \quad e = 0$
- $0 = a_{3,1} = y \cdot a \quad \Longrightarrow \quad y = 0$
- $1 = a_{3,2} = y \cdot b + z \cdot d \quad \Longrightarrow \quad z = 1$
- $1 = a_{3,3} = y \cdot c + z \cdot e + 1 \cdot f \quad \Longrightarrow \quad f = 1$

$\rightsquigarrow$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

MATLAB® `[L,U] = lu(A)`

3.12 Rank of a Matrix

Determine the rank of the matrix

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ 3 & 4 & 1 \\ 5 & 7 & 1 \end{pmatrix}.$$

Resources: Rank

Problem Variants

■ $\begin{pmatrix} 2 & 6 & 8 & 6 \\ 1 & 7 & 5 & 9 \\ 1 & 3 & 4 & 3 \end{pmatrix}$

?:
[ ] [check]

■ $\begin{pmatrix} 3 & 1 & 3 & 3 \\ 2 & 1 & 2 & 2 \\ 1 & 3 & 1 & 1 \\ 3 & 2 & 3 & 1 \\ 2 & 1 & 2 & 4 \end{pmatrix}$

?:
[ ] [check]

■ $\begin{pmatrix} 2 & 0 & 1 & 2 & 0 \\ -1 & -2 & 2 & 1 & -1 \\ 5 & 2 & 0 & 3 & 1 \\ 6 & 4 & -2 & 2 & 2 \end{pmatrix}$

?:
[ ] [check]

Solution

transformation of the matrix A to triangular form with Gauß transformations (which leave the rank of A invariant)

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 5 & -1 \\ 3 & 4 & 1 \\ 5 & 7 & 1 \end{pmatrix} \underset{(1)}{\rightarrow} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & -2 & 4 \\ 0 & -3 & 6 \end{pmatrix} \underset{(2)}{\rightarrow} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(1):
row 2 $\leftarrow$ row 2 $-$ 3 $\cdot$ row 1
row 3 $\leftarrow$ row 3 $-$ 3 $\cdot$ row 1
row 4 $\leftarrow$ row 4 $-$ 5 $\cdot$ row 1

(2):
row 3 $\leftarrow$ row 3 $-$ 2 $\cdot$ row 2
row 4 $\leftarrow$ row 4 $-$ 3 $\cdot$ row 2

$\Longrightarrow$ rank $A = 2$ (number of nonzero diagonal elements)

3.13 Rank of a Matrix and Orthogonal Basis of Its Kernel

Determine the rank of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 & 2 \end{pmatrix}$$

and construct an orthogonal basis for its kernel.

Resources: Rank, Range and Kernel, Gram-Schmidt Algorithm

Problem Variants

■ $A = \begin{pmatrix} -2 & 2 & 0 & -3 \\ 3 & -1 & -3 & 2 \\ 1 & 1 & -3 & -1 \end{pmatrix}$

---

rank A =?:

[ ] check

■ $A = \begin{pmatrix} 3 & 3 & 0 & 3 & -2 \\ 1 & 2 & 2 & -1 & -3 \\ -1 & 0 & -1 & -2 & -3 \\ 1 & 1 & -3 & 2 & -2 \end{pmatrix}$

---

rank A =?:

[ ] check

■ $A = \begin{pmatrix} -2 & 1 & -1 & 3 \\ 2 & -1 & 1 & -2 \end{pmatrix}$

---

rank A =?:

[ ] check

Solution

Rank

transformation of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 2 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 2 & 1 & 2 & 1 & 2 \end{pmatrix}$$

to row echelon form E (rank and kernel invariant) with Gauß transformations:

$$A \stackrel{(1)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & -2 & -1 & 1 & -3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix} \stackrel{(2)}{\longrightarrow} \begin{pmatrix} 1 & 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} =: E$$

(1): row 4 $\leftarrow$ row 4 $-$ row 2, row 2 $\leftarrow$ row 2 $-$ 2 $\times$ row 1

(2): row 4 $\leftarrow$ row 4 $-$ row 3, row 2 $\leftarrow$ row 2 + 2 $\times$ row 3, permutation of row 2 and row 3

3 nontrivial rows of $E \quad \Longrightarrow \quad \operatorname{rank} A = \operatorname{rank} E = 3$

Orthogonal basis for $\ker A$

general solution x of $Ex = (0, 0, 0, 0)^{\mathrm{t}}$ via backward substitution:

$$x_5 = s,\ x_4 = t \quad \text{(arbitrary)}$$
$$x_3 = -t + s,\ x_2 = -x_3 - s = t - 2s,\ x_1 = -x_2 - x_3 - 2s = -s$$

i.e.

$$\begin{aligned} x &= (-s, t-2s, -t+s, t, s)^{\mathrm{t}} \\ &= s\,\underbrace{(-1,-2,1,0,1)^{\mathrm{t}}}_{u} + t\,\underbrace{(0,1,-1,1,0)^{\mathrm{t}}}_{v} \end{aligned}$$

$\rightsquigarrow \quad$ basis $\{u, v\}$ for the kernel of A

orthogonalization ($v \to \tilde{v} \perp u$) with one step of the Gram-Schmidt algorithm:

$$\begin{aligned} \tilde{v} &= v - \frac{v^{\mathrm{t}}u}{u^{\mathrm{t}}u}u = (0,1,-1,1,0)^{\mathrm{t}} - \frac{-3}{7}(-1,-2,1,0,1)^{\mathrm{t}} \\ &= \frac{1}{7}(-3,1,-4,7,3)^{\mathrm{t}} \end{aligned}$$

$\rightsquigarrow \quad$ orthogonal basis $\{(-1,-2,1,0,1)^{\mathrm{t}},\ (-3,1,-4,7,3)^{\mathrm{t}}\}$

Alternative solution

computation of the singular value decomposition

$$A = USV^{\mathrm{t}}, \quad U^{-1} = U^{\mathrm{t}},\, V^{-1} = V^{\mathrm{t}},\, S = \mathrm{diag}(s_1, \ldots, s_r, 0, \ldots)$$

The number r of singular values is equal to $\operatorname{rank} A$. An orthogonal basis for $\ker A$ consists of the columns $r+1, r+2, \ldots$ of V.

This procedure is used by the MATLAB® function `B = null(A)`, which computes an orthogonal basis for $\ker A$ directly.

3.14 Rank of a Matrix and Orthogonal Basis for Its Range

Determine the rank of the Matrix

$$A = \begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

and construct an orthogonal basis for its range.

Resources: Rank, Range and Kernel, Gram-Schmidt Algorithm

Problem Variants

- $A = \begin{pmatrix} -3 & -2 & -3 \\ 2 & -4 & 3 \\ -1 & -1 & 1 \\ 0 & 1 & 4 \end{pmatrix}$

---

rank A =?:

[ ] check

- $A = \begin{pmatrix} -1 & -2 & 3 \\ 4 & 0 & -4 \\ 1 & 3 & -4 \\ 2 & -3 & 1 \\ 4 & -1 & -3 \end{pmatrix}$

---

rank A =?:

[ ] check

- $A = \begin{pmatrix} -4 & 1 & -1 & 4 \\ 4 & 2 & 3 & 0 \\ 1 & -2 & -3 & 3 \\ 0 & -3 & -2 & -4 \end{pmatrix}$

---

rank A =?:

[ ] check

Solution

Rank

transformation of the Matrix

$$A = \begin{pmatrix} 2 & 2 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 \\ 2 & 0 & 1 & 0 \\ 1 & 1 & 2 & 2 \end{pmatrix}$$

to column echelon form (rank and range invariant) with Gauß transformations, applied to the columns:

$$\stackrel{(1)}{\to} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & -2 & -1 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix} \stackrel{(2)}{\to} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 2 \\ 2 & -2 & 1 & 2 \\ 1 & 0 & 1 & 2 \end{pmatrix} \stackrel{(3)}{\to} \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix} =: E$$

(1): column 2 ← column 2 − column 1, column 3 ← column 3 − column 1

(2): column 3 ← column 3 − column 2, column 4 ← column 4 − column 2

(3): column 4 ← column 4 − 2 × column 3

3 nontrivial columns $\implies$ $\operatorname{rank} A = \operatorname{rank} E = 3$

Orthogonal basis for the range

basis of the range: nontrivial columns of E (scaled)

$$u = (0,0,1,1,1)^{\mathrm{t}}, \quad v = (0,1,0,-2,0)^{\mathrm{t}}, \quad w = (2,0,1,2,1)^{\mathrm{t}}$$

orthogonalization ($v \to \tilde{v}$, $w \to \tilde{w}$) with the Gram-Schmidt algorithm

$$\begin{aligned} \tilde{v} &= v - \frac{v^{\mathrm{t}}u}{u^{\mathrm{t}}u}u = (0,1,0,-2,0)^{\mathrm{t}} - \frac{-2}{3}(0,0,1,1,1)^{\mathrm{t}} \\ &= (0,1,2/3,-4/3,2/3)^{\mathrm{t}} \\ \tilde{w} &= w - \frac{w^{\mathrm{t}}u}{u^{\mathrm{t}}u}u - \frac{w^{\mathrm{t}}\tilde{v}}{\tilde{v}^{\mathrm{t}}\tilde{v}}\tilde{v} \\ &= (2,0,1,2,1)^{\mathrm{t}} - \frac{4}{3}(0,0,1,1,1)^{\mathrm{t}} - \frac{-4/3}{33/9}(0,1,2/3,-4/3,2/3)^{\mathrm{t}} \\ &= (2,4/11,-1/11,2/11,-1/11)^{\mathrm{t}} \end{aligned}$$

orthogonal basis (scaled): $(0,0,1,1,1)^{\mathrm{t}}$, $(0,3,2,-4,2)^{\mathrm{t}}$, $(22,4,-1,2,-1)^{\mathrm{t}}$

3.15 Matrix with Prescribed Range and Kernel

Construct a matrix A with the plane $E:\ x_1 - x_2 + x_3 = 0$ as range and the straight line $g:\ t(1,1,1)^{\mathrm{t}}$ as kernel.

Resources: Range and Kernel

Problem Variants

■ range $g:\ t(1,-2,3)$, kernel $E:\ 3x_1 - 2x_2 + x_3 = 0$

$[3,-2,?;-?,?,-?;?,-?,?]$ (MATLAB® notation):

[ ] check

■ range $g:\ t(1,2)$, kernel $g:\ t(-3,4)$,

$[4,?;?,?]$ (MATLAB® notation):

[ ] check

■ range and kernel $\mathrm{span}\{(1,1,1,0)^{\mathrm{t}},(0,1,1,1)^{\mathrm{t}}\}$

$[-2,1,?,-?;-?,2,0,-?;-?,?,0,-2;?,?,?,?]$ (MATLAB® notation):

[ ] check

Solution

choosing two vectors which span the plane $E:\, x_1 - x_2 + x_3 = 0$, e.g.,

$$u = (1,1,0)^{\mathrm{t}}, \quad v = (0,1,1)^{\mathrm{t}}$$

$\rightsquigarrow$ ansatz[2] $A = (u, v, w)$ with $w = su + tv$

$\ker A = \operatorname{span}(1,1,1)^{\mathrm{t}} \iff$

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = u + v + w = (1+s)u + (1+t)v = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

i.e. $s = t = -1$ and $A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix}$

[2]In general, a matrix with range E has the form $\begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} x^{\mathrm{t}} \\ y^{\mathrm{t}} \end{pmatrix}$ with $\{u, v\}$ a basis for E and two linearly independent vectors x, y

Chapter 4

Determinants

4.1 Determinants of 3×3 Matrices

Compute the determinants

$$\begin{vmatrix} 1 & 1 & 3 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 4 & 3 \\ 2 & 4 & 2 \\ 3 & 4 & 1 \end{vmatrix}, \qquad \begin{vmatrix} 2 & 5 & 6 \\ 4 & 4 & 4 \\ 6 & 5 & 2 \end{vmatrix}.$$

Resources: Determinant, Properties of Determinants, Expansion of Determinants

Problem Variants

- $\begin{vmatrix} -3 & 0 & 3 \\ -2 & -1 & 2 \\ 1 & 4 & -4 \end{vmatrix}, \begin{vmatrix} -6 & 0 & 6 \\ 4 & 2 & -4 \\ -3 & 0 & 3 \end{vmatrix}, \begin{vmatrix} 3 & 0 & -3 \\ -6 & -3 & 6 \\ 4 & 4 & -7 \end{vmatrix}$

−?, ?, ??:

[ ] check

- $\begin{vmatrix} -4 & -2 & 0 \\ 4 & 2 & -3 \\ 2 & 1 & -1 \end{vmatrix}, \begin{vmatrix} -2 & -2 & 2 \\ 2 & 2 & -5 \\ 2 & 1 & -2 \end{vmatrix}, \begin{vmatrix} 6 & -4 & 2 \\ 6 & 4 & -8 \\ 5 & 2 & -3 \end{vmatrix}$

?, ?, ??:

[ ] check

- $\begin{vmatrix} 2 & 4 & -2 \\ -1 & -4 & 3 \\ 0 & 1 & -3 \end{vmatrix}, \begin{vmatrix} 4 & 8 & -4 \\ 2 & 3 & 1 \\ 0 & 2 & -6 \end{vmatrix}, \begin{vmatrix} -2 & -4 & 2 \\ -3 & -7 & 2 \\ 0 & -1 & 3 \end{vmatrix}$

?, ?, ?:

[ ] check

Solution

application of the rules for a determinant $\det(a,b,c)$ with columns a, b, c

$a=(1,2,3)^{\mathrm{t}}$, $b=(1,0,1)^{\mathrm{t}}$, $c=(3,2,1)^{\mathrm{t}}$

Sarrus scheme

$$\begin{aligned}\det(a,b,c) &= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \\ &= a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1\end{aligned}$$

$\rightsquigarrow \quad \det(a,b,c) = (0+6+6)-(2+2+0) = 8$

$a'=a=(1,2,3)^{\mathrm{t}}$, $b'=(4,4,4)^{\mathrm{t}}$, $c'=c=(3,2,1)^{\mathrm{t}}$

Since $b'=a'+c'$, the columns of the determinant are linearly dependent, and, consequently, $\det(a',b',c')=0$.

$a''=(2,4,6)^{\mathrm{t}}$, $b''=(5,4,5)^{\mathrm{t}}$, $c''=(6,4,2)^{\mathrm{t}}$

Since $a''=a+a'=2a$, $b''=b+b'$, $c''=c+c'=2c$, linearity of the determinant implies

$$\begin{aligned}\det(a'',b'',c'') &= \det(2a,b+b',2c) = 2\cdot 2\cdot \det(a,b+b',c) \\ &= 4\underbrace{\det(a,b,c)}_{=8} + 4\underbrace{\det(a,b',c)}_{=\det(a',b',c')=0} = 32\end{aligned}$$

⚠ Note that

$$\det(a+a',b+b',c+c') \neq \det(a,b,c)+\det(a',b',c')$$

and

$$\det(sa,sb,sc) \neq s\det(a,b,c)\,.$$

4.2 Sarrus Scheme and Rules for Determinants

Compute the determinants $|A|$, $|2A|$, $|A^2|$, $|A^{\mathrm{t}}A^{-1}|$ for the matrix

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{pmatrix}.$$

Resources: Determinant, Properties of Determinants

Problem Variants

- $\begin{pmatrix} -2 & -1 & 2 \\ 3 & 4 & -3 \\ -4 & -5 & 5 \end{pmatrix}$

$\det A = -?$:

[ ] check

- $\begin{pmatrix} 1 & -1 & -2 \\ -5 & -3 & 3 \\ 5 & 2 & -4 \end{pmatrix}$

$\det A = ?$:

[ ] check

- $\begin{pmatrix} -5 & -4 & -3 \\ 3 & 5 & 1 \\ -1 & 2 & -2 \end{pmatrix}$

$\det A = ?$:

[ ] check

Solution

Determinant

Sarrus scheme

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2}$$
$$-a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$

$\rightsquigarrow$

$$\underbrace{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}}_{|A|} = 1 \cdot 1 \cdot 1 + 2 \cdot 2 \cdot 2 + 3 \cdot 3 \cdot 3 - 3 \cdot 1 \cdot 2 - 1 \cdot 2 \cdot 3 - 2 \cdot 3 \cdot 1$$
$$= 18$$

Application of Rules for Determinants

multilinearity $\quad\Longrightarrow$

$$|2A| = |(2A(:,1), 2A(:,2), 2A(:,3))| = 2 \cdot 2 \cdot 2\,|A| = 8 \cdot 18 = 144$$

$|AB| = |A||B|$, $|A^{-1}| = 1/|A|$, $|A^{\mathrm{t}}| = |A| \quad\Longrightarrow$

$$|A^2| = 18^2 = 324, \quad |A^{\mathrm{t}} A^{-1}| = 18/18 = 1$$

4.3 Determinant of Matrix Powers

Compute the determinants $d_n = |A^n + A|$, $n \in \mathbb{N}$, for the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Resources: Eigenvalues and eigenvectors, Sum and Product of Eigenvalues

Problem Variants

■ $\begin{pmatrix} -7 & 4 \\ -8 & 5 \end{pmatrix}$, $d_n = |A^n + A|$

---

d_{10} =??????:

[ ] check

■ $\begin{pmatrix} 4 & 6 \\ -3 & -5 \end{pmatrix}$, $d_n = |A^n - 2A|$

$d_{10} = -????$:

[ ] check

■ $\begin{pmatrix} 4 & -1 \\ 6 & -1 \end{pmatrix}$, $d_n = |A^n - 3A|$

$d_{10} = -????$:

[ ] check

Solution

Eigenvalues

characteristic polynomial of the matrix A

$$\begin{aligned} p_A(\lambda) &= \det(A - \lambda E) = \begin{pmatrix} 1-\lambda & 2 \\ 1 & 0-\lambda \end{pmatrix} \\ &= (1-\lambda)(0-\lambda) - 1 \cdot 2 = \lambda^2 - \lambda - 2 \end{aligned}$$

$p_A(\lambda) = 0 \quad \rightsquigarrow \quad$ eigenvalues $\lambda_1 = -1$, $\lambda_2 = 2$

Determinant

Eigenvalues of the matrix polynomial $q(A) = A^n + A$:

$$\varrho_k = q(\lambda_k) = \lambda_k^n + \lambda_k, \quad k = 1, 2$$

computing the determinant as product of the eigenvalues $\quad \rightsquigarrow$

$$|A^n + A| = \varrho_1 \varrho_2 = ((-1)^n - 1)(2^n + 2) = \begin{cases} -2^{n+1} - 4, & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

4.4 Taylor Approximation of a Determinant

Determine the linear Taylor approximation of the determinant

$$d(\varepsilon) = \begin{vmatrix} 2+\varepsilon & 1 & 1 \\ 1 & 3+\varepsilon & 1 \\ 1 & 1 & 4+\varepsilon \end{vmatrix}$$

for $\varepsilon \approx 0$.

Resources: Properties of Determinants, Expansion of Determinants

Problem Variants

- $d(\varepsilon) = \begin{pmatrix} 1+\varepsilon & -1+\varepsilon & 1+\varepsilon \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$d(0) = -?$, $d'(0) = ?$:

[ ] check

- $d(\varepsilon) = \begin{pmatrix} 1 & 2+\varepsilon & 3 \\ 3 & 1 & 2 \\ 2+\varepsilon & 3 & 1+\varepsilon \end{pmatrix}$

$d(0) = ??$, $d'(0) = -?$:

[ ] check

- $d(\varepsilon) = \begin{pmatrix} 2 & 2 & 1+\varepsilon \\ 2 & 1+\varepsilon & 3 \\ 1+\varepsilon & 3 & 3 \end{pmatrix}$

$d(0) = -??$, $d'(0) = ??$:

[ ] check

Solution

$$d(\varepsilon) = \begin{vmatrix} 2+\varepsilon & 1 & 1 \\ 1 & 3+\varepsilon & 1 \\ 1 & 1 & 4+\varepsilon \end{vmatrix} = \det(u+\varepsilon e_1, v+\varepsilon e_2, w+\varepsilon e_3)$$

with $u = (2,1,1)^{\mathrm{t}}$, $v = (1,3,1)^{\mathrm{t}}$, $w = (1,1,4)^{\mathrm{t}}$ and unit vectors e_k
expanding the determinant, using linearity with respect to the columns $\rightsquigarrow$
sum of 8 determinants, where terms of order $O(\varepsilon^2)$ can be neglected:

$$\begin{aligned} d(\varepsilon) &= \det(u,v,w) + \varepsilon\det(e_1,v,w) + \varepsilon\det(u,e_2,w) + \varepsilon\det(u,v,e_3) + O(\varepsilon^2) \\ &= \begin{vmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{vmatrix} + \varepsilon\left(\begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 1 \\ 0 & 1 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 4 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix}\right) + O(\varepsilon^2) \\ &\underset{(*)}{=} (24+1+1-2-4-3) + \varepsilon(12-1) + \varepsilon(8-1) + \varepsilon(6-1) + O(\varepsilon^2) \\ &= 17 + 23\varepsilon + O(\varepsilon^2) \end{aligned}$$

$(*)$ computation of $\det(u,v,w) = d(0)$ with the Sarrus scheme and using the expansion rule for determinants involving unit vectors, e.g.,

$$\det(u,e_2,w) = \begin{vmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 4 \end{vmatrix} = (-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} = 8 - 1 = 7$$

⚠ Note that

$$\det(u+\varepsilon a, v+\varepsilon b) \neq \det(u,v) + \varepsilon^2 \det(a,b)\,.$$

The correct application of linearity yields

$$\det(u+\varepsilon a, v+\varepsilon b) = \det(u,v) + \varepsilon\det(a,v) + \varepsilon\det(u,b) + \varepsilon^2\det(a,b)\,,$$

the reason for the 8 terms in the expansion of $\det(u+\varepsilon a, v+\varepsilon, w+\varepsilon c)$.

4.5 Expansion of a 4×4 Determinant

Compute

$$\begin{vmatrix} 3 & 0 & 2 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 3 \end{vmatrix}.$$

Resources: Determinant, Expansion of Determinants

Problem Variants

- $\begin{vmatrix} 7 & 6 & -5 & -8 \\ 4 & 5 & 0 & 0 \\ 2 & 0 & -3 & -2 \\ 8 & 3 & -7 & -1 \end{vmatrix}$

$-?$:

[ ] check

- $\begin{vmatrix} 2 & 6 & -1 & -6 \\ 5 & 8 & 0 & 0 \\ 4 & 0 & -3 & -4 \\ 1 & 3 & -2 & -8 \end{vmatrix}$

?:

[ ] check

- $\begin{vmatrix} 2 & 4 & -1 & -6 \\ 8 & 7 & 0 & -5 \\ 1 & 5 & 0 & -4 \\ 0 & 3 & -2 & -8 \end{vmatrix}$

$-?$:

[ ] check

Solution

expansion with respect to the j-th row:

$$d := \det A = \begin{vmatrix} 3 & 0 & 2 & 1 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 2 & 1 & 3 \end{vmatrix} = \sum_k (-1)^{j+k} a_{j,k} \det A_{j,k}$$

with $A_{j,k}$ the matrix after deletion of the j-th row and the k-th column of A and the sign chosen according to a chessboard pattern (+ for the upper left corner ($j = k = 1$))

analogously: expansion with respect to a column

choosing $j = 3$ (maximal number of zeros) $\rightsquigarrow$

$$d = (-1)^{3+1} \cdot 1 \cdot \begin{vmatrix} 0 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 1 & 3 \end{vmatrix} + 0 + 0 + (-1)^{3+4} \cdot 2 \cdot \begin{vmatrix} 3 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{vmatrix} \qquad (1)$$

expansion of the 3×3 determinants (alternative: Sarrus rule):

- first 3×3 determinant: expansion with respect to column $k = 1$ $\rightsquigarrow$

$$\begin{aligned} \begin{vmatrix} 0 & 2 & 1 \\ 1 & 3 & 0 \\ 2 & 1 & 3 \end{vmatrix} &= (-1)^{2+1} \cdot 1 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} + (-1)^{3+1} \cdot 2 \cdot \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} \\ &= -1 \cdot (6 - 1) + 2 \cdot (0 - 3) = -11 \end{aligned}$$

- second 3×3 determinant: expansion with respect to column $k = 2$ $\rightsquigarrow$

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 2 \\ 2 & 1 & 3 \\ 0 & 2 & 1 \end{vmatrix} &= (-1)^{2+2} \cdot 1 \cdot \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} + (-1)^{3+2} \cdot 2 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 3 \end{vmatrix} \\ &= 1 \cdot (3 - 0) - 2 \cdot (9 - 4) = -7 \end{aligned}$$

substituting the computed values into (1) $\rightsquigarrow$ $d = 1 \cdot (-11) - 2(-7) = 3$

Alternative solution

transforming the matrix with Gauß operations to triangular form and computing the determinant as product of the diagonal entries

4.6 Determinant of a Sparse Matrix

Compute the determinant

$$\begin{vmatrix} 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -4 \\ 0 & 5 & -6 & 0 \\ 0 & 0 & 7 & -8 \end{vmatrix}.$$

Resources: Expansion of Determinants

Problem Variants

- $$\begin{vmatrix} -7 & 0 & 5 & 8 \\ 0 & -2 & 0 & -6 \\ 3 & -9 & 0 & 0 \\ 0 & -4 & 1 & 0 \end{vmatrix}$$

−??:

[ ] check

- $$\begin{vmatrix} 9 & 0 & -2 & -1 \\ 8 & -3 & 0 & 6 \\ -4 & 0 & 5 & 0 \\ 0 & 0 & -7 & 0 \end{vmatrix}$$

??:

[ ] check

- $$\begin{vmatrix} 0 & 7 & 0 & 0 \\ -5 & 0 & 2 & 0 \\ 3 & -9 & -6 & -1 \\ -4 & 8 & 0 & 0 \end{vmatrix}$$

−??:

[ ] check

Solution

expansion with respect to the first row:

$$|A| = \sum_{k=1}^{4} (-1)^{1+k} a_{1,k} |A_k|$$

with A_k the matrix without the first row and the k-th column of A
application to the given matrix $\quad \rightsquigarrow$

$$\begin{vmatrix} 1 & -2 & 0 & 0 \\ 3 & 0 & 0 & -4 \\ 0 & 5 & -6 & 0 \\ 0 & 0 & 7 & -8 \end{vmatrix} = 1 \cdot \underbrace{\begin{vmatrix} 0 & 0 & -4 \\ 5 & -6 & 0 \\ 0 & 7 & -8 \end{vmatrix}}_{d_1} - (-2) \cdot \underbrace{\begin{vmatrix} 3 & 0 & -4 \\ 0 & -6 & 0 \\ 0 & 7 & -8 \end{vmatrix}}_{d_2}$$

expansion of d_k with respect to the first column,

$$|B| = \sum_{j=1}^{3} (-1)^{j+1} b_{j,1} |B_j| \,,$$

with B_j the matrix without the first column and the j-th row od $B \quad \rightsquigarrow$

$$d_1 = (-5) \cdot \begin{vmatrix} 0 & -4 \\ 7 & -8 \end{vmatrix} = (-5) \cdot 28, \quad d_2 = 3 \cdot \begin{vmatrix} -6 & 0 \\ 7 & -8 \end{vmatrix} = 3 \cdot 48$$

adding the terms $\quad \rightsquigarrow \quad 1 \cdot (-5) \cdot 28 - (-2) \cdot 3 \cdot 48 = 148$

4.7 Equation of a Plane, Determined by Three Points

Represent the plane E, containing the three points

$$a = (0, 1, -2), \quad b = (2, 0, 1), \quad c = (-1, 2, 0)\,,$$

by an equation

$$E:\ n_1x_1 + n_2x_2 + n_3x_3 = d\,.$$

Resources: Determinant, Properties of Determinants

Problem Variants

■ $a = (-2, -1, 6)$, $b = (-3, 2, 1)$, $c = (-4, 7, -9)$

$E: ?x_1 + ?x_2 + ?x_3 = -3$:

check

■ $a = (-2, 4, 1)$, $b = (6, -6, 8)$, $c = (5, -5, 7)$

$E:\ -?x_1 - ?x_2 + ?x_3 = 4$:

check

■ $a = (5, -7, -2)$, $b = (3, -5, -3)$, $c = (1, -4, -8)$

$E: ?x_1 + ?x_2 - ?x_3 = -7$:

check

Solution

A point x belongs to the plane, containing the points a, b, c (assumed to be not colinear), if $x^{\mathrm{t}} - a^{\mathrm{t}}$ is a linear combination of the vectors $b^{\mathrm{t}} - a^{\mathrm{t}}$, $c^{\mathrm{t}} - a^{\mathrm{t}}$, i.e. if $x^{\mathrm{t}} - a^{\mathrm{t}}$ does not extend these vectors, spanning the plane, to a basis of $\mathbb{R}^3$, i.e. if

$$0 = \det(x^{\mathrm{t}} - a^{\mathrm{t}}, b^{\mathrm{t}} - a^{\mathrm{t}}, c^{\mathrm{t}} - a^{\mathrm{t}}) \iff 0 = \begin{vmatrix} a_1 & b_1 & c_1 & x_1 \\ a_2 & b_2 & c_2 & x_2 \\ a_3 & b_3 & c_3 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

in view of the invariance of determinants under subtraction of columns (In the 4×4 determinant, we subtract the first from the three other columns and then expand with respect to the last row, equal to $(1, 0, 0, 0)$.)

substituting $a = (0, 1, -2)$, $b = (2, 0, 1)$, $c = (-1, 2, 0)$ $\rightsquigarrow$

$$0 = \begin{vmatrix} 0 & 2 & -1 & x_1 \\ 1 & 0 & 2 & x_2 \\ -2 & 1 & 0 & x_3 \\ 1 & 1 & 1 & 1 \end{vmatrix}$$

expanding with respect to the last column $\rightsquigarrow$

$$0 = -x_1 \begin{vmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} + x_2 \begin{vmatrix} 0 & 2 & -1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} - x_3 \begin{vmatrix} 0 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} + \underbrace{\begin{vmatrix} 0 & 2 & -1 \\ 1 & 0 & 2 \\ -2 & 1 & 0 \end{vmatrix}}_{-d}$$

Sarrus rule $\rightsquigarrow$

$$-d = 0 + 2 \cdot 2 \cdot (-2) + (-1) \cdot 1 \cdot 1 - 0 - 0 - 0 = -9$$

subtracting the first from the last two columns for the other determinants $\rightsquigarrow$

$$\begin{vmatrix} 1 & 0 & 2 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 3 & 2 \\ 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 3 & 2 \end{vmatrix} = -5$$

and, analogously,

$$\begin{vmatrix} 0 & 2 & -1 \\ -2 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 7, \qquad \begin{vmatrix} 0 & 2 & -1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{vmatrix} = 1$$

resulting equation for the plane:

$$E:\ 5x_1 + 7x_2 - x_3 = 9$$

Alternative solution

computing a normal vector for the plane as scalar product:

$$n = (b - a)^{\mathrm{t}} \times (c - a)^{\mathrm{t}}$$

and setting $d = a^{\mathrm{t}} n$

4.8 Determinant of a 5×5 Matrix

Compute the determinant

$$\begin{vmatrix} 1 & 2 & 2 & 3 & 1 \\ 1 & 2 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 2 \\ 2 & 4 & 4 & 4 & 5 \\ 3 & 4 & 5 & 4 & 3 \end{vmatrix}.$$

Resources: Properties of Determinants, Expansion of Determinants

Problem Variants

■ $\begin{vmatrix} 2 & 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 & 3 \end{vmatrix}$

$-$?:

[ ] check

■ $\begin{vmatrix} 2 & 2 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 & 2 \\ 1 & 1 & 2 & 2 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 2 & 2 & 1 & 2 & 1 \end{vmatrix}$

?:

[ ] check

■ $\begin{vmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 3 & 1 \\ 2 & 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 2 \end{vmatrix}$

?:

[ ] check

Solution

simplification of the determinant with Gauß transformations (adding multiples of rows and columns), which leave the determinant unchanged, in conjunction with expansion with respect to rows and columns with few nonzero entries

subtracting (2 × column 1) from column 3 and expanding with respect to the modified third column $\rightsquigarrow$

$$\begin{vmatrix} 1 & 2 & 2 & 3 & 1 \\ 1 & 2 & 2 & 2 & 3 \\ 2 & 3 & 4 & 3 & 2 \\ 2 & 4 & 4 & 4 & 5 \\ 3 & 4 & 5 & 4 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 0 & 3 & 1 \\ 1 & 2 & 0 & 2 & 3 \\ 2 & 3 & 0 & 3 & 2 \\ 2 & 4 & 0 & 4 & 5 \\ 3 & 4 & -1 & 4 & 3 \end{vmatrix} = (-1) \cdot \begin{vmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 2 \\ 2 & 4 & 4 & 5 \end{vmatrix}$$

subtracting (2 × row 2) from row 4 and expanding with respect to the modified fourth row $\rightsquigarrow$

$$(-1) \cdot \begin{vmatrix} 1 & 2 & 3 & 1 \\ 1 & 2 & 2 & 3 \\ 2 & 3 & 3 & 2 \\ 0 & 0 & 0 & -1 \end{vmatrix} = (-1) \cdot (-1) \cdot \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 2 & 3 & 3 \end{vmatrix}$$

subtracting (2 × column 1) from column 2 and expanding with respect to the modified second column $\rightsquigarrow$

$$\begin{vmatrix} 1 & 0 & 3 \\ 1 & 0 & 2 \\ 2 & -1 & 3 \end{vmatrix} = -(-1) \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1$$

Chapter 5

Linear Systems

5.1 Cramer's Rule for a Linear System with 2 Unknowns

Solve the equations

$$x_1 + 2x_2 = 4, \quad 3x_1 + 4x_2 = -2$$

with Cramer's rule.

Resources: Cramer's Rule

Problem Variants

■ $-3x_1 - 6x_2 = -3$, $4x_1 + 7x_2 = 6$

$x_1 = ?$, $x_2 = -?$:

[ ] check

■ $4x_1 - 6x_2 = 2$, $3x_1 - 3x_2 = 3$

$x_1 = ?$, $x_2 = ?$:

[ ] check

■ $x_1 + 5x_2 = 7$, $-6x_1 - 8x_2 = 2$

$x_1 = -?$, $x_2 = ?$:

[ ] check

Solution

Cramer's rule expresses the solution of a linear system $Ax = b$ as quotient of determinants:

$$x_k = \det(C_k)/\det(A)$$

with C_k the matrix, where the k-th column of A is replaced by the right-hand side b.

application to the given linear system

$$\underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 4 \\ -2 \end{pmatrix}}_{b}$$

$\leadsto$

$$x_1 = \frac{\begin{vmatrix} 4 & 2 \\ -2 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{20}{-2} = -10, \quad x_2 = \frac{\begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}} = \frac{-14}{-2} = 7$$

5.2 Cramer's Rule for a Linear System with 3 Unknowns

Solve the linear system

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & 3x_3 & = & -3 \\ 3x_1 & + & 2x_2 & - & x_3 & = & 1 \\ -4x_1 & - & 2x_2 & + & 3x_3 & = & 3 \end{array}$$

with Cramer's rule.

Resources: Cramer's Rule

Problem Variants

■ $$\begin{array}{rcrcrcr} -6x_1 & - & 7x_2 & + & 4x_3 & = & -5 \\ 3x_1 & - & 2x_2 & + & x_3 & = & 2 \\ -8x_1 & - & 9x_2 & + & 5x_3 & = & -8 \end{array}$$

$x = (?, ?, ?)^t$:

[ ] check

■ $$\begin{array}{rcrcrcr} 5x_1 & - & 7x_2 & - & 6x_3 & = & 4 \\ 4x_1 & - & 8x_2 & + & 3x_3 & = & -1 \\ 2x_1 & - & 1x_2 & - & 9x_3 & = & 4 \end{array}$$

$x = (?, ?, ?)^t$:

[ ] check

■ $$\begin{array}{rcrcrcr} -x_1 & - & 2x_2 & + & 3x_3 & = & -2 \\ -3x_1 & + & 2x_2 & - & x_3 & = & 4 \\ x_1 & - & 4x_2 & + & 5x_3 & = & -4 \end{array}$$

$x = (?, ?, ?)^t$:

[ ] check

Solution

matrix form of the linear system:

$$\underbrace{\begin{pmatrix} 2 & 1 & -3 \\ 3 & 2 & -1 \\ -4 & -2 & 3 \end{pmatrix}}_{A=(a_1,a_2,a_3)} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -3 \\ 1 \\ 3 \end{pmatrix}}_{b}$$

with a_k the columns of the matrix A

Cramer's rule $\implies$

$$x_1 = \frac{\det(b, a_2, a_3)}{\det(a_1, a_2, a_3)} = \frac{\begin{vmatrix} -3 & 1 & -3 \\ 1 & 2 & -1 \\ 3 & -2 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 1 & -3 \\ 3 & 2 & -1 \\ -4 & -2 & 3 \end{vmatrix}}$$

computing the determinants with the Sarrus scheme $\leadsto$

$$x_1 = \frac{(-3)\cdot 2\cdot 3 + (-3) + 6 - (-3)\cdot 2\cdot 3 - (-6) - 3}{2\cdot 2\cdot 3 + 4 + 18 - (-3)\cdot 2\cdot(-4) - 4 - 9} = \frac{6}{-3} = -2$$

similarly:

$$x_2 = \frac{\det(a_1, b, a_3)}{\det(a_1, a_2, a_3)} = \frac{\begin{vmatrix} 2 & -3 & -3 \\ 3 & 1 & -1 \\ -4 & 3 & 3 \end{vmatrix}}{-3} = \frac{-12}{-3} = 4$$

$$x_3 = \frac{\det(a_1, a_2, b)}{\det(a_1, a_2, a_3)} = \frac{\begin{vmatrix} 2 & 1 & -3 \\ 3 & 2 & 1 \\ -4 & -2 & 3 \end{vmatrix}}{-3} = \frac{-3}{-3} = 1$$

5.3 Rational Interpolation

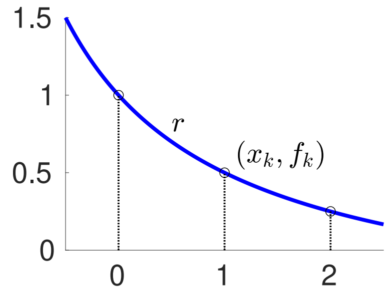
Interpolate the data

$$(x_k, f_k) = (k, 2^{-k}), \quad k = 0, 1, 2\,,$$

with a rational function

$$r(x) = \frac{a + bx}{1 + cx}\,.$$

Do there exist function values f_k, for which the interpolation problem cannot be solved?



Resources: Linear System

Problem Variants

■ $r(x) = a + \dfrac{b}{x - c}$,

x_k	1	2	3
f_k	4	2	1

$a + b + c =$?:

[ ] check

■ $r(x) = \dfrac{a}{1 + bx + cx^2}$,

x_k	0	2	3
f_k	5	2	1

$a + b + c =$??/6:

[ ] check

■ $r(x) = ax + b + \dfrac{c}{x}$,

x_k	1	2	3
f_k	3	1	2

$a + b + c =$?:

[ ] check

Solution

Construction of the interpolant

interpolation conditions for the rational function $r(x) = \dfrac{a+bx}{1+cx}$:

$$\begin{aligned} f_0 &= 1 &= r(0) &\underset{x=0}{=} a \\ f_1 &= \frac{1}{2} &= r(1) &\underset{a=1}{=} \frac{1+b}{1+c} \\ f_2 &= \frac{1}{4} &= r(2) &= \frac{1+2b}{1+2c} \end{aligned}$$

multiplying the last two equations with their denominators $\rightsquigarrow$ linear system

$$1/2 + c/2 = 1 + b, \quad 1/4 + c/2 = 1 + 2b$$

subtracting the first equation from the second $\rightsquigarrow$ $-1/4 = b$
substituting into the first equation $\rightsquigarrow$ $1/2 + c/2 = 1 - 1/4$, i.e. $c = 1/2$

$$\rightsquigarrow \quad r(x) = \frac{1 - x/4}{1 + x/2}$$

Failure of interpolation

If the rational function $r(x) = (a+bx)/(1+cx)$ is not constant, it has either no zero ($a \neq 0$, $b = 0$) or exactly one zero ($b \neq 0$). This implies that, e.g., the data

$$f_0 = 1,\ f_1 = 0,\ f_2 = 0$$

cannot be interpolated.

5.4 Solution of a 4×4 Linear System

Solve the linear system

$$\begin{pmatrix} 2 & 4 & 0 & 2 \\ 3 & 6 & 3 & 0 \\ 0 & 2 & 3 & 1 \\ 2 & 0 & -4 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -3 \\ 9 \end{pmatrix}$$

with the Gauß-Jordan algorithm.

Resources: Linear System, Gauß Elimination

Problem Variants

Note that $x_k \in \{-4, -3, \ldots, 4\}$ for all variants.

■ $$\begin{pmatrix} 2 & 2 & -3 & 0 \\ 1 & 0 & 1 & -2 \\ 0 & -3 & -1 & 4 \\ 4 & -1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -3 \\ 1 \end{pmatrix}$$

$x = (?, ?, ?, ?)^{\mathrm{t}}$:

[ ] check

■ $$\begin{pmatrix} 0 & -2 & 3 & 1 \\ -1 & -4 & 0 & -2 \\ 4 & 3 & 1 & 0 \\ -2 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -1 \\ 2 \end{pmatrix}$$

$x = (?, -?, -?, ?)^{\mathrm{t}}$:

[ ] check

■ $$\begin{pmatrix} 0 & -3 & -1 & -3 \\ 4 & 0 & -4 & 1 \\ 2 & -3 & -2 & 0 \\ 1 & 1 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \\ -1 \end{pmatrix}$$

$x = (-?, -?, -?, ?)^{\mathrm{t}}$:

[ ] check

Solution

transformation of the tableau of the linear system $Ax = b$,

$$(A|b) = \left(\begin{array}{cccc|c} 2 & 4 & 0 & 2 & 0 \\ 3 & 6 & 3 & 0 & -3 \\ 0 & 2 & 3 & 1 & -3 \\ 2 & 0 & -4 & 3 & 9 \end{array}\right),$$

to diagonal form with Gauß operations:

- permutation of rows (if necessary)
- scaling and adding multiples of rows

step 1 (generation of zeros in the first column):
row 1 $\leftarrow$ row 1 / 2
row 2 $\leftarrow$ row 2 - 3 $\cdot$ (scaled) row 1
row 4 $\leftarrow$ row 4 - 2 $\cdot$ (scaled) row 1

$$(A|b) \to \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 3 & -3 & -3 \\ 0 & 2 & 3 & 1 & -3 \\ 0 & -4 & -4 & 1 & 9 \end{array}\right)$$

step 2
(1)
permutation of row 2 and row 3, to obtain a nonzero pivot in the diagonal position
row 2 $\leftarrow$ row 2 / 2
(2)
row 1 $\leftarrow$ row 1 $-$ 2 $\cdot$ (modified) row 2
row 4 $\leftarrow$ row 4 + 4 $\cdot$ (modified) row 2

$$\underset{(1)}{\to} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3/2 & 1/2 & -3/2 \\ 0 & 0 & 3 & -3 & -3 \\ 0 & -4 & -4 & 1 & 9 \end{array}\right) \underset{(2)}{\to} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 3 \\ 0 & 1 & 3/2 & 1/2 & -3/2 \\ 0 & 0 & 3 & -3 & -3 \\ 0 & 0 & 2 & 3 & 3 \end{array}\right)$$

In a similar fashion, the final two steps scale the third and fourth diagonal element to one and annihilate the remaining nonzero off-diagonal elements:

$$\underset{\textbf{step 3}}{\to} \left(\begin{array}{cccc|c} 1 & 0 & 0 & -3 & 0 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 5 & 5 \end{array}\right) \underset{\textbf{step 4}}{\to} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

Having transformed A to the unit matrix, the last column of the tableau contains the solution $x = (3, -2, 0, 1)^t$.

Remark
Generating zeros only below the diagonal of the tableau (standard Gauß algorithm) and computing x with backward substitution after the elimination step requires slighty less operations. However, from a programming point of view, the Gauß-Jordan algorithm is appealing, in particular, if one can take advantage of vector operations[1].

[1] Counting only vector operations, the Gauß-Jordan version is more efficient than the standard Gauß procedure.

5.5 Inverse of a Symmetric 3×3 Matrix

Determine the inverse $C = A^{-1}$ of the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 2 \end{pmatrix}$$

with Cramer's rule.

Resources: Cramer's Rule

Problem Variants

- $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$

$\max_{j,k} c_{j,k}$ =?:

[ ] check

- $A = \begin{pmatrix} 1 & -3 & -1 \\ -3 & 2 & 2 \\ -1 & 2 & 1 \end{pmatrix}$

$\max_{j,k} c_{j,k}$ =?:

[ ] check

- $A = \begin{pmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{pmatrix}$

$\max_{j,k} c_{j,k}$ =?.??:

[ ] check

Solution

Cramer's rule $\quad\rightsquigarrow\quad$ formula for the elements of $C = A^{-1}$:

$$c_{j,k} = \underbrace{(-1)^{j+k} \det \tilde{A}_{k,j}}_{\text{cofactor}} \,/ \det A \tag{1}$$

with $\tilde{A}_{k,j}$ obtained from A by deleting row k and column j

⚠ Note the permutation of indices: $c_{j,k} \leftrightarrow \tilde{A}_{k,j}$.

- determinant of A: expanding with respect to the first row $\quad\rightsquigarrow$

$$\det A = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 2 & 3 \\ 0 & 3 & 2 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = -1 \cdot (1 \cdot 2 - 0 \cdot 3) = -2$$

- application of formula (1):

$$c_{1,1} = (-1)^{1+1} |\tilde{A}_{1,1}| / (-2) = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} / (-2) = 5/2$$

($\tilde{A}_{1,1} = A$ without row 1 and column 1)

$$c_{1,2} = (-1)^{1+2} |\tilde{A}_{2,1}| / (-2) = - \begin{vmatrix} 1 & 0 \\ 3 & 2 \end{vmatrix} / (-2) = 1$$

($\tilde{A}_{2,1} = A$ without row 2 and column 1)
analogously

$$c_{1,3} = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} / (-2) = -\frac{3}{2},\ c_{2,2} = \begin{vmatrix} 0 & 0 \\ 0 & 2 \end{vmatrix} / (-2) = 0,\ c_{2,3} = 0,\ c_{3,3} = \frac{1}{2}$$

symmetry of $A \quad\Longrightarrow\quad C^{\mathrm{t}} = (A^{-1})^{\mathrm{t}} = (A^{\mathrm{t}})^{-1} = A^{-1} = C$ and, consequently, $c_{1,2} = c_{2,1}$, $c_{1,3} = c_{3,1}$, $c_{2,3} = c_{3,2}$, i.e.,

$$C = \begin{pmatrix} 5/2 & 1 & -3/2 \\ 1 & 0 & 0 \\ -3/2 & 0 & 1/2 \end{pmatrix}$$

Verification with MATLAB®

```
A = [0 1 0; 1 2 3; 0 3 2], C = inv(A)
```

5.6 Gauß Elimination for a Linear System with 3 Unknowns

Solve the linear system

$$\begin{array}{rcrcrcr} 2x_1 & + & 2x_2 & + & 2x_3 & = & 0 \\ x_1 & + & 3x_2 & + & x_3 & = & 4 \\ 3x_1 & + & x_2 & + & x_3 & = & 2 \end{array}$$

with Gauß elimination.

Problem Variants

■ $$\begin{array}{rcrcrcr} -2x_1 & - & 4x_2 & - & 3x_3 & = & 1 \\ 9x_1 & + & 6x_2 & - & x_3 & = & -5 \\ & - & 7x_2 & - & 8x_3 & = & 2 \end{array}$$

$x_1 =?$, $x_2 = -?$, $x_3 =?$:

[ ] check

■ $$\begin{array}{rcrcrcr} -3x_1 & + & 4x_2 & - & 6x_3 & = & -4 \\ -8x_1 & & & - & 9x_3 & = & 7 \\ 3x_1 & - & 5x_2 & + & 9x_3 & = & 8 \end{array}$$

$x_1 = -?$, $x_2 = -?$, $x_3 =?$:

[ ] check

■ $$\begin{array}{rcrcrcr} & & 2x_2 & - & x_3 & = & -6 \\ -8x_1 & - & 2x_2 & - & 7x_3 & = & 6 \\ x_1 & - & 3x_2 & + & 3x_3 & = & 7 \end{array}$$

$x_1 =?$, $x_2 = -?$, $x_3 = -?$:

[ ] check

Solution

Transformation to triangular form

combining the coefficient matrix A and the right-hand side b of the linear system

$$\underbrace{\begin{pmatrix} 2 & 2 & 2 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 \\ 4 \\ 2 \end{pmatrix}}_{b}$$

to a tableau $(A|b)$, and annihilating the entries below the diagonal with Gauß transformations (interchanging rows, if necessary, and adding multiples of rows) $\rightsquigarrow$

$$\left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 1 & 3 & 1 & 4 \\ 3 & 1 & 1 & 2 \end{array}\right) \underset{(1)}{\rightarrow} \left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & -2 & -2 & 2 \end{array}\right) \underset{(2)}{\rightarrow} \left(\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 0 & 2 & 0 & 4 \\ 0 & 0 & -2 & 6 \end{array}\right)$$

(1):
row 2 $\longleftarrow$ row 2 $-$ $(1/2) \cdot$ row 1, row 3 $\longleftarrow$ row 3 $-$ $(3/2) \cdot$ row 1
(2):
row 3 $\longleftarrow$ row 3 $+$ row 2

Backward substitution

$$\begin{array}{lrcl} \text{row}\,3: & -2x_3 = 6 & \Longrightarrow & x_3 = -3 \\ \text{row}\,2: & 2x_2 + 0x_3 = 4 & \Longrightarrow & x_2 = 2 \\ \text{row}\,1: & 2x_1 + 2x_2 + 2x_3 = 0 & \Longrightarrow & x_1 = \frac{0-2x_2-2x_3}{2} = \frac{-4+6}{2} = 1 \end{array}$$

5.7 Solving a Linear System with MATLAB®

Solve the linear system

$$\underbrace{\begin{pmatrix} p & 1 & -1 \\ -1 & p & 1 \\ 1 & -1 & p \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} =: b$$

for $p = 1$ and $p = 0$.

Resources: Linear System, Matrix Operations with MATLAB®

Problem Variants

■ $\displaystyle\sum_{k=1}^{5} \frac{1}{j+k-1}\, x_k = 1,\ j = 1, \ldots, 5$

$\sum_{k=1}^{5} x_k =??$:

[ ] check

■ $\displaystyle\sum_{k=1}^{5} 2^{jk}\, x_k = j,\ j = 1, \ldots, 5$

$\sum_{k=1}^{5} x_k =?.??$:

[ ] check

■ $\displaystyle\sum_{k=1}^{5} j^{k}\, x_k = 2^j,\ j = 1, \ldots, 5$

$\sum_{k=1}^{5} x_k =?$:

[ ] check

Solution

$p = 1$

```
% matrix, right-hand side, determinant
A = [1 1 -1; -1 1 1; 1 -1 1]; b = [0; 1; -1];
det(A)
   4
```

$\Longrightarrow$ unique solution for all b, solution using the \-operator

```
x = A \ b
x = -0.5000
     0.5000
     0
```

$p = 0$

```
A = [0 1 -1; -1 0 1; 1 -1 0]; b = [0; 1; -1];
% determinant and comparison of ranks
det(A), rank(A), rank([A, b])
   0       2        2
```

$\operatorname{rank}(A) = \operatorname{rank}((A, b)) = 2 \quad \Longrightarrow \quad$ affine one-dimensional set of solutions:

$$x = u + tv, \quad t \in \mathbb{R},\, v \in \ker A$$

The \-operator cannot be applied. Instead, a particular solution can be obtained with the pseudoinverse A^+.

```
Ap = pinv(A)
Ap =  0         -0.3333    0.3333
      0.3333   -0.0000   -0.3333
     -0.3333    0.3333    0.0000
u = Ap*b, v = null(A)
u = -0.6667   v = -0.5774
     0.3333       -0.5774
     0.3333       -0.5774
```

5.8 Solving Linear Systems with Maple$^{\text{TM}}$

Solve the parameter dependent linear system

$$\begin{pmatrix} 0 & 1 & 2 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ p \\ 3 \end{pmatrix},$$

in different ways.

Resources: Linear System, Matrix Operations with Maple

Problem Variants

■ $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & p \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}$

---

solution for $p = 7$: $(?, -?, ?)^{\text{t}}$:

[ ] check

■ $A = \begin{pmatrix} p & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 4 & 4 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \\ q \end{pmatrix}$

---

solution for $p = 0$, $q = 1$: $(-?, -?, ?)^{\text{t}}$:

[ ] check

■ $A = \begin{pmatrix} -1 & 2 & 0 \\ 2 & -1 & 2 \\ 0 & 2 & -1 \end{pmatrix}$, $b = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix}$

---

solution for $p_k = 7$: $(?, ?, ?)^{\text{t}}$:

[ ] check

Solution

Definition of the coefficient matrix and right-hand side of the linear system

```
with(LinearAlgebra):   # loading relevant functions
A := Matrix([[0,1,2],[2,3,1],[1,2,0]]):
b := Vector([0,p,3]):
```

Gauß elimination

```
# transformation to upper triangular form, backward substitution
Ab := Matrix([A,b]):   # coefficients and right-hand side
U := GaussianElimination(Ab); x := BackwardSubstitute(U);
```

$$U := \begin{bmatrix} 2 & 3 & 1 & p \\ 0 & 1 & 2 & 0 \\ 0 & 0 & -3/2 & 3-p/2 \end{bmatrix}, \quad x := \begin{bmatrix} -5+4p/3 \\ 4-2p/3 \\ -2+p/3 \end{bmatrix},$$

Alternative: combining Gauß elimination and backward substitution

```
x := LinearSolve(A,b);
```

Cramer's rule

```
d := Determinant(A);
```

$$d := 3$$

```
x[1] := Determinant(Matrix([b,A[1..3,2..3]]))/d;
```

$x_2 = \det(A(:,1), b, A(:,3))/\det A$, $x_3 = \det(A(:,1:2), b)/\det A$

Multiplication with the inverse matrix

```
C := MatrixInverse(A);
```

$$C := \begin{bmatrix} -2/3 & 4/3 & -5/3 \\ 1/3 & -2/3 & 4/3 \\ 1/3 & 1/3 & -2/3 \end{bmatrix}$$

```
x := MatrixVectorMultiply(C,b);
```

5.9 Gauß-Seidel Iteration with MATLAB®

Write a MATLAB® function `x = gauss-seidel(A,b,x,tol)`, which solves the linear system $Ax = b$ with the Gauß-Seidel iteration. Test your program with the 4×4 matrix $A: a_{j,k} = 1/(j+k-1)$ and $b_j = 1$.

Resources: Gauß-Seidel Iteration

Problem Variants

■ `x = gauss-seidel_odd_even(A,b,x,tol)`, update x_j first for odd and then for even j
test data: $a_{j,k} = 2^{-|j-k|}$, $b_j = 1$, $1 \le j \le 8$

$\max_k x_k =$?.??:

[ ] check

■ `x = gauss-seidel_tridiagonal(T,b,x,tol)`, $A(j, j-1 : j+1) = T(j, 1:3)$
test data: $T(j,:) = [1/2, 1, 1/2]$, $b_j = 1$, $1 \le j \le 8$

$\max_k x_k =$?.??:

[ ] check

■ `x = gauss-seidel_sparse(S,b,x,tol)`, $A(S(j,1), S(j,2)) = S(j,3)$
test data: $S(2j-1,:) = [j,j,j]$, $S(2j,:) = [j, 9-j, 1]$, $b_j = 1$, $1 \le j \le 8$

$\max_k x_k =$?:

[ ] check

Solution

Gauß-Seidel iteration

iteration step $x \to y$

$$\begin{aligned} y_j &= (b_j - \sum_{k<j} a_{j,k} y_k - \sum_{k>j} a_{j,k} x_k)/a_{j,j} \\ &= x_j + \underbrace{(b_j - \sum_{k<j} a_{j,k} y_k - \sum_{k\geq j} a_{j,k} x_k)/a_{j,j}}_{\text{update/error}\, e_j}, \quad j = 1, 2, \ldots \end{aligned}$$

MATLAB® implementation

```
function x = gauss_seidel(A,b,x,tol)
max_iterations = 1000;  % avoid an endless loop
for n=1:max_iterations
  for j=1:length(b)
    e(j) = (b(j)-A(j,:)*x)/A(j,j);
    % no split of the sum A(j,:)*x since
    % x(1:j-1) already contains the updated values
    x(j) = x(j)+e(j);
  end
    % terminate if the relative error is less than the tolerance
    if norm(e,inf)/norm(x) < tol; return; end;
  end
  display('no convergence after 1000 iterations')
end
```

Test example

```
A = hilb(4); b = ones(4,1); x = zeros(4,1);
x = gauss_seidel(A,b,x,1.0e-3)
     -3.9049
     58.9781
   -177.6030
    138.4677
norm(A*x-b,inf)  % check the residuum
      2.8435e-04
```

5.10 Matrix from Prescribed Images

Determine the matrix A which maps the vectors $(1,3)^t$ and $(2,4)^t$ to $(3,-1,2)^t$ and $(-2,0,-4)^t$.

Resources: Matrix of a Linear Map, Cramer's Rule

Problem Variants

■ $\begin{pmatrix} -3 \\ -4 \end{pmatrix} \mapsto \begin{pmatrix} 6 \\ -2 \\ -4 \end{pmatrix}, \quad \begin{pmatrix} 3 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} -6 \\ 5 \\ -8 \end{pmatrix}$

$A = [-?, ?; ?, -?; -?; ?]$ (MATLAB® notation):

[ ] check

■ $\begin{pmatrix} -3 \\ 4 \end{pmatrix} \mapsto \begin{pmatrix} -6 \\ 8 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -2 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

$A = [?, ?; -?, -?]$ (MATLAB® notation):

[ ] check

■ $\begin{pmatrix} -4 \\ 3 \end{pmatrix} \mapsto \begin{pmatrix} 2 \\ 0 \\ 8 \\ 6 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ 1 \\ -3 \\ -2 \end{pmatrix}$

$A = [-?, -?; -?, -?; ?, ?; ?, ?]$ (MATLAB® notation):

[ ] check

Solution

combining the assertions

$$A \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}, \quad A \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 \\ -4 \end{pmatrix}$$

$\rightsquigarrow$ matrix form of the problem:

$$\underbrace{\begin{pmatrix} 3 & -2 \\ -1 & 0 \\ 2 & -4 \end{pmatrix}}_{V} = A \underbrace{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}}_{U} \quad \Longleftrightarrow \quad A = VU^{-1}$$

formula for the inverse of a 2×2 matrix,

$$U^{-1} = \frac{1}{\det U} \begin{pmatrix} u_{2,2} & -u_{1,2} \\ -u_{2,1} & u_{1,1} \end{pmatrix} = \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$$

$\rightsquigarrow$

$$A = \begin{pmatrix} 3 & -2 \\ -1 & 0 \\ 2 & -4 \end{pmatrix} \frac{1}{-2} \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -9 & 4 \\ 2 & -1 \\ -10 & 4 \end{pmatrix}$$

5.11 Echelon Form and General Solution of a Linear System

Transform the linear system

$$\begin{pmatrix} 1 & -1 & 2 & 0 & -2 \\ -3 & 5 & -5 & -2 & 5 \\ 0 & -4 & -2 & 5 & 4 \\ 1 & -1 & 2 & 1 & 0 \end{pmatrix} x = \begin{pmatrix} 2 \\ -8 \\ 6 \\ 4 \end{pmatrix}$$

to echelon form and determine the general solution.

Resources: Gauß Elimination, Echelon Form

Problem Variants

- $$\begin{pmatrix} 1 & 1 & -2 & 1 & -2 \\ 1 & 0 & -1 & 3 & 2 \\ 2 & 2 & -3 & 0 & -2 \\ -2 & -2 & 3 & -1 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \\ -4 \\ 6 \end{pmatrix}$$

---

particular solution $x = (-3, -?, -?, ?, -?)^t$:

check

- $$\begin{pmatrix} 0 & 2 & -3 & 1 \\ 1 & -2 & 3 & -1 \\ 1 & -3 & 2 & -3 \end{pmatrix} x = \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix}$$

---

particular solution $x = (1, -?, -?, -?)^t$:

check

- $$\begin{pmatrix} -3 & -1 & 1 & -2 \\ -1 & 1 & -3 & 0 \\ -1 & 0 & -3 & 3 \end{pmatrix} x = \begin{pmatrix} 3 \\ -3 \\ 2 \end{pmatrix}$$

---

particular solution $x = (-2, ?, ?, ?)^t$:

check

Solution

Transformation to echelon form

application of Gauß transformations to the tableau $(A|b)$, with A the matrix and b the right-hand side of the linear system:

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -2 & 2 \\ -3 & 5 & -5 & -2 & 5 & -8 \\ 0 & -4 & -2 & 5 & 4 & 6 \\ 1 & -1 & 2 & 1 & 0 & 4 \end{array}\right) \underset{(1)}{\longrightarrow} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -2 & 2 \\ 0 & 2 & 1 & -2 & -1 & -2 \\ 0 & -4 & -2 & 5 & 4 & 6 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right)$$

$$\underset{(2)}{\longrightarrow} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & -2 & 2 \\ 0 & 2 & 1 & -2 & -1 & -2 \\ 0 & 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1 & 2 & 2 \end{array}\right) \underset{(3)}{\longrightarrow} \left(\begin{array}{ccccc|c} \boxed{1} & -1 & 2 & 0 & -2 & 2 \\ 0 & \boxed{2} & 1 & -2 & -1 & -2 \\ 0 & 0 & 0 & \boxed{1} & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

(1) row 2 $\longleftarrow$ row 2 + 3 $\cdot$ row 1, row 4 $\longleftarrow$ row 4 − row 1
(2) row 3 $\longleftarrow$ row 3 + 2 $\cdot$ row 2
(3) row 4 $\longleftarrow$ row 4 − row 3

General solution

The unknowns, corresponding to columns which do not contain pivots (framed matrix entries), can be chosen arbitrarily, i.e., $x_3 = s$, $x_5 = t$ with $s, t \in \mathbb{R}$. The remaining unknowns are uniquely determined and can be computed with backward substitution:

$$\begin{aligned} x_4 &\underset{\text{row 3}}{=} 2 - 2x_5 = 2 - 2t \\ x_2 &\underset{\text{row 2}}{=} (-2 - x_3 + 2x_4 + x_5)/2 \\ &= (-2 - s + 2(2 - 2t) + t)/2 = 1 - s/2 - 3t/2 \\ x_1 &\underset{\text{row 1}}{=} 2 + x_2 - 2x_3 + 2x_5 \\ &= 2 + (1 - s/2 - 3t/2) - 2s + 2t = 3 - 5s/2 + t/2 \end{aligned}$$

or, using vector notation,

$$x = \begin{pmatrix} 3 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} + \tilde{s} \begin{pmatrix} -5 \\ -1 \\ 2 \\ 0 \\ 0 \end{pmatrix} + \tilde{t} \begin{pmatrix} 1 \\ -3 \\ 0 \\ -4 \\ 2 \end{pmatrix}, \quad \tilde{s} = s/2,\ \tilde{t} = t/2$$

5.12 Reduced Echelon Form and General Solution of a Linear System

Transform the linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & + & & & 3x_4 & = & 1 \\ 2x_1 & + & 3x_2 & - & x_3 & + & 4x_4 & = & 3 \\ & & x_2 & + & 4x_3 & - & x_4 & = & 2 \end{array}$$

to reduced echelon form and determine the general solution x.

Resources: Gauß Elimination, Echelon Form

Problem Variants

■

:

check

■

:

check

■

:

check

Solution

Gauß elimination

The tableau $C := (A|b)$ of a linear system $Ax = b$ with an $m \times n$ matrix A is transformed to reduced echelon form with at most m steps.

step k
(1) Determine the column with smallest index i_k having a nonzero entry c_{j,i_k} with $j \geq k$. If $j \neq k$, permute row j with row k.
(2) Divide row k by c_{k,i_k} to normalize the pivot element to 1.
(3) Add multiples of row k to rows $j \neq k$ to annihilate the entries c_{j,i_k} above and below the pivot position (k, i_k).

Reduced echelon form for the given linear system

$$C := (A|b) = \left(\begin{array}{cccc|c} 1 & 2 & 0 & 3 & 1 \\ 2 & 3 & -1 & 4 & 3 \\ 0 & 1 & 4 & -1 & 2 \end{array}\right)$$

step 1 $(i_1 = 1)$
(3) row 2 $\leftarrow$ row 2 $-$ 2 $\cdot$ row 1

$$\underset{(3)}{\rightarrow} \left(\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 3 & 1 \\ 0 & -1 & -1 & -2 & 1 \\ 0 & 1 & 4 & -1 & 2 \end{array}\right)$$

(pivot element boxed) **step 2** $(i_2 = 2)$
(2) row 2 $\leftarrow$ row 2 / (-1)
(3) row 1 $\leftarrow$ row 1 $-$ 2 $\cdot$ row 2
(3) row 3 $\leftarrow$ row 3 $-$ row 2

$$\underset{(2)}{\rightarrow} \left(\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 3 & 1 \\ 0 & 1 & 1 & 2 & -1 \\ 0 & 1 & 4 & -1 & 2 \end{array}\right) \underset{(3)}{\rightarrow} \left(\begin{array}{cccc|c} \boxed{1} & 0 & -2 & -1 & 3 \\ 0 & \boxed{1} & 1 & 2 & -1 \\ 0 & 0 & 3 & -3 & 3 \end{array}\right)$$

step 3 $(i_3 = 3)$
(2) row 3 $\leftarrow$ row 3 / 3
(3) row 1 $\leftarrow$ row 1 + 2 $\cdot$ row 3
(3) row 2 $\leftarrow$ row 2 $-$ row 3

$$\underset{(2)}{\rightarrow} \left(\begin{array}{cccc|c} \boxed{1} & 0 & -2 & -1 & 3 \\ 0 & \boxed{1} & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & 1 \end{array}\right) \underset{(3)}{\rightarrow} \left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & -3 & 5 \\ 0 & \boxed{1} & 0 & 3 & -2 \\ 0 & 0 & \boxed{1} & -1 & 1 \end{array}\right) =: D$$

MATLAB® `D = rref(C)`

General solution

Computing the solution x with backward substitution, the unknowns x_k, which do not correspond to pivots (boxed), can be chosen arbitrarily; in the case considered, x_4. Setting $x_4 = s$ (free parameter), the backward substitution proceeds as follows:

$$\begin{array}{rrcll} \text{row3}: & x_3 - x_4 & = & 1 & \Longrightarrow \; x_3 = 1 + s \\ \text{row2}: & x_2 + 3x_4 & = & -2 & \Longrightarrow \; x_2 = -2 - 3s \\ \text{row1}: & x_1 - 3x_4 & = & 5 & \Longrightarrow \; x_1 = 5 + 4s \end{array}$$

one-dimensional affine solution set:

$$x = \begin{pmatrix} 5 \\ -2 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 4 \\ -3 \\ 1 \\ 1 \end{pmatrix}$$

5.13 Solvability of a Linear System

For which vectors b does the linear system $Ax = b$ with

$$A = \begin{pmatrix} 1 & 0 & -3 & 2 \\ -2 & 2 & -3 & 1 \\ -3 & 2 & 0 & -1 \end{pmatrix}$$

have a solution?

Resources: Gauß Elimination, Echelon Form

Problem Variants

- $A = \begin{pmatrix} 4 & -4 \\ 2 & 0 \\ -3 & -1 \end{pmatrix}$

$-b_1 + ?b_2 + ?b_3 = 0$:

[ ] check

- $A = \begin{pmatrix} 3 & 1 & -2 \\ -4 & 1 & 0 \\ 2 & -4 & 4 \\ -1 & -3 & -2 \end{pmatrix}$

$2b_1 + ?b_2 + ?b_3 + ?b_4 = 0$:

[ ] check

- $A = \begin{pmatrix} -2 & 0 & 4 & 0 \\ -2 & -3 & 2 & -1 \\ -4 & -4 & -1 & -2 \\ 3 & 3 & -4 & 1 \end{pmatrix}$

$b_1 + ?b_2 + ?b_3 + ?b_4 + b_5 = 0$:

[ ] check

Solution

transformation of the tableau

$$(A|b) = \left(\begin{array}{cccc|c} 1 & 0 & -3 & 2 & b_1 \\ -2 & 2 & -3 & 1 & b_2 \\ -3 & 2 & 0 & -1 & b_3 \end{array}\right),$$

associated with the linear system $Ax = b$, with Gauß operations to echelon form $(\tilde{A}|\tilde{b})$:

$$(A|b) \underset{(1)}{\rightarrow} \left(\begin{array}{cccc|c} 1 & 0 & -3 & 2 & b_1 \\ 0 & 2 & -9 & 5 & b_2 + 2b_1 \\ 0 & 2 & -9 & 5 & b_3 + 3b_1 \end{array}\right) \underset{(2)}{\rightarrow} \underbrace{\left(\begin{array}{cccc|c} 1 & 0 & -3 & 2 & b_1 \\ 0 & 2 & -9 & 5 & b_2 + 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_2 + b_1 \end{array}\right)}_{(\tilde{A}|\tilde{b})}$$

(1) row 2 $\longleftarrow$ row 2 + 2 $\cdot$ row 1, row 3 $\longleftarrow$ row 3 + 3 $\cdot$ row 1
(2) row 3 $\longleftarrow$ row 3 - row 2

existence of solutions
$\iff$ $\tilde{b}_k = 0$ for $k > \operatorname{rank} A = 2$
$\iff$ $\tilde{b}_3 = b_3 - b_2 + b_1 = 0$

5.14 Linear System with Parameter (3×3)

Determine the solution x of the linear system

$$x_1 + tx_2 = -2, \quad 2x_2 + tx_3 = 1, \quad tx_1 + 4x_3 = 2\,,$$

depending on the parameter $t \in \mathbb{R}$.

Resources: Linear System, Cramer's Rule, Gauß Elimination

Problem Variants

■ $2x_2 + tx_3 = 0, \quad x_1 + tx_2 + x_3 = 3, \quad tx_1 + 2x_2 = 0$

$t = 1 \implies x = (?, -?, ?)^{\mathrm{t}}$:

check

■ $tx_1 + x_2 = 2, \quad -x_1 + x_2 + 2x_3 = 4, \quad x_2 + tx_3 = 2$

$t = 1 \implies x = (?, ?, ?)^{\mathrm{t}}$:

check

■ $x_1 - x_3 = 1, \quad 2x_1 + tx_2 = 4, \quad -x_2 + 2x_3 = 2$

$t = -2 \implies x = (?, ?, ?)^{\mathrm{t}}$:

check

Solution

matrix form of the linear system

$$\underbrace{\begin{pmatrix} 1 & t & 0 \\ 0 & 2 & t \\ t & 0 & 4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underbrace{\begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}}_{b}$$

determinant of the coefficient matrix A

$$d = \begin{vmatrix} 1 & t & 0 \\ 0 & 2 & t \\ t & 0 & 4 \end{vmatrix} = t^3 + 8$$

$= 0$ for $t = -2 \quad \rightsquigarrow \quad$ different cases

- $t \neq -2$ (unique solution):
  Cramer's rule $\quad \Longrightarrow$

$$\begin{aligned} x_2 &= \begin{vmatrix} a_{1,1} & b_1 & a_{1,3} \\ a_{2,1} & b_2 & a_{2,3} \\ a_{3,1} & b_3 & a_{3,3} \end{vmatrix} \Big/ d = \begin{vmatrix} 1 & -2 & 0 \\ 0 & 1 & t \\ t & 2 & 4 \end{vmatrix} \Big/ d \\ &= \frac{-2t^2 - 2t + 4}{t^3 + 8} \underset{\text{factoring and dividing by } t+2}{=} \frac{2 - 2t}{t^2 - 2t + 4} \end{aligned}$$

  substituting into equations 1 and 2 $\quad \Longrightarrow$

$$x_1 = -2 - tx_2 = \frac{2t - 8}{t^2 - 2t + 4}, \quad x_3 = \frac{1 - 2x_2}{t} = \frac{t + 2}{t^2 - 2t + 4}$$

- $t = -2$ (singular system):
  Gauß elimination $\quad \rightsquigarrow$

$$\left(\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & 2 & -2 & 1 \\ -2 & 0 & 4 & 2 \end{array}\right) \underset{(1)}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & 2 & -2 & 1 \\ 0 & -4 & 4 & -2 \end{array}\right) \underset{(2)}{\rightarrow} \left(\begin{array}{ccc|c} 1 & -2 & 0 & -2 \\ 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

  (1) row 3 $\leftarrow$ row 3 + 2 $\cdot$ row 1

  (2) row 3 $\leftarrow$ row 3 + 2 $\cdot$ row 2

  backward substitution $\quad \Longrightarrow \quad x_3 = s \in \mathbb{R}$ arbitrary, $x_2 = 1/2 + s$, $x_1 = -1 + 2s$

Chapter 6

Eigenvalues and Normal Forms

6.1 Eigenvalues and Eigenvectors of a 2×2 Matrix

Determine the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$

Resources: Eigenvalues and Eigenvectors, Characteristic Polynomial

Problem Variants

■ $A = \begin{pmatrix} 3 & 5 \\ 4 & 2 \end{pmatrix}$

---

eigenvectors corresponding to the smallest and largest eigenvalues: $(4, -?)^{\mathrm{t}}$,$(5, ?)^{\mathrm{t}}$:

[ ] check

■ $A = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$

---

eigenvectors corresponding to the smallest and largest eigenvalues: $(3, -?)^{\mathrm{t}}$,$(1, ?)^{\mathrm{t}}$:

[ ] check

■ $A = \begin{pmatrix} 4 & 7 \\ 1 & -2 \end{pmatrix}$

---

eigenvectors corresponding to the smallest and largest eigenvalues: $(1, -?)^{\mathrm{t}}$,$(7, ?)^{\mathrm{t}}$:

[ ] check

Solution

Eigenvalues

characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$:

$$\det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 0-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6$$

zeros $\quad\rightsquigarrow\quad$ eigenvalues $\lambda_1 = -2$, $\lambda_2 = 3$

Eigenvectors

solutions v of the linear system $(A - \lambda E)v = (0,0)^{\mathrm{t}}$

- $\lambda_1 = -2$

$$\begin{pmatrix} 1-(-2) & 2 \\ 3 & 0-(-2) \end{pmatrix} v = \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightsquigarrow \quad v_1 = s \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

- $\lambda_2 = 3$

$$\begin{pmatrix} -2 & 2 \\ 3 & -3 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \rightsquigarrow \quad v_2 = s \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$s \in \mathbb{R}$

Remark Because of $\operatorname{rank}(A - \lambda E) = 1$, i.e., in view of the linear dependence of the two equations, only one of the equations needs to be solved.

6.2 2×2 Matrix with Prescribed Complex Eigenvalue and Eigenvector

Determine the real matrix A which has the eigenvector $v = (1, \mathrm{i})^{\mathrm{t}}$ with the eigenvalue $\lambda = 2 + \mathrm{i}$.

Resources: Eigenvalues and Eigenvectors, Characteristic Polynomial

Problem Variants

■ $v = (-3 + 5\mathrm{i}, 1 + 4\mathrm{i})^{\mathrm{t}}$, $\lambda = 2 + \mathrm{i}$

$A = [?, -?; ?, ?]$ (MATLAB® notation):

[ ] check

■ $v = (1 + 5\mathrm{i}, 3 + 2\mathrm{i})^{\mathrm{t}}$, $\lambda = -1 + 2\mathrm{i}$

$A = [?, -?; ?, -?]$ (MATLAB® notation):

[ ] check

■ $v = (5 + 2\mathrm{i}, 3 + 7\mathrm{i})^{\mathrm{t}}$, $\lambda = -7 + 2\mathrm{i}$

$A = [-?, ?; -?, -?]$ (MATLAB® notation):

[ ] check

Solution

A real $\quad\Longrightarrow\quad$ The second eigenvalue and a second eigenvector are complex conjugates to $\lambda_1 = 2 + \mathrm{i}$ and $v_1 = (1, \mathrm{i})^{\mathrm{t}}$, i.e.

$$\lambda_2 = 2 - \mathrm{i}, \quad v_2 = (1, -\mathrm{i})^{\mathrm{t}}\,.$$

Setting

$$V = (v_1, v_2) = \begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}, \quad \Lambda = \operatorname{diag}(\lambda_1, \lambda_2) = \begin{pmatrix} 2+\mathrm{i} & 0 \\ 0 & 2-\mathrm{i} \end{pmatrix},$$

the diagonal form of A is obtained with the similarity transformation

$$V^{-1}AV = \Lambda \quad\Longleftrightarrow\quad A = V\Lambda V^{-1}$$

formula for the inverse of a 2×2 matrix $\quad\Longrightarrow$

$$V^{-1} = \frac{1}{\det V}\begin{pmatrix} v_{2,2} & -v_{1,2} \\ -v_{2,1} & v_{1,1} \end{pmatrix} = -\frac{1}{2\mathrm{i}}\begin{pmatrix} -\mathrm{i} & -1 \\ -\mathrm{i} & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix}$$

similarity transformation $\quad\rightsquigarrow$

$$A = V\Lambda V^{-1} = \underbrace{\begin{pmatrix} 1 & 1 \\ \mathrm{i} & -\mathrm{i} \end{pmatrix}\begin{pmatrix} 2+\mathrm{i} & 0 \\ 0 & 2-\mathrm{i} \end{pmatrix}}_{\begin{pmatrix} 2+\mathrm{i} & 2-\mathrm{i} \\ -1+2\mathrm{i} & -1-2\mathrm{i} \end{pmatrix}}\frac{1}{2}\begin{pmatrix} 1 & -\mathrm{i} \\ 1 & \mathrm{i} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

6.3 Eigenvalue and Eigenvector of a 2×2 Matrix with Parameter

For which $t > 0$ does the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix}$$

have only one eigenvalue? Determine a corresponding eigenvector.

Resources: Eigenvalues and Eigenvectors, Characteristic Polynomial

Problem Variants

■ $A = \begin{pmatrix} 8 & t \\ -2 & 4 \end{pmatrix}$

---

eigenvector: $(?, -1)^{\mathrm{t}}$:

[ ] check

■ $A = \begin{pmatrix} -5 & 1 \\ -4 & t \end{pmatrix}$

---

eigenvector: $(-1, ?)^{\mathrm{t}}$:

[ ] check

■ $A = \begin{pmatrix} 9 & 2 \\ t & 1 \end{pmatrix}$

---

eigenvector: $(1, -?)^{\mathrm{t}}$:

[ ] check

Solution

Eigenvalue

characteristic polynomial of $A = \begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix}$:

$$\det\left(\begin{pmatrix} 0 & 1 \\ -1 & t \end{pmatrix} - \lambda \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{E}\right) = \begin{vmatrix} 0-\lambda & 1 \\ -1 & t-\lambda \end{vmatrix} = \lambda^2 - t\lambda + 1$$

zeros $\quad\rightsquigarrow\quad$ eigenvalues $\lambda_{1,2} = t/2 \pm \sqrt{t^2/4 - 1}$
double eigenvalue $\lambda_1 = \lambda_2 = 1$ for $\sqrt{\ldots} = 0$, i.e. for $t = 2$

Eigenvector

linear system for an eigenvector v:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - \lambda E)v = \begin{pmatrix} 0-1 & 1 \\ -1 & 2-1 \end{pmatrix} v = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} v \quad \Longrightarrow \quad v \parallel \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

6.4 Eigenvalues and Eigenvectors of a 3×3 Matrix

The matrix

$$A = \begin{pmatrix} 2 & 9 & 5 \\ -1 & -8 & -5 \\ -1 & 9 & 8 \end{pmatrix}$$

has the eigenvector $v_1 = (1, -1, 1)^{\mathrm{t}}$ and $\lambda_2 = 1$ is an eigenvalue. Determine all eigenvalues and corresponding eigenvectors.

Resources: Eigenvalues and Eigenvectors, Characteristic Polynomial

Problem Variants

- $A = \begin{pmatrix} 5 & 4 & 4 \\ 2 & 8 & 4 \\ -5 & 4 & 8 \end{pmatrix}$, $v_1 = (0, 1, -1)^{\mathrm{t}}$, $\lambda_2 = 9$

---

eigenvector v_3: $(4, ?, -?)^{\mathrm{t}}$:

check

- $A = \begin{pmatrix} 7 & 4 & -7 \\ 1 & 4 & 7 \\ -2 & -2 & 4 \end{pmatrix}$, $v_1 = (7, -7, -1)^{\mathrm{t}}$, $\lambda_2 = 3$

---

eigenvector v_3: $(-3, ?, ?)^{\mathrm{t}}$:

check

- $A = \begin{pmatrix} 7 & 1 & -1 \\ -3 & 7 & -1 \\ -3 & -1 & 7 \end{pmatrix}$, $v_1 = (0, 1, 1)^{\mathrm{t}}$, $\lambda_2 = 7$

---

eigenvector v_3: $(-2, ?, ?)^{\mathrm{t}}$:

check

Solution

Eigenvalues

The eigenvalue corresponding to the eigenvector $v_1 = (1, -1, 1)^{\mathrm{t}}$ is determined by computing Av:

$$\underbrace{\begin{pmatrix} 2 & 9 & 5 \\ -1 & -8 & -5 \\ -1 & 9 & 8 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}}_{v_1} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix} = -2v_1 \quad \Longrightarrow \quad \lambda_1 = -2$$

$$\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{trace} A = 2 - 8 + 8 = 2,\ \lambda_1 = -2,\ \lambda_2 = 1 \quad \Longrightarrow \quad \lambda_3 = 3$$

Eigenvectors

solving the linear systems $Av - \lambda v = (0,0,0)^{\mathrm{t}}$ $\quad \rightsquigarrow \quad$ eigenvectors to $\lambda_2 = 1$ and $\lambda_3 = 3$

$$\begin{aligned} Av_2 - v_2 &= \begin{pmatrix} 2-1 & 9 & 5 \\ -1 & -8-1 & -5 \\ -1 & 9 & 8-1 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad v_2 \parallel \begin{pmatrix} 3 \\ -2 \\ 3 \end{pmatrix} \\ Av_3 - 3v_3 &= \begin{pmatrix} 2-3 & 9 & 5 \\ -1 & -8-3 & -5 \\ -1 & 9 & 8-3 \end{pmatrix} v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad v_3 \parallel \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} \end{aligned}$$

In both cases, one equation is redundant, since $\operatorname{rank}(A - \lambda E) = 2$. Hence, a solution can be obtained as cross product of two linearly independent rows. For example

$$v_2 \parallel \begin{pmatrix} 1 \\ 9 \\ 5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 9 \\ 7 \end{pmatrix} = \begin{pmatrix} 18 \\ -12 \\ 18 \end{pmatrix}.$$

6.5 Eigenvalues and Eigenvectors of a Symmetric 3×3 Matrix

Determine the eigenvalues of the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

and corresponding eigenvalues.

Resources: Diagonal Form of Hermitian Matrices, Characteristic Polynomial

Problem Variants

■ $A = \begin{pmatrix} -4 & 6 & -3 \\ 6 & -1 & -2 \\ -3 & -2 & -4 \end{pmatrix}$

---

eigenvector to the smallest eigenvalue: $(3, -?, ?)^t$:

[ ] check

■ $A = \begin{pmatrix} 2 & 2 & -2 \\ 2 & 5 & -4 \\ -2 & -4 & -2 \end{pmatrix}$

---

eigenvector to the largest eigenvalue: $(1, ?, -?)^t$:

[ ] check

■ $A = \begin{pmatrix} 1 & -3 & 4 \\ -3 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

---

eigenvector to the largest eigenvalue: $(5, -?, ?)^t$:

[ ] check

Solution

Eigenvalues

Since the sum of the columns of the matrix

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix}$$

equals $(0,0,0)^{\mathrm{t}}$, $v_1 = (1,1,1)^{\mathrm{t}}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 0$.
characteristic polynomial

$$\begin{aligned} p_A(\lambda) \quad &= \quad \det\left(A - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right) = \begin{vmatrix} 0-\lambda & 1 & -1 \\ 1 & -2-\lambda & 1 \\ -1 & 1 & 0-\lambda \end{vmatrix} \\ &\underset{\text{Sarrus rule}}{=} \lambda^2(-2-\lambda) - 1 - 1 + \lambda + \lambda + (2+\lambda) = -\lambda(\lambda^2 + 2\lambda - 3) \end{aligned}$$

zeros $\quad\rightsquigarrow\quad$ eigenvalues $\lambda_1 = 0$, $\lambda_2 = 1$, $\lambda_3 = -3$

Eigenvectors

An eigenvector v is a solution of the linear system $Av - \lambda v = (0,0,0)^{\mathrm{t}}$.

$$Av_2 - \underbrace{\lambda_2}_{=1} v_2 = \begin{pmatrix} 0-1 & 1 & -1 \\ 1 & -2-1 & 1 \\ -1 & 1 & 0-1 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \implies v_2 \parallel \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

orthogonality of the eigenvectors of a symmetric matrix $\quad\implies$

$$v_3 \parallel \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

6.6 Normality of a 2×2 Matrix with Parameter, Eigenvalues, and Eigenvectors

For which value of $p \in \mathbb{C}$ is the matrix

$$\begin{pmatrix} 0 & 2 \\ p & 3\mathrm{i} \end{pmatrix}$$

normal? Determine the eigenvalues and eigenvectors for this parameter.

Resources: Unitary Diagonalization

Problem Variants

■ $\begin{pmatrix} p & 2 \\ -2 & 1 \end{pmatrix}$

---

eigenvalues: $? \pm ??$:

[ ] check

■ $\begin{pmatrix} 2 & 1+\mathrm{i} \\ 1+\mathrm{i} & p \end{pmatrix}$

---

eigenvalues: $1??, ??\mathrm{i}$:

[ ] check

■ $\begin{pmatrix} 2+\mathrm{i} & 1 \\ p & 2+\mathrm{i} \end{pmatrix}$

---

eigenvalues: $2??\mathrm{i}, ?$:

[ ] check

Solution

Normality

characterization:

$$AA^* = A^*A, \quad a^*_{j,k} = \overline{a}_{k,j}$$

application to the given matrix

$$\underbrace{\begin{pmatrix} 0 & 2 \\ p & 3\mathrm{i} \end{pmatrix}}_{A} \begin{pmatrix} 0 & \overline{p} \\ 2 & -3\mathrm{i} \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0 & \overline{p} \\ 2 & -3\mathrm{i} \end{pmatrix} \begin{pmatrix} 0 & 2 \\ p & 3\mathrm{i} \end{pmatrix}$$

comparison of the matrix elements

$$\begin{array}{llcl} (1,1): & 4 & = & |p|^2, \\ (2,1): & 6\mathrm{i} & = & -3\mathrm{i}p, \end{array} \qquad \begin{array}{llcl} (1,2): & -6\mathrm{i} & = & 3\mathrm{i}\overline{p} \\ (2,2): & |p|^2+9 & = & 13 \end{array}$$

equation (2,1) $\implies$ $p = -2$
consistent with the other equations

$$\rightsquigarrow \quad A = \begin{pmatrix} 0 & 2 \\ -2 & 3\mathrm{i} \end{pmatrix}$$

Eigenvalues and eigenvectors

characteristic polynomial

$$p_A(\lambda) = \det(A - \lambda E) = \begin{vmatrix} 0-\lambda & 2 \\ -2 & 3\mathrm{i}-\lambda \end{vmatrix} = \lambda^2 - 3\mathrm{i}\lambda + 4$$

zeros $\rightsquigarrow$ eigenvalues $\lambda_\pm = (3/2)\mathrm{i} \pm \sqrt{-(9/4)-4} = (3/2)\mathrm{i} \pm (5/2)\mathrm{i}$, i.e. $\lambda_- = -\mathrm{i}$, $\lambda_+ = 4\mathrm{i}$

- eigenvector to $\lambda_- = -\mathrm{i}$:

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (A - \lambda_- E)v_- = \begin{pmatrix} \mathrm{i} & 2 \\ -2 & 4\mathrm{i} \end{pmatrix} v_- \quad \rightsquigarrow \quad v_- = \begin{pmatrix} 2 \\ -\mathrm{i} \end{pmatrix}$$

- eigenvector to $\lambda_+ = 4\mathrm{i}$: $v_+ \perp v_-$, in view of the normality of A, definition of the complex scalar product, $\langle x, y \rangle = \sum_k \overline{x}_k y_k$ $\rightsquigarrow$

$$v_+ = \begin{pmatrix} \mathrm{i} \\ -2 \end{pmatrix}, \quad \langle v_-, v+ \rangle = 2 \cdot \mathrm{i} + \mathrm{i} \cdot (-2) = 0 \quad \checkmark$$

6.7 Eigenvalues and Inverse of a Cyclic Matrix

Determine the eigenvalues and the inverse of the matrix

$$\begin{pmatrix} 1 & 0 & 1 & 3 \\ 3 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix}.$$

Resources: Diagonal Form of Cyclic Matrices

Problem Variants

■ first column of the cyclic matrix: $(0, -1, 2)^{\mathrm{t}}$

---

first column of the inverse: $(?, ?, ?)^{\mathrm{t}}/7$:

[ ] check

■ first column of the cyclic matrix: $(1, 2, -1, 3)^{\mathrm{t}}$

---

first column of the inverse: $(?, ?, -?, ?)^{\mathrm{t}}/5$:

[ ] check

■ first column of the cyclic matrix: $(1, -1, 0, 0, 0, 0, 0, -1)^{\mathrm{t}}$

---

first column of the inverse: $(-?, -?, -?, ?, ?, ?, -?, ?)^{\mathrm{t}}/3$:

[ ] check

Solution

Eigenvalues

The eigenvalues of a cyclic $n \times n$ matrix A, generated by $a = (a_0, a_1, \ldots)^{\mathrm{t}}$, can be computed with the discrete Fourier transform:

$$(\lambda_0, \lambda_1, \ldots)^{\mathrm{t}} = \overline{W}(a_0, a_1, \ldots)^{\mathrm{t}} \tag{1}$$

with the Fourier matrix

$$W : \, w_{k,\ell} = w^{k\ell}, \, 0 \le k, \ell < n, \quad w = \exp(2\pi\mathrm{i}/n)$$

(complex conjugate $\overline{W} : \, w^{-k\ell}$)

$a = (1, 3, 1, 0)^{\mathrm{t}}$, $n = 4$, $w = \mathrm{i} \quad \Longrightarrow$

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \end{pmatrix}}_{\overline{W}} \begin{pmatrix} 1 \\ 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ -3\mathrm{i} \\ -1 \\ 3\mathrm{i} \end{pmatrix}$$

Inverse

The inverse $B = A^{-1}$ of a cyclic matrix A is also cyclic. Since the eigenvalues of B are reciprocals of the eigenvalues λ_k of A, the generator b of B can be computed by inverting the formula (1):

$$(1/\lambda_0, 1/\lambda_1, \ldots)^{\mathrm{t}} = \overline{W} b \quad \Longleftrightarrow \quad b = \frac{1}{n} W \, (1/\lambda_0, 1/\lambda_1, \ldots)^{\mathrm{t}} \, ,$$

since W is unitary up to normalization of the columns ($\overline{W}^{-1} = W/n$) substituting $\lambda = (5, -3\mathrm{i}, -1, 3\mathrm{i})^{\mathrm{t}} \quad \rightsquigarrow$

$$b = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \mathrm{i} & -1 & -\mathrm{i} \\ 1 & -1 & 1 & -1 \\ 1 & -\mathrm{i} & -1 & \mathrm{i} \end{pmatrix} \begin{pmatrix} 1/5 \\ \mathrm{i}/3 \\ -1 \\ -\mathrm{i}/3 \end{pmatrix} = \frac{1}{15} \begin{pmatrix} -3 \\ 2 \\ -3 \\ 7 \end{pmatrix}$$

Confirmation

$$AB = \begin{pmatrix} 1 & 0 & 1 & 3 \\ 3 & 1 & 0 & 1 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 3 & 1 \end{pmatrix} \frac{1}{15} \begin{pmatrix} -3 & 7 & -3 & 2 \\ 2 & -3 & 7 & -3 \\ -3 & 2 & -3 & 7 \\ 7 & -3 & 2 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \checkmark$$

6.8 Symmetric Matrix with Prescribed Eigenvalues and Eigenvectors

Determine the symmetric matrix A, which has the eigenvalues $\lambda_1 = -4$, $\lambda_2 = 8$, $\lambda_3 = -7$ and corresponding eigenvectors $v_1 = (1, -1, 0)^t$, $v_2 = (1, 1, 2)^t$.

Resources: Diagonal Form of Hermitian Matrices

Problem Variants

■ $\lambda_1 = 8$, $\lambda_2 = -9$, $\lambda_3 = -6$
$v_1 = (1, -1, 0)^t$, $v_2 = (1, 1, 1)^t$

$A = [?, -?, -?; -?, ?, -?; -?, -?, -?]$ (MATLAB® notation):

[ ] check

■ $\lambda_1 = -1$, $\lambda_2 = 4$, $\lambda_3 = -3$
$v_1 = (2, 0, 1)^t$, $v_2 = (-4, 5, 8)^t$

$A = [-?, -?, -?; -?, -?, ?; -?, ?, ?]$ (MATLAB® notation):

[ ] check

■ $\lambda_1 = -3$, $\lambda_2 = 2$, $\lambda_3 = 4$
$v_1 = (1, 2, 0)^t$, $v_2 = (2, -1, 5)^t$

$A = [?, -?, -?; -?, -?, ?; -?, ?, ?]/3$ (MATLAB® notation):

[ ] check

Solution

Eigenvectors

Since the matrix A is assumed to be symmetric, the eigenvectors to different eigenvalues are orthogonal. Hence, an eigenvector to λ_3 can be computed by taking the cross product of the eigenvectors v_1 and v_2 to λ_1 and λ_2:

$$v_3 \parallel \underbrace{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}_{v_1} \times \underbrace{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}}_{v_2} = \begin{pmatrix} -2 \\ -2 \\ 2 \end{pmatrix},$$

and $v_3 = (1, 1, -1)^{\mathrm{t}}$ is a convenient choice.

Construction of the matrix

Combining the three eigenvalue equations, $Av_k = \lambda_k v_k$, $k = 1, 2, 3$,

$$A \underbrace{(v_1, v_2, v_3)}_{=:V} = (v_1, v_2, v_3) \underbrace{\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)}_{\Lambda} \quad \Longleftrightarrow \quad A = V\Lambda V^{-1}.$$

To compute V^{-1}, we note that $Q^{-1} = Q^{\mathrm{t}}$ for an orthogonal matrix Q (orthogonal column vectors (the case also for V) of length one). Setting

$$Q := (v_1/|v_1|, v_2/|v_2|, v_3/|v_3|) =: VS, \quad S := \operatorname{diag}(1/|v_1|, 1/|v_2|, 1/|v_3|)$$

yields

$$\underbrace{(VS)^{-1}}_{S^{-1}V^{-1}} = (VS)^{\mathrm{t}} \quad \Longleftrightarrow \quad V^{-1} = \underbrace{SS^{\mathrm{t}}}_{=S^2} V^{\mathrm{t}}.$$

substituting the concrete vectors, noting that $|v_1| = \sqrt{2}$, $|v_2| = \sqrt{6}$, $|v_3| = \sqrt{3}$ $\rightsquigarrow$

$$\begin{aligned} A &= V\Lambda \underbrace{S^2 V^{\mathrm{t}}}_{V^{-1}} \\ &= \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} -4 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -7 \end{pmatrix} \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/6 & 0 \\ 0 & 0 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix} \\ &= \ldots \text{MATLAB}^{\circledR} \ldots = \begin{pmatrix} -3 & 1 & 5 \\ 1 & -3 & 5 \\ 5 & 5 & 3 \end{pmatrix} \end{aligned}$$

6.9 Cube Root of a 2×2 Matrix

Determine a real matrix R with

$$R^3 = \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} .$$

Resources: Basis of Eigenvectors

Problem Variants

■ $\begin{pmatrix} -3 & 4 \\ -2 & 3 \end{pmatrix}^{1/3}$

$[-?, ?; -?, ?]$ (MATLAB® notation):

[ ] check

■ $\begin{pmatrix} 7 & -4 & -1 \\ -2 & 0 & -2 \\ -6 & 4 & 2 \end{pmatrix}^{1/3}$

$[?, -?, -?; -?, ?, -?; ?, ?, ?]$ (MATLAB® notation):

[ ] check

■ $\begin{pmatrix} -2 & -6 \\ 3 & 7 \end{pmatrix}^{1/2}$

---

sum of the absolute values of the entries of all 4 possible square roots ??:

[ ] check

Solution

General Procedure

For a diagonal matrix $\Lambda = \operatorname{diag}(\lambda, \varrho, \ldots)$, we can define a cube root simply as $\Lambda^{1/3} := \operatorname{diag}(\lambda^{1/3}, \varrho^{1/3}, \ldots)$. Clearly, $(\Lambda^{1/3})^3 = \Lambda$.

Hence, diagonalization is the key to define a cube root for a general matrix A. If $Q = (u, v, \ldots)$ is a matrix of eigenvectors (columns) for A and Λ a diagonal matrix of corresponding eigenvalues $\lambda, \varrho, \ldots$, then $A = Q\Lambda Q^{-1}$ and, as expected,

$$R := Q\Lambda^{1/3}Q^{-1} \tag{1}$$

is a cube root of A:

$$\begin{aligned} R^3 &= (Q\Lambda^{1/3}Q^{-1})(Q\Lambda^{1/3}Q^{-1})(Q\Lambda^{1/3}Q^{-1}) \\ &= Q\Lambda^{1/3}\Lambda^{1/3}\Lambda^{1/3}Q^{-1} = Q\Lambda Q^{-1} = A \quad \checkmark \end{aligned}$$

The key observation is that the factors QQ^{-1} are equal to the unit matrix and thus can be omitted in the product.

Application to the given matrix

Since both rows of the matrix

$$A = \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix}$$

sum to the same value, $2 + 6 = 3 + 5 = 8$, $\lambda = 8$ is an eigenvalue with eigenvector $u = (1,\, 1)^{\mathrm{t}}$. Using that the sum of the eigenvalues is equal to the trace,

$$\varrho = \operatorname{trace} A - 8 = (2 + 5) - 8 = -1$$

is the second eigenvalue. An eigenvector v is obtained by solving the homogeneous linear system

$$Av - \varrho v = \begin{pmatrix} 2-(-1) & 6 \\ 3 & 5-(-1) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The vector $v = (2, -1)^{\mathrm{t}}$ is a convenient choice. With

$$Q = (u,\, v) = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$$

we can apply the formula (1):

$$R = Q\Lambda^{1/3}Q^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \underbrace{\begin{pmatrix} 1/3 & 2/3 \\ 1/3 & -1/3 \end{pmatrix}}_{Q^{-1}} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix}.$$

Confirmation

$$R^3 = \underbrace{\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}}_{R^2} \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 6 \\ 3 & 5 \end{pmatrix} = A \quad \checkmark$$

6.10 Eigenvalues and Eigenvectors with MATLAB®

Determine the eigenvalues λ_k and a matrix V of eigenvectors for the following matrices:

$$\begin{pmatrix} 2 & -1 & 0 \\ 1 & 2 & -1 \\ 0 & 1 & 2 \end{pmatrix}, \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ 2/3 & -2/3 & 1/3 \end{pmatrix}, \begin{pmatrix} 0 & 2 & 1 \\ 1 & 0 & 2 \\ 2 & 1 & 0 \end{pmatrix}.$$

Which of the properties

Hermitian (symmetric), normal, unitary (orthogonal), cyclic

do the matrices possess?

Resources: Eigenvalues and Eigenvectors

Problem Variants

Describe the properties with a 4-digit 0/1-sequence. For example, 0110 means that the matrix is unitary and, hence, also normal.

■ $a_{j,k} = 2^{-jk}$, $1 \leq j,k \leq 4$

---

largest absolute value of the eigenvalues: ?.??, properties: ????:

[ ] check

■ $a_{j,k} = \mathrm{sign}(j-k)$, $1 \leq j,k \leq 4$

---

largest absolute value of the eigenvalues: ?.??, properties: ????:

[ ] check

■ $a_{j,k} = \delta_{j,k} - 1/2$, $1 \leq j,k \leq 4$

---

largest absolute value of the eigenvalues: ?, properties: ????:

[ ] check

Solution

$A = [2\ -1\ 0;\ 1\ 2\ -1;\ 0\ 1\ 2]$

```
[V,Lambda] = eig(A); V, diag(Lambda)
   -0.000+0.500i -0.000-0.500i 0.707+0.000i   2.000+1.414i
    0.707+0.000i  0.707+0.000i 0.000+0.000i   2.000-1.414i
    0.000-0.500i  0.000+0.500i 0.707+0.000i   2.000+0.000i
```

normal matrix: $A^\star A = AA^\star$, V unitary

$A = [2\ -1\ 1;\ -1\ 2\ 1;\ 1\ 1\ 0]$

```
[V,Lambda] = eig(A); V, diag(Lambda)
   0.408 0.577  0.707   -1.000
   0.408 0.577 -0.707    2.000
  -0.816 0.577  0.000    3.000
```

symmetric matrix ($\implies$ normal): $\lambda_k \in \mathbb{R}$, V orthogonal

$A = [1\ 2\ 2;\ -2\ -1\ 2;\ 2\ -2\ 1]/3$

```
[V,Lambda] = eig(A); V, diag(Lambda)
   -0.707+0.000i  0.000-0.500i  0.000+0.500i    1.000+0.000i
   -0.000+0.000i  0.707+0.000i  0.707+0.000i   -0.333+0.942i
   -0.707+0.000i -0.000+0.500i -0.000-0.500i    0.333-0.942i
```

rotation: A orthogonal ($\implies$ normal), $\det A = 1$, $|\lambda_k| = 1$, V unitary

$A = [0\ 2\ 1;\ 1\ 0\ 2;\ 2\ 1\ 0]$

```
[V,Lambda] = eig(A); V, diag(Lambda)
   0.577+0.000i -0.577+0.000i -0.577+0.000i    3.000+0.000i
   0.577+0.000i  0.288-0.500i  0.288+0.500i   -1.500+0.866i
   0.577+0.000i  0.288+0.500i  0.288-0.500i   -1.500-0.866i
```

cyclic matrix ($\implies$ normal) $v_{j,k} = \mathrm{e}^{2\pi\mathrm{i}(j-1)(k-1)/3}/\sqrt{3}$ (Fourier matrix, unitary),
$\lambda_k = \sum_{\ell=0}^{2} a_{\ell,1}\mathrm{e}^{-2\pi\mathrm{i}(k-1)\ell/3}$ (discrete Fourier transform)

6.11 Eigenvalues and Jordan Form with Maple™

Determine the eigenvalues, eigenvectors, and the Jordan form of the stochastic matrix[1]

$$A = \frac{1}{10}\begin{pmatrix} 8 & 1 & 5 \\ 1 & 7 & 0 \\ 1 & 2 & 5 \end{pmatrix}.$$

Moreover, compute $\lim_{n\to\infty} A^n$.

Resources: Eigenvalues and Eigenvectors, Matrix Operations with Maple™

Problem Variants

■ $A = \frac{1}{14}\begin{pmatrix} 6 & 4 & 4 \\ 4 & 7 & 5 \\ 4 & 3 & 5 \end{pmatrix}$

$x_\infty = (?, ?, ?)^t/18$:

[ ] check

■ $A = \frac{1}{13}\begin{pmatrix} 8 & 5 & 5 \\ 4 & 5 & 2 \\ 1 & 3 & 6 \end{pmatrix}$

$x_\infty = (?, ?, ?)^t/10$:

[ ] check

■ $A = \frac{1}{24}\begin{pmatrix} 18 & 6 & 10 \\ 3 & 14 & 2 \\ 3 & 4 & 12 \end{pmatrix}$

$x_\infty = (?, ?, ?)^t/14$:

[ ] check

[1] stochastic $A \iff a_{j,k} \ge 0,\ \sum_j a_{j,k} = 1$

Solution

Definition of the matrix

```
with(LinearAlgebra):   # loading relevant functions
A := ScalarMultiply(Matrix([[8,1,5],[1,7,0],[1,2,5]]),1/10):
```

Eigenvalues and eigenvectors

compute a vector Λ and a matrix V with $AV(:,k) = \Lambda(k)V(:,k)$

```
LambdaV := Eigenvectors(A);
```

$$LambdaV := \begin{bmatrix} 1 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad \begin{bmatrix} 3 & -2 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

last column of V equal to $(0,0,0)^{\mathrm{t}}$ $\implies$ one-dimensional eigenspace for $\Lambda(2) = \Lambda(3) = 1/2$ $\implies$ A is not diagonalizable

Jordan form

compute a matrix Q of eigenvectors and an additional principal vector, corresponding to the eigenvalue 1/2 and a matrix J with Jordan blocks ($AQ = QJ$)

```
JQ := JordanForm(A,output=['J','Q']):
J := JQ[1]; Q := JQ[2];   # Jordan form and transformation matrix
```

$$J := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1 \\ 0 & 0 & 1/2 \end{bmatrix}, \quad Q := \begin{bmatrix} 3/5 & 1/5 & -3/5 \\ 1/5 & -1/10 & -1/5 \\ 1/5 & -1/10 & 4/5 \end{bmatrix}$$

Limit

$A^n = (QJQ^{-1})^n = QJ^nQ^{-1}$, $\lim_{n\to\infty} J^n = \operatorname{diag}(1,0,0) =: J_\infty$ $\implies$

$$\lim_{n\to\infty} A^n = QJ_\infty Q^{-1}$$

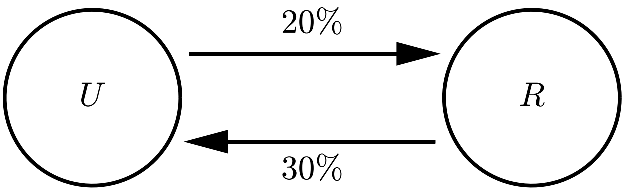
```
Jinf := DiagonalMatrix(Vector([1,0,0])):
Qi := MatrixInverse(Q):
Ainf := MatrixMatrixMultiply(Q,MatrixMatrixMultiply(Jinf,Qi));
```

$$A_\infty := \begin{bmatrix} 3/5 & 3/5 & 3/5 \\ 1/5 & 1/5 & 1/5 \\ 1/5 & 1/5 & 1/5 \end{bmatrix}$$

$\Longrightarrow$ $x_\infty := \lim_{n\to\infty} A^n x = (3/5,\, 1/5,\, 1/5)^{\mathrm{t}}$ for every vector x with $\sum_k x_k = 1$

6.12 Migration of Urban and Rural Population

Each year, 20% of the urban population (U) moves to the countryside and 30% of the rural population (R) moves to the city.



Determine the asymptotic ratio of the two populations.

Resources: Convergence of Matrix Powers

Problem Variants

■ $U \xrightarrow{10\%} R$, $R \xrightarrow{20\%} U$

$U/R \to ?/1$:

[ ] check

■ $U \xrightarrow{20\%} R$, $R \xrightarrow{50\%} U$

$U/R \to ?/2$:

[ ] check

■ $U \xrightarrow{30\%} R$, $R \xrightarrow{10\%} U$

$U/R \to 1/?$:

[ ] check

Solution

Matrix formulation

annual population change (urban $U \leftrightarrow$ rural R):

$$U \;\to\; 0.8U + 0.3R, \quad R \;\to\; 0.2U + 0.7R\,,$$

i.e.

$$\begin{pmatrix} U_{n+1} \\ R_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{pmatrix}}_{A} \begin{pmatrix} U_n \\ R_n \end{pmatrix}$$

Asymptotic behavior

Since A is a stochastic matrix (positive entries, columns summing to $= 1$), the limit of the ratio $U_n : R_n$ for $n \to \infty$ equals the ratio $v_1 : v_2$ of the components of an eigenvector v corresponding to the dominant eigenvalue 1. An eigenvector v for an eigenvalue λ is a solution of the linear system $Av - \lambda v = (0,0)^{\mathrm{t}}$.
$\lambda = 1 \quad \rightsquigarrow$

$$\underbrace{\begin{pmatrix} 0.8-1 & 0.3 \\ 0.2 & 0.7-1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}}_{Av-v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \text{i.e.} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \parallel \begin{pmatrix} 3 \\ 2 \end{pmatrix},$$

and, hence, $\lim_{n\to\infty} U_n/R_n = 3/2$

6.13 Three Term Recursion

Write the recursion

$$x_{k+1} = 3x_k - 2x_{k-1}, \quad x_0 = 0,\, x_1 = 1\,,$$

in matrix form and derive a formula for x_n.

Resources: Basis of Eigenvectors

Problem Variants

■ $x_{k+1} = -6x_k + 7x_{k-1},\, x_0 = 1,\, x_1 = 9$

$x_n = -?(-?)^n + ?$:

[ ] check

■ $x_{k+1} = x_k + 6x_{k-1},\, x_0 = 8,\, x_1 = 9$

$x_n = ??^n + ?(-?)^n$:

[ ] check

■ $x_{k+1} = -5x_k - 6x_{k-1},\, x_0 = 0,\, x_1 = -4$

$x_n = ?(-?)^n - ?(-?)^n$:

[ ] check

Solution

Matrix form of the recursion

$x_{k+1} = 3x_k - 2x_{k-1}$, $x_0 = 0$, $x_1 = 1 \quad \Longleftrightarrow$

$$\begin{pmatrix} x_k \\ x_{k+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}}_{A} \begin{pmatrix} x_{k-1} \\ x_k \end{pmatrix}, \quad \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$\rightsquigarrow$ representation of the sequence $x_0, x_1, \ldots$ with the eigenvectors v_ℓ and eigenvalues λ_ℓ of A:

$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = c_1 v_1 + c_2 v_2 \implies \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix} = A^n \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = c_1 \lambda_1^n v_1 + c_2 \lambda_2^n v_2 \quad (1)$$

Eigenvalues and eigenvectors of A

Since the entries of the rows of A sum to 1, $v_1 = (1,1)^{\mathrm{t}}$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$.
Since $\lambda_1 + \lambda_2 = \operatorname{trace} A = 3$, $\lambda_2 = 2$ is the second eigenvalue with eigenvector determined by solving the linear system

$$\begin{pmatrix} 0-2 & 1 \\ -2 & 3-2 \end{pmatrix} v_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

i.e. $v_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Representation of the sequence

expressing the initial values $x_0 = 0$, $x_1 = 1$ in terms of the eigenvectors $\quad \rightsquigarrow$

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{i.e. } c_1 = -1,\ c_2 = 1$$

and (1) $\quad \Longrightarrow$

$$x_n = -1 \cdot 1^n \cdot \underbrace{1}_{(v_1)_1} + 1 \cdot 2^n \cdot \underbrace{1}_{(v_2)_1} = -1 + 2^n$$

6.14 Jordan Form of a 2×2 Matrix

Determine a matrix Q which transforms the matrix

$$A = \begin{pmatrix} 4 & 4 \\ -9 & -8 \end{pmatrix}$$

to Jordan form: $J = Q^{-1}AQ$.

Resources: Jordan Form

Problem Variants

■ $A = \begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}$

$Q = [1, 5; -?, -?]$ (MATLAB® notation):

[ ] check

■ $A = \begin{pmatrix} 4 & 1 \\ -4 & 8 \end{pmatrix}$

$Q = [1, -4; ?, -?]$ (MATLAB® notation):

[ ] check

■ $A = \begin{pmatrix} 2 & -4 \\ 1 & 6 \end{pmatrix}$

$Q = [-6, 9; ?, -?]$ (MATLAB® notation):

[ ] check

Solution

Eigenvalues, eigenvectors, and Jordan form

characteristic polynomial of the matrix $A = \begin{pmatrix} 4 & 4 \\ -9 & -8 \end{pmatrix}$:

$$\det(A - \lambda E) = \begin{vmatrix} 4-\lambda & 4 \\ -9 & -8-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 4 = (\lambda+2)^2$$

double zero $-2 \quad \Longrightarrow \quad$ eigenvalue $\lambda = -2$ with algebraic multiplicity $m_\lambda = 2$

linear system for an eigenvector v:

$$\underbrace{\begin{pmatrix} 4-(-2) & 4 \\ -9 & -8-(-2) \end{pmatrix}}_{Av-\lambda v} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad v \parallel \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Since the eigenspace is one-dimensional, i.e. the geometric multiplicity d_λ equals 1 and is less than the algebraic multiplicity m_λ, the matrix A is not diagonalizable. Hence, A has the Jordan form

$$J = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}.$$

Transformation matrix

The matrix Q of the similarity transformation $A \to J = Q^{-1}AQ$ has the form $Q = (v, w)$ with a generalized eigenvector w, which can be determined by considering the second column of the matrix equation

$$\underbrace{\begin{pmatrix} 4 & 4 \\ -9 & -8 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} 2 & w_1 \\ -3 & w_2 \end{pmatrix}}_{Q} = \begin{pmatrix} 2 & w_1 \\ -3 & w_2 \end{pmatrix} \underbrace{\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}}_{J},$$

i.e.

$$\begin{pmatrix} 4 & 4 \\ -9 & -8 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} - 2 \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

rewriting this equation $\quad \rightsquigarrow$

$$\begin{pmatrix} 6 & 4 \\ -9 & -6 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Noting that $(2, -3)^{\mathrm{t}}$ is a solution of the homogeneous equation and $(1, -1)^{\mathrm{t}}$ is a particular solution,

$$w = \begin{pmatrix} 1 \\ -1 \end{pmatrix} + s \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \quad s \in \mathbb{R}\,.$$

Chapter 7

Least Squares and Singular Value Decomposition

7.1 Least Squares Estimation of a Parameter

Interpolate the data

$$(x_k, y_k):\ (2, 2.2),\ (4, 3.9),\ (5, 5.0)$$

approximately by a straight line $g:\ y = px$ by minimizing the least squares error

$$e(p) = \sum_k (y_k - px_k)^2 \,.$$

Resources: Least Squares Line

Problem Variants

■ approximation by a circle: $(x_k, y_k):\ (3, 4),\ (-2, 5),\ (-4, -2)$, $e(p) = \sum_k (x_k^2 + y_k^2 - p)^2$

$p =$??.??:

check

■ approximation by a parabola: $(x_k, y_k):\ (2, 2),\ (3, 5),\ (4, 8)$, $e(p) = \sum_k (y_k - px_k^2)^2$

$p =$?.??:

check

■ approximation by a hyperbola: $(x_k, y_k):\ (2, 5),\ (3, 4),\ (4, 2)$, $e(p) = \sum_k (y_k - p/x_k)^2$

$p =$??.??:

check

Solution

necessary condition for a minimum of $e(p) = \sum_k (y_k - px_k)^2$:

$$0 = e'(p) = \sum_k 2(y_k - px_k)(-x_k) \quad \Longleftrightarrow \quad p = \left(\sum_k y_k x_k\right) \Big/ \left(\sum_k x_k x_k\right)$$

substituting the data (x_k, y_k): $(2, 2.2)$, $(4, 3.9)$, $(5, 5.0)$ $\quad \rightsquigarrow$

$$p = \frac{2.2 \cdot 2 + 3.9 \cdot 4 + 5.0 \cdot 5}{2 \cdot 2 + 4 \cdot 4 + 5 \cdot 5} = \frac{45}{45} = 1$$

7.2 Least Squares Estimation of Parameters

Determine the constants c and λ of the physical law $N(t) = ce^{-\lambda t}$ numerically from the data

t_k	1	2	4	5	8
N_k	8.1	6.7	4.4	3.6	2.1

by minimizing the squared sum of the logarithmic errors

$$e(c,\lambda) = \sum_k |\ln N_k - \ln N(t_k)|^2 .$$

Resources: Least Squares Line

Problem Variants

■ $p(t) = c\exp(qt)$,

t_k	0	1	2	3	4
p_k	2	5	15	40	120

$c = ?.??$, $q = ?.??$:

check

■ $d(h) = ch^m$,

h_k	1	0.9	0.8	0.6	0.4
d_k	1.01	0.80	0.65	035	017

$c = ?.??$, $m = ?.??$:

check

■ $y(x) = c/x^s$,

x_k	1	2	3	4	5
y_k	11.0	8.1	6.8	6.0	5.5

$c = ??.??$, $s = ?.??$:

check

Solution

Formulation as standard least squares problem

logarithmizing the physical law $N(t) = c\mathrm{e}^{-\lambda t}$ $\quad\rightsquigarrow$

$$\underbrace{\ln N_k}_{y_k} \stackrel{!}{=} \ln N(t_k) = \ln c - \lambda t_k, \quad k = 1, 2, \ldots$$

overdetermined equations for a straight line with y-intercept $\ln c$ and slope $-\lambda$

applying the formulas for the line parameters which minimize $\sum_k (y_k - \ln c - (-\lambda)t_k)^2$ $\quad\rightsquigarrow$

$$\begin{aligned} \ln c &= \frac{(\sum t_k^2)(\sum y_k) - (\sum t_k)(\sum t_k y_k)}{n(\sum t_k^2) - (\sum t_k)^2} \\ -\lambda &= \frac{n(\sum t_k y_k) - (\sum t_k)(\sum y_k)}{n(\sum t_k^2) - (\sum t_k)^2} \end{aligned}$$

MATLAB® implementation

```
t = [1; 2; 4; 5; 8];
N = [8.1; 6.7; 4.4; 3.6; 2.1];
n = length(t);
y = log(N);
d = n*sum(t.^2)-sum(t)^2;
c = exp((sum(t.^2)*sum(y)-sum(t)*sum(t.*y))/d)
lambda = - (n*sum(t.*y)-sum(t)*sum(y))/d
```

$\rightsquigarrow$ $\quad c \approx 9.7491$, $\lambda \approx 0.1944$

7.3 Overdetermined Linear System

Solve the least squares problem

$$\left| \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right| \to \min$$

and compute the error $|Ax - b|$.

Resources: Normal Equation

Problem Variants

■ $\left| \begin{pmatrix} 4 & -5 \\ 2 & 6 \\ 9 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -3 \\ -9 \\ 0 \end{pmatrix} \right| \to \min$

$|Ax - b|$ =?.??:

[ ] check

■ $\left| \begin{pmatrix} -5 & 7 & 3 \\ -6 & -1 & 6 \\ -7 & -9 & 9 \\ 5 & -2 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 4 \\ 1 \\ -8 \\ 0 \end{pmatrix} \right| \to \min$

$|Ax - b|$ =?.??:

[ ] check

■ $\left| \begin{pmatrix} -9 & 9 \\ 4 & -7 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} - \begin{pmatrix} -3 \\ 0 \\ 6 \end{pmatrix} \right| \to \min$

$|Ax - b|$ =?.??:

[ ] check

Solution

The vector x, which minimizes $|Ax - b|$, satisfies the normal equation

$$A^{\mathrm{t}}Ax = A^{\mathrm{t}}b\,. \tag{1}$$

substituting the given data

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$\rightsquigarrow$

$$\begin{aligned} A^{\mathrm{t}}A &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \\ A^{\mathrm{t}}b &= \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \end{aligned}$$

solving the linear system (1),

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

with Gauß elimination or Cramer's rule $\quad \rightsquigarrow \quad x = (3, -2, 1)^{\mathrm{t}}/4$
error:

$$\begin{aligned} |Ax - b| &= \left| \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3/4 \\ -1/2 \\ 1/4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right| \\ &= \left| \begin{pmatrix} 3/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right| = \left| \begin{pmatrix} -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{pmatrix} \right| = \sqrt{4 \cdot (1/16)} = \frac{1}{2} \end{aligned}$$

7.4 Least Squares Plane

Approximate the data

x	2	−1	0	1
y	0	1	1	0
z	2	−1	0	−3

with a plane $E:\ z = p_1 x + p_2 y + p_3$ by minimizing

$$S = \sum_{k=1}^{4} |p_1 x_k + p_2 y_k + p_3 - z_k|^2 \,.$$

Resources: Normal Equation

Problem Variants

■

x	−2	0	−2	−2	2
y	2	1	2	1	0
z	1	0	−1	−2	0

$p = (?, ?, -?)^{\mathrm{t}}$:

[ ] check

■

x	3	−3	−2	−3
y	−1	2	1	2
z	1	−3	2	−1

$p = (-?, -?, ?)^{\mathrm{t}}$:

[ ] check

■

x	1	0	−1	1	−1	0
y	2	1	2	−1	2	1
z	2	1	0	2	1	−1

$p = (-?, -?, ?)^{\mathrm{t}}$:

[ ] check

Solution

Formulation as least squares problem in standard form

abbreviating the kth component of the error by

$$e_k := p_1 x_k + p_2 y_k + p_3 - z_k, \quad k = 1, \ldots, 4\,,$$

$S = |e|^2 = |Ap - z|^2$ with

$$A = \begin{pmatrix} x_1 & y_1 & 1 \\ & \vdots & \\ & \vdots & \\ x_4 & y_4 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad z = \begin{pmatrix} 2 \\ -1 \\ 0 \\ -3 \end{pmatrix}$$

Solution of the normal equation

Each minimizer p of $S = |Ap - z|^2$ satisfies the normal equation $A^{\mathrm{t}}Ap = A^{\mathrm{t}}z$.
substituting the concrete data $\rightsquigarrow$

$$\begin{aligned} A^{\mathrm{t}}A &= \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 6 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & 4 \end{pmatrix} \\ A^{\mathrm{t}}z &= \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix} \end{aligned}$$

$\rightsquigarrow$ normal equation

$$\begin{pmatrix} 6 & -1 & 2 \\ -1 & 2 & 2 \\ 2 & 2 & 4 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$$

with the solution $p = (3, 6, -5)^{\mathrm{t}}$

resulting sum of the squared errors

$$S = \left| \begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 6 \\ -5 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 0 \\ -3 \end{pmatrix} \right|^2 = \left| \begin{pmatrix} -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} \right|^2 = 4$$

7.5 Weighted Least Squares

Determine the parameters (u, v) which minimize

$$\sum_{k=1}^{3} k\,(b_k - u - v/k)^2$$

for $b = (9, 6, 5)^{\mathrm{t}}$.

Resources: Normal Equation

Problem Variants

- $\displaystyle\sum_{k=1}^{3} 2^k\,(b_k - u - v/2^k)^2 \to \min, \quad b = (1, -2, 3)^{\mathrm{t}}$

$(u, v) = (?, -?)$:

[ ] check

- $\displaystyle\sum_{k=1}^{3} k\,(b_k - u - (-1)^k v)^2 \to \min, \quad b = (0, -1, 4)^{\mathrm{t}}$

$(u, v) = (?, -?)$:

[ ] check

- $\displaystyle\sum_{k=1}^{3} (b_k - u - kv)^2/k \to \min, \quad b = (1, -1, 3)^{\mathrm{t}}$

$(u, v) = (?, -?)$:

[ ] check

Solution

<u>Matrix formulation</u>

$e_k = 1 - u - v/k,\ k = 1,2,3 \quad \Longleftrightarrow$

$$e = \begin{pmatrix} 9 \\ 6 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1/2 \\ 1 & 1/3 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} =: b - Ax$$

weighted norm

$$|e|_w^2 = \sum_k w_k e_k^2 = |De|^2, \quad D = \mathrm{diag}(\sqrt{w_1}, \sqrt{w_2}, \ldots)$$

with $w_k = k$ for the problem considered
$\rightsquigarrow$ least squares problem in standard form (minimization of the Euclidean norm)

$$|De|^2 = |D(b - Ax)|^2 = |(Db) - (DA)x|^2 \to \min$$

<u>Minimization</u>

matrix DA, right-hand side Db $\rightsquigarrow$ normal equation for the optimal parameters $x = (u, v)^{\mathrm{t}}$:

$$(DA)^{\mathrm{t}}(DA)\, x = (DA)^{\mathrm{t}}\,(Db) \quad \text{i.e.} \quad A^{\mathrm{t}} D^2 A\, x = A^{\mathrm{t}} D^2\, b$$

substituting the concrete data $\rightsquigarrow$

$$A^{\mathrm{t}} D^2 A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1/2 \\ 1 & 1/3 \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ 3 & 11/6 \end{pmatrix}$$

$$A^{\mathrm{t}} D^2\, b = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1/2 & 1/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 9 \\ 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 36 \\ 20 \end{pmatrix}$$

and

$$\begin{pmatrix} 6 & 3 \\ 3 & 11/6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 36 \\ 20 \end{pmatrix} \quad \Longrightarrow \quad x = \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

7.6 Fitting Data with MATLAB®

Write a MATLAB® function, which fits the data (x_k, y_k) with a quadratic polynomial $Q: \; y = c_1 + c_2 x + c_3 x^2$, by solving a least squares problem (minimizing the sum of the squared interpolation errors).

Test your program for `x = [0:0.1:2]`, `y=0.1+1.1*x+x.^2.1`.

Resources: Normal Equation

Problem Variants

■ fit with a rational function $y = \dfrac{ax+b}{cx+1}$ by minimizing $\sum_k |y_k(cx_k+1) - ax_k - b|^2$
test data: $y_k = 1/(x_k^{1.1}+1)$, $x_k = 1,\ldots,10$

$a+b+c = ?.??$:

[ ] check

■ fit with a trigonometric polynomial $y = p(x) = a + b\cos x + c\sin x$ by minimizing $\sum_k |y_k - p(x_k)|^2$
test data: $y_k = \cos(1.1x_k)$, $x_k = 1,\ldots,10$

$a+b+c = ?.??$:

[ ] check

■ fit with exponential functions $y = e(x) = a\mathrm{e}^{-x} + b\mathrm{e}^{-2x} + c\mathrm{e}^{-3x}$ by minimizing $\sum_k |y_k - e(x_k)|^2$
test data: $y_k = 2\mathrm{e}^{-1.3x}$, $x_k = 1,\ldots,10$

$a+b+c = -?.??$:

[ ] check

Solution

Formulation as least squares problem

interpolating the data $\rightsquigarrow$ overdetermined linear system for the coefficients c:

$$c_1 + c_2 x_k + c_3 x_k^2 \stackrel{!}{=} y_k, \quad k = 1, 2, \ldots$$

matrix form

$$\underbrace{\begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ & \vdots & \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \end{pmatrix}$$

The coefficients c, which minimize $|Ax - b|$ satisfy the normal equation $A^{\mathrm{t}} A x = A^{\mathrm{t}} y$.

Matlab® function

```
function c = parabola(x,y)
x = x(:); y = y(:);    % conversion to column vectors
% y_k = c_1 + c_2 x_k + c_3 x_k^2 -> interpolation matrix
A = [ones(size(x)) x x.^2];
% solution of the normal equations
c = (A'*A) \ (A'*y);    % equivalent: c = A\y;
```

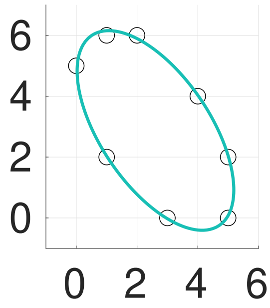
call of the function with the test data `x = [0:0.1:2], y=0.1+1.1*x+x.^2.1` $\rightsquigarrow$

$$c \approx (0.12, 0.94, 1.14)^{\mathrm{t}}$$

7.7 Fitting by Curves with MATLAB®

Interpolate the points

x	1	2	4	5	5	3	1	0
y	6	6	4	2	0	0	2	5



with an ellipse

$$E:\ c_1x^2 + c_2xy + c_3y^2 + c_4x + c_5y = 1$$

by solving a suitable least squares problem.

Resources: Matrix Operations with MATLAB® , Normal Equation, Pseudoinverse

Problem Variants

■ fit with a circle $C:\ x^2 + y^2 + ax + by + c = 0$
test data: $x_k = 3 + 1.2\cos(2\pi k/10)$, $y_k = 4 + \sin(2\pi k/10)$, $k = 0, \ldots, 9$

---

radius: ?.??:

check

■ fit with a hyperbola $H:\ axy + bx + cy = 1$
test data: $y_k = 1 + 1/x_k^{1.1}$, $x_k = 1, \ldots, 5$

$a + b + c = -?.???$:

check

■ fit with a parabola $P:\ ax^2 + bxy + cy^2 + dx + ey = 1$ by first computing a least squares approximation by a general quadratic curve and then correcting the coefficient b: $b = \sqrt{4ac}$
test data: $x_k = 1 + k + 2|k|^{2.1}$, $y_k = 2 + 2k - |k|^{2.1}$, $k = -3, \ldots, 3$

$b =?.??$:

check

Solution

assembly of the overdetermined linear system $Ac \stackrel{!}{=} b$:

$$c_1 x_k^2 + c_2 x_k y_k + c_3 y_k^2 + c_4 x_k + c_5 y_k \stackrel{!}{=} 1, \quad k = 1, \ldots 8$$

$$\iff \quad Ac \stackrel{!}{=} b$$

```
A = [x.^2 x.*y y.^2 x y], b = ones(size(x))
A =  1    6   36    1    6     b = 1
     4   12   36    2    6         1
    16   16   16    4    4         1
    25   10    4    5    2         1
    25    0    0    5    0         1
     9    0    0    3    0         1
     1    2    4    1    2         1
```

solution of $|Ac - b| \to \min$ with the \ operator

```
c = A\b, error = norm(A*c-b)
c = -0.0683   error = 0.0376
    -0.0640
    -0.0423
     0.5401
     0.4099
```

<u>Alternative solution</u>

using the normal equations ($A^\mathrm{t}Ac = A^\mathrm{t}b$) or the pseudoinverse ($c = A^+b$), generated with the singular value decomposition

```
c_NG = (A'*A)\(A'*b); c_PI = pinv(A)*b;
```

7.8 Singular Value Decomposition of a 2×2 Matrix

Determine the singular value decomposition of the matrix

$$\begin{pmatrix} 0 & 4 \\ 5 & -3 \end{pmatrix}.$$

Resources: Singular Value Decomposition

Problem Variants

■ $\begin{pmatrix} 7 & 4 \\ -4 & -1 \end{pmatrix}$

---

singular values: ? >?:

[ ] check

■ $\begin{pmatrix} 6 & -4 \\ -1 & -6 \end{pmatrix}$

---

singular values: ? >?:

[ ] check

■ $\begin{pmatrix} -2 & -4 \\ 7 & 2 \end{pmatrix}$

---

singular values: ? >?:

[ ] check

Solution

singular value decomposition:

$$A = USV^{\mathrm{t}}$$

with orthogonal matrices U, V, and a diagonal matrix S

- determination of V and S:
  The columns of V are normalized orthogonal eigenvectors of

  $$A^{\mathrm{t}}A = \begin{pmatrix} 0 & 5 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 0 & 4 \\ 5 & -3 \end{pmatrix} = \begin{pmatrix} 25 & -15 \\ -15 & 25 \end{pmatrix}$$

  and the singular values $s_1 \geq s_2$ are square roots of the corresponding eigenvalues.

  Since the entries of both rows sum to the same value, $(1,1)^{\mathrm{t}}$ is an eigenvector. Because of the symmetry of $A^{\mathrm{t}}A$, a second eigenvector is orthogonal to $(1,1)^{\mathrm{t}}$, hence, parallel to $(1,-1)^{\mathrm{t}}$. The corresponding eigenvalues are 10 and 40.
  normalizing the eigenvectors and labelling the eigenvalues according to their magnitude $\rightsquigarrow$

  $$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \; s_1 = \sqrt{40}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \; s_2 = \sqrt{10}\,,$$

  i.e.

  $$V = (v_1, v_2) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad S = \operatorname{diag}(s_1, s_2) = \begin{pmatrix} 2\sqrt{10} & 0 \\ 0 & \sqrt{10} \end{pmatrix}$$

- determination of U:
  The orthogonal matrix U is obtained by normalizing the columns of

  $$US = AV = \begin{pmatrix} 0 & 4 \\ 5 & -3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -4 & 4 \\ 8 & 2 \end{pmatrix}.$$

  dividing column 1 by $s_1 = 2\sqrt{10}$ and column 2 by $s_2 = \sqrt{10}$ $\rightsquigarrow$

  $$U = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

summarizing $\rightsquigarrow$

$$\underbrace{\begin{pmatrix} 0 & 4 \\ 5 & -3 \end{pmatrix}}_{A} = \underbrace{\frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} 2\sqrt{10} & 0 \\ 0 & \sqrt{10} \end{pmatrix}}_{S} \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_{V^{\mathrm{t}}}$$

7.9 Singular Value Decomposition and Pseudoinverse of a 3 × 2 Matrix

Determine the singular value decomposition and the pseudoinverse of the matrix

$$\begin{pmatrix} 2 & 0 \\ -1 & 3 \\ 2 & 0 \end{pmatrix}.$$

Resources: Singular Value Decomposition, Pseudoinverse

Problem Variants

■ $\begin{pmatrix} 1 & 3 \\ -4 & 4 \\ -3 & -1 \end{pmatrix}$

$A^+ = [?, -?, -??; ??, ?, -?]/72$ (MATLAB® notation):

[ ] check

■ $\begin{pmatrix} -1 & 3 \\ 8 & -8 \\ -3 & 1 \end{pmatrix}$

$A^+ = [??, ?, -??; ??, -?, -??]/72$ (MATLAB® notation):

[ ] check

■ $\begin{pmatrix} 5 & 4 \\ -4 & -5 \\ 2 & -2 \end{pmatrix}$

$A^+ = [?, ?, ?; ?, -?, -?]/9$ (MATLAB® notation):

[ ] check

Solution

Singular value decomposition

$$A = USV^{\mathrm{t}}$$

with S a 3×2 diagonal matrix of (nonnegative) singular values in descending order and U, V orthogonal matrices (unitary if A has complex entries)

- $V = (v_1, v_2)$ and $S\,\mathrm{diag}(s_1, s_2)$:

$$V\,(S^{\mathrm{t}}S)\,V^{\mathrm{t}} = A^{\mathrm{t}}A = \begin{pmatrix} 9 & -3 \\ -3 & 9 \end{pmatrix}, \quad S^{\mathrm{t}}S = \begin{pmatrix} s_1^2 & 0 \\ 0 & s_2^2 \end{pmatrix}$$

$\Longrightarrow$ The columns v_k of V are normalized eigenvectors of $A^{\mathrm{t}}A$, and the singular values are the nonnegative square roots of the corresponding eigenvalues λ_k.
forming the characteristic polynomial, $p(\lambda) = \quad (9-\lambda)^2 - 9 \quad \rightsquigarrow$

$$\lambda_1 = s_1^2 = 12,\ s_1 = 2\sqrt{3},\ \lambda_2 = s_2^2 = 6,\ s_2 = \sqrt{6}, \quad V = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

- $U = (u_1, u_2, u_3)$:

$$AV = US = (u_1, u_2, u_3)\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \\ 0 & 0 \end{pmatrix} = (u_1 s_1, u_2 s_2)$$

$\Longrightarrow$ The first two columns u_k of U can be computed by dividing the columns of AV by the singular values.

$$AV = \begin{pmatrix} 2 & 0 \\ -1 & 3 \\ 2 & 0 \end{pmatrix} \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 2 \\ -4 & 2 \\ 2 & 2 \end{pmatrix}$$

dividing column 1 by $s_1 = 2\sqrt{3}$ and column 2 by $\sqrt{6} \quad \rightsquigarrow U$

$$u_1 = \frac{1}{\sqrt{6}}(1,-2,1)^{\mathrm{t}}, \quad u_2 = \frac{1}{\sqrt{3}}(1,1,1)^{\mathrm{t}}$$

completion of the orthogonal basis with $u_3 = (1,0,-1)^{\mathrm{t}}/\sqrt{2} \quad \rightsquigarrow$

$$U = \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ -2/\sqrt{6} & 1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix}$$

Pseudoinverse

$$A^+ = VS^+U^{\mathrm{t}}, \quad S^+ = \operatorname{diag}(1/s_1, 1/s_2)$$

substituting the computed matrices $\quad\rightsquigarrow$

$$\begin{aligned} A^+ &= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{12}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{12} \begin{pmatrix} 3 & 0 & 3 \\ 1 & 4 & 1 \end{pmatrix} \end{aligned}$$

Alternative solution
Since A has maximal rank, $x = A^+b$ is the unique solution of the least squares problem $|Ax - b| \to \min$, which can also be computed by solving the normal equations $A^{\mathrm{t}}Ax = A^{\mathrm{t}}b$. Hence,

$$A^+ = (A^{\mathrm{t}}A)^{-1}A^{\mathrm{t}}\,.$$

7.10 Minimal Norm and General Solution of a Least Squares Problem

Determine all solutions x of the least squares problem

$$|Ax - (1,2,0,0)^{\mathrm{t}}| \to \min\,,$$

as well as the solution $x^\star$ with minimal norm using the singular value decomposition

$$A = USV^{\mathrm{t}} = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 & -2 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & -2 \end{pmatrix}\Big/3\,.$$

Resources: Singular Value Decomposition, Pseudoinverse

Problem Variants

In the following variants, determine first the singular value decomposition from the given representation of the matrix A.

■ $A = \begin{pmatrix} -2 \\ -1 \\ 2 \end{pmatrix}\begin{pmatrix} 1 & -2 \end{pmatrix}$, $b = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$

$|Ax - b|^2 =$?.??:

[ ] check

■ $A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -1 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$|Ax - b|^2 =$?.??:

[ ] check

■ $A = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$|Ax - b|^2$ =?.??:

check

Solution

$|e| = |U^{\mathrm{t}} e|$, $U^{\mathrm{t}} = U^{-1}$ for an orthogonal matrix U $\quad\Longrightarrow$

$$|\underbrace{USV^{\mathrm{t}}}_{A} x - b| \underset{\cdot U^{\mathrm{t}}}{=} |S \underbrace{V^{\mathrm{t}} x}_{=:y} - \underbrace{U^{\mathrm{t}} b}_{=:c}| = |Sy - c|$$

substituting the given data $\quad\rightsquigarrow$

$$\begin{aligned} Sy - c &= \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{S} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} - \underbrace{\frac{1}{2} \begin{pmatrix} 1 & -1 & 1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \end{pmatrix}}_{U^{\mathrm{t}}} \underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}}_{b} \\ &= \begin{pmatrix} 2y_1 + 1/2 \\ y_2 - 3/2 \\ 0 + 1/2 \\ 0 - 3/2 \end{pmatrix} \end{aligned}$$

$\Longrightarrow \quad |Ax - b| = |Sy - c|$ is minimal for $y_1 = -1/4$, $y_2 = 3/2$, and $y_3 \in \mathbb{R}$ arbitrary; the choice for which the first two components of $Sy - c$ are zero. backtransformation $y = V^{\mathrm{t}} x \to x$

$$x \underset{V^{\mathrm{t}} = V^{-1}}{=} Vy = \frac{1}{3} \begin{pmatrix} 2 & 1 & 2 \\ -2 & 2 & 1 \\ 1 & 2 & -2 \end{pmatrix} \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{2} \\ y_3 \end{pmatrix} = \begin{pmatrix} 1/3 \\ 7/6 \\ 11/12 \end{pmatrix} + y_3 \begin{pmatrix} 2/3 \\ 1/3 \\ -2/3 \end{pmatrix}$$

$|x| = |Vy| = |y|$ in view of the orthogonality of V $\quad\Longrightarrow\quad$ $y_3 = 0$ for the solution $x^\star$ of minimal norm

norm of the error:

$$|Ax - b|^2 = |Sy - c| = |(0, 0, -c_3, -c_4)^{\mathrm{t}}| = |(0, 0, 1/2, -3/2)^{\mathrm{t}}| = \frac{\sqrt{10}}{2}$$

Alternative solution

computation of the minimum norm solution with the pseudoinverse:

$$x^\star = A^+ b = V \underbrace{\begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}}_{S^+} U^{\mathrm{t}} b$$

7.11 Pseudoinverse via Singular Value Decomposition

Determine the pseudoinverse of the matrix

$$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} .$$

Resources: Pseudoinverse, Singular Value Decomposition

Problem Variants

■ $A = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 3 & -3 \end{pmatrix} + \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \end{pmatrix}$

$A^+ = [?, -?, ?; ?, ?, ?]/6$ (MATLAB® notation):

[ ] check

■ $A = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \\ -1 \end{pmatrix} \begin{pmatrix} -1 & -1 \end{pmatrix}$

$A^+ = [?, ?, -?; ?, -?, ?]/6$ (MATLAB® notation):

[ ] check

■ $A = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \begin{pmatrix} -1 & 1 \end{pmatrix}$

$A^+ = [?, -?, ?; ?, ?, ?]/6$ (MATLAB® notation):

[ ] check

Solution

normalizing the orthogonal vectors in the definition of

$$A = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$\rightsquigarrow$ singular value decomposition: $A = u_1 s_1 v_1^{\mathrm{t}} + +u_2 s_2 v_2^{\mathrm{t}}$, i.e.

$$A = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \underbrace{\sqrt{12}}_{s_1} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \underbrace{2}_{s_2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \end{pmatrix}$$

formula for the pseudoinverse, $A^+ = v_1 s_1^{-1} u_1^{\mathrm{t}} + v_2 s_2^{-1} u_2^{\mathrm{t}}$ $\rightsquigarrow$

$$\begin{aligned} A^+ &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \frac{1}{\sqrt{12}} \frac{1}{\sqrt{6}} \begin{pmatrix} 1 & 2 & -1 \end{pmatrix} + \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \frac{1}{2} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 1 \end{pmatrix} \\ &= \frac{1}{12} \begin{pmatrix} 1 & 2 & -1 \\ -1 & -2 & 1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \end{pmatrix} \end{aligned}$$

Chapter 8

Reflections, Rotations, and Projections

8.1 Reflection on a Straight Line in Implicit Form

Determine the affine map $x \mapsto y = Ax + b$ which represents the reflection on the straight line $g: -2x_1 + x_2 = 1$.

Resources: Affine Map, Reflection

Problem Variants

- $3x_1 + x_2 = 2$

---

largest entry of $(A|b)$: ?.?:

[ ] check

- $2x_1 - 6x_2 = 5$

---

largest entry of $(A|b)$: ?.?:

[ ] check

- $4x_1 - 3x_2 = 2$

---

largest entry of $(A|b)$: ?.??:

[ ] check

Solution

geometric description of the reflection $x \mapsto y$ on the line $g:\ x_2 = 1 + 2x_1$:

$$x - y \perp g \iff x - y \parallel \underbrace{(-2,1)^{\mathrm{t}}}_{\text{normal vector of g}} \quad \wedge \quad (x+y)/2 \in g\,.$$

i.e.

$$x_1 - y_1 = -2\lambda,\ x_2 - y_2 = \lambda, \quad (x_2 + y_2)/2 = 1 + (x_1 + y_1)$$

eliminating λ using the second equation, and solving the resulting linear system

$$x_1 - y_1 = -2\underbrace{(x_2 - y_2)}_{\lambda}, \quad (x_2 + y_2)/2 = 1 + (x_1 + y_1)$$

for (y_1, y_2) $\quad \rightsquigarrow$

$$y_1 = (-3x_1 + 4x_2 - 4)/5, \quad y_2 = (4x_1 + 3x_2 + 2)/5$$

matrix form:

$$y = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix} x + \frac{1}{5}\begin{pmatrix} -4 \\ 2 \end{pmatrix}$$

8.2 Reflection on a Straight Line in Parametric Form

Determine the affine map $x \mapsto y = Ax + b$ which represents the reflection on the straight line

$$g : \begin{pmatrix} 3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad t \in \mathbb{R}\,.$$

Resources: Affine Map, Reflection

Problem Variants

■ $g : \begin{pmatrix} 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \end{pmatrix}, \quad t \in \mathbb{R}$

---

largest entry of $(A|b)$: ?.?:

[ ] check

■ $g : \begin{pmatrix} 3 \\ 5 \end{pmatrix} + t \begin{pmatrix} 6 \\ -2 \end{pmatrix}, \quad t \in \mathbb{R}$

---

largest entry of $(A|b)$: ??.?:

[ ] check

■ $g : \begin{pmatrix} 3 \\ 5 \end{pmatrix} + t \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \quad t \in \mathbb{R}$

---

largest entry of $(A|b)$: ?.??:

[ ] check

Solution

Represent the reflection on the straight line

$$g : \underbrace{\begin{pmatrix} 3 \\ 4 \end{pmatrix}}_{p} + t \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{d}, \quad t \in \mathbb{R},$$

as composition of a translation $x \mapsto x' = x - p$, which moves the straight line g to a parallel straight line passing through the origin, a reflection $x' \mapsto y' = Ax$ on the shifted straight line, and a translation $y' \mapsto y = y' + p$ undoing the initial shift. Applying the formula for the matrix of a reflection in terms of a normal vector $u = (1, -2)^{\mathrm{t}} \perp d$ for g,

$$A = E - 2\frac{uu^{\mathrm{t}}}{u^{\mathrm{t}}u} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{2}{5} \underbrace{\begin{pmatrix} 1 \\ -2 \end{pmatrix} \begin{pmatrix} 1 & -2 \end{pmatrix}}_{\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}} = \frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix},$$

and, hence,

$$\begin{aligned} y &= y' + p = Ax' + p = A(x - p) + p = Ax + \underbrace{p - Ap}_{b} \\ &= \frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \left(\begin{pmatrix} 3 \\ 4 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right) \\ &= \underbrace{\frac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} -2 \\ 4 \end{pmatrix}}_{b}. \end{aligned}$$

8.3 Projection onto a Plane

Determine the matrix A representing the projection onto the plane

$$E:\ 2x_1 - x_2 + 2x_3 = 0\,.$$

Resources:

Problem Variants

■ $E:\ 3x_1 + x_2 - 3x_3 = -2$

$A = [??, -?, ?; -?, ??, ?; ?, ?, ??]/19$ (MATLAB® notation):

check

■ $E:\ x_1 - 4x_2 - x_3 = 3$

$A = [??, ?, ?; ?, ?, -?; ?, -?, ??]/18$ (MATLAB® notation):

check

■ $E:\ 2x_1 + 3x_2 - x_3 = -4$

$A = [??, -?, ?; -?, ?, ?; ?, ?, ??]/14$ (MATLAB® notation):

check

Solution

General formula

projection $x \mapsto y = Ax \quad \Longleftrightarrow \quad y = x + tu$ with u the normal vector of the plane $E:\ u^{\mathrm{t}}x = 0$

$y \in E \quad \Longleftrightarrow \quad u^{\mathrm{t}}(x + tu) = 0$, i.e. $t = -u^{\mathrm{t}}x/u^{\mathrm{t}}u$ and

$$y = x - u\,\frac{u^{\mathrm{t}}x}{u^{\mathrm{t}}u}, \quad A = E - \frac{1}{u^{\mathrm{t}}u}\,uu^{\mathrm{t}}$$

Application to the given data

substituting $u = (2, -1, 2)^{\mathrm{t}} \quad \rightsquigarrow$

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \underbrace{\frac{2x_1 - x_2 + 2x_3}{2^2 + 1 + 2^2}}_{-t}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \underbrace{\frac{1}{9} \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix}}_{uu^{\mathrm{t}}/u^{\mathrm{t}}u} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 5 & 2 & -4 \\ 2 & 8 & 2 \\ -4 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

8.4 Axis and Angle of a Rotation

Show that the matrix

$$Q = \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}$$

represents a rotation and determine the direction u of the rotation axis and the rotation angle $\varphi \in [0, \pi]$.

Resources: Rotation

Problem Variants

- $Q = \frac{1}{9}\begin{pmatrix} 4 & -4 & 7 \\ 8 & 1 & -4 \\ 1 & 8 & 4 \end{pmatrix}$

$u = (?.??, ?.??, ?.??)^{\mathrm{t}}$, $\varphi =?.??$:

[ ] check

- $Q = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 3 & -1 \end{pmatrix}$

$u = \pm(?.??, ?.??, ?.??)^{\mathrm{t}}$, $\varphi =?.??$:

[ ] check

- $Q = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$

$u = (?.??, -?.??, ?.??)^{\mathrm{t}}$, $\varphi =?.??$:

[ ] check

Solution

<u>Verification of the criteria for a rotation matrix Q</u>

- The columns of Q form an orthonormal basis:

$$Q^{\mathrm{t}}Q = E \quad \checkmark$$

- The determinant of Q is equal to 1:

$$\det Q = \left(\frac{1}{3}\right)^3 \begin{vmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{vmatrix} \underset{\text{Sarrus}}{=} \frac{1}{27}(8+8-1+4+4+4) = 1 \quad \checkmark$$

⚠ Note that $\det(sa, sb, sc) \neq s\det(a, b, c)$. Instead, scaling all columns yields a factor s^3.

<u>Rotation axis</u>

The direction u of the rotation axis is an eigenvector 1, i.e. a solution of the homogeneous linear system

$$Qx = x \quad \Longleftrightarrow \quad \underbrace{\frac{1}{3}\begin{pmatrix} 2-3 & 2 & -1 \\ -1 & 2-3 & 2 \\ 2 & -1 & 2-3 \end{pmatrix}}_{Q-E} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Since the entries of each row of $Q - E$ sum to 0, the direction u of the axis is parallel to $(1, 1, 1)^{\mathrm{t}}$.

<u>Rotation angle φ</u>

application of the formula $1 + 2\cos\varphi = \operatorname{trace} Q \quad \rightsquigarrow$

$$1 + 2\cos\varphi = \operatorname{trace} Q = q_{1,1} + q_{2,2} + q_{3,3} = 2, \quad \text{i.e. } \varphi = \arccos(1/2) = \pi/3$$

To determine the sign σ of the rotation axis, to ensure that (u, σ) represent a right-hand rotation (oriented like a right-hand screw), we insert the normalized direction vector $u = \sigma(1, 1, 1)^{\mathrm{t}}/\sqrt{3}$ in the formula

$$q_{j,k} = \cos\varphi\,\delta_{j,k} + (1 - \cos\varphi)\,u_j u_k + \sin\varphi \sum_{\ell} \varepsilon_{j,\ell,k} u_\ell \,.$$

It suffices to compare one entry for which the last summand is nonzero with the given matrix Q. Choosing, e.g., $(j,k) = (1,2)$,

$$\frac{2}{3} \stackrel{!}{=} 0 + \left(1 - \frac{1}{2}\right)\frac{1}{3} + \underbrace{\frac{\sqrt{3}}{2}}_{\sin(\pi/3)} \underbrace{\left(-\sigma\frac{1}{\sqrt{3}}\right)}_{\varepsilon_{1,3,2}u_3},$$

i.e. $\sigma = -1$ and $u = -(1,1,1)^{\mathrm{t}}/\sqrt{3}$.

8.5 Axis and Angle of the Composition of Two Rotations

Denote by $R(u, \varphi)$ a right-hand (counterclockwise) rotation with axis in the direction of a vector u by the angle φ. Determine u and φ for a rotation R resulting from a composition of rotations in the x_1x_2 and x_2x_3 plane:

$$R = R((1,0,0)^{\mathrm{t}}, \pi/4) \circ R((0,0,1)^{\mathrm{t}}, \pi/4)\,.$$

Resources: Rotation

Problem Variants

■ $R = R((1,0,0)^{\mathrm{t}}, \pi/6) \circ R((0,1,0)^{\mathrm{t}}, \pi/3)$

$u^\circ = (?.??, ?.??, ?.??)$, $\varphi =$?.??:

[ ] check

■ $R = R((0,1,0)^{\mathrm{t}}, 2\pi/3) \circ R((0,0,1)^{\mathrm{t}}, \pi/4)$

$u^\circ = (?.??, ?.??, ?.??)$, $\varphi =$?.??:

[ ] check

■ $R = R((0,0,1)^{\mathrm{t}}, 3\pi/4) \circ R((1,0,0)^{\mathrm{t}}, \pi/6)$

$u^\circ = (?.??, ?.??, ?.??)$, $\varphi =$?.??:

[ ] check

Solution

Rotation matrix

matrix of a counterclockwise plane rotation by an angle ϑ

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

composition of rotations by $\vartheta = \pi/4$ ($\cos\vartheta = \sin\vartheta = 1/\sqrt{2} \approx 0.7071$) in the x_1x_2 and x_2x_3 planes $\quad\rightsquigarrow$

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & s & -s \\ 0 & s & s \end{pmatrix} \begin{pmatrix} s & -s & 0 \\ s & s & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} s & -s & 0 \\ 1/2 & 1/2 & -s \\ 1/2 & 1/2 & s \end{pmatrix}, \quad s = 1/\sqrt{2}$$

Rotation angle

$$\cos\varphi = (\operatorname{trace} R - 1)/2 \quad \implies \quad \varphi = \arccos(s - 1/4) \approx 1.096$$

Rotation axis

The rotation axis is parallel to an eigenvector u to the eigenvalue 1, i.e. a solution of

$$\begin{pmatrix} s-1 & -s & 0 \\ 1/2 & 1/2-1 & -s \\ 1/2 & 1/2 & s-1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

setting, e.g., $u_1 = 1 \quad\rightsquigarrow\quad u = (1, 1-2s, 1)^{\mathrm{t}} = (1, 1-\sqrt{2}, 1)^{\mathrm{t}}$

Orientation

To determine the direction of the eigenvector u we compare the already computed matrix R with the formula

$$r_{j,k} = \cos\varphi\,\delta_{j,k} + (1-\cos\varphi)\,u_j u_k/|u|^2 + \sin\varphi \sum_\ell \varepsilon_{j,\ell,k} u_\ell/|u|$$

for the entries of the rotation matrix. Considering, e.g., the entry $(1,3)$, a MATLAB® computation shows that the chosen direction of u is correct.
normalizing $\quad\rightsquigarrow\quad u^\circ \approx (0.67, -0.28, 0.67)^{\mathrm{t}}$

8.6 Matrix of a Reflection

Determine the matrix of the reflection which maps the vector $(4, -1, 8)^t$ to a positive multiple of the first unit vector[1].

Resources: Reflection

Problem Variants

■ $(4, 3)^t$

---

largest matrix element: ?.?:

check

■ $(1, 2, 2)^t$

---

largest matrix element: ?.??:

check

■ $(1, 1, 1, 1)^t$

---

largest matrix element: ?.?:

check

[1]Reflections mapping to unit vectors are referred to as *Householder reflections* and can be used, similarly as Gauß operations, to transform a linear system to triangular form.

Solution

The matrix of a reflection on a plane with normal vector u is

$$Q = E - \frac{2}{|u|^2} u u^{\mathrm{t}} \tag{1}$$

with E the unit matrix. If Q maps x to $s(1,0,0)^{\mathrm{t}} =: se$, then $s = |x|$ because of the invariance of the norm under an orthogonal transformation. Moreover, from

$$|x|e = Qx = x - \frac{2u^{\mathrm{t}}x}{u^{\mathrm{t}}}\, u$$

it follows that $x - |x|e \parallel u$. Since the length of u is irrelevant in (1), one can set $u = x - |x|e$.

substituting $x = (4, -1, 8)^{\mathrm{t}}$ $\quad\rightsquigarrow$

$$|x| = \sqrt{16+1+64} = 9, \quad u = \begin{pmatrix} 4 \\ -1 \\ 8 \end{pmatrix} - 9 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -5 \\ -1 \\ 8 \end{pmatrix}$$

substituting into (1), noting that $u^{\mathrm{t}}u = 25 + 1 + 64 = 90$ $\quad\rightsquigarrow$

$$\begin{aligned} Q &= E - \frac{2}{|u|^2}\, u\, u^{\mathrm{t}} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{45} \begin{pmatrix} -5 \\ -1 \\ 8 \end{pmatrix} \begin{pmatrix} -5 & -1 & 8 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \frac{1}{45} \begin{pmatrix} 25 & 5 & -40 \\ 5 & 1 & -8 \\ -40 & -8 & 64 \end{pmatrix} \\ &= \frac{1}{45} \begin{pmatrix} 20 & -5 & 40 \\ -5 & 44 & 8 \\ 40 & 8 & -19 \end{pmatrix} \end{aligned}$$

8.7 Matrix of a Rotation

Determine the matrix R of the right-hand (counterclockwise) rotation by $\varphi = 2\pi/3$ with axis in the direction of $u = (2, -1, 2)^t$.

Resources: Rotation

Problem Variants

■ $u = (1, 2, 2)^t$, $\varphi = \pi/6$

$\max_{j,k} r_{j,k}$ =?.??:

[ ] check

■ $u = (1, 4, 8)^t$, $\varphi = \pi/4$

$\max_{j,k} r_{j,k}$ =?.??:

[ ] check

■ $u = (4, -7, 4)^t$, $\varphi = \pi/3$

$\max_{j,k} r_{j,k}$ =?.??:

[ ] check

Solution

application of the formula

$$r_{j,k} = \cos\varphi\,\delta_{j,k} \;+\; (1-\cos\varphi)\,u_j u_k/|u|^2 \;+\; \sin\varphi \sum_{\ell} \varepsilon_{j,\ell,k} u_\ell/|u|$$

for the matrix R of a right-hand (counterclockwise) rotation by φ with the direction vector u

substituting $\varphi = 2\pi/3$ and $u = (2,-1,2)^{\mathrm{t}}$ $\quad\rightsquigarrow$

$$c := \cos\varphi = -1/2, \quad s := \sin\varphi = \sqrt{3}/2, \quad |u| = 3$$

$$uu^{\mathrm{t}}/(u^{\mathrm{t}}u) = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & 2 \end{pmatrix} \Big/ 9 = \begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} \Big/ 9$$

$$\begin{pmatrix} 0 & \varepsilon_{1,3,2}u_3 & \varepsilon_{1,2,3}u_2 \\ \varepsilon_{2,3,1}u_3 & 0 & \varepsilon_{2,1,3}u_1 \\ \varepsilon_{3,2,1}u_2 & \varepsilon_{3,1,2}u_1 & 0 \end{pmatrix} \Big/ 3 = \begin{pmatrix} 0 & -2 & -1 \\ 2 & 0 & -2 \\ 1 & 2 & 0 \end{pmatrix} \Big/ 3$$

adding the terms $\quad\rightsquigarrow$

$$\begin{aligned} R \;&=\; \begin{pmatrix} 5c+4 & -2+2c-6s & 4-4c-3s \\ -2+2c+6s & 8c+1 & -2+2c-6s \\ 4-4c+3s & -2+2c+6s & 5c+4 \end{pmatrix} \Big/ 9 \\ &\approx\; \begin{pmatrix} 0.16 & -0.91 & 0.37 \\ 0.24 & -0.33 & -0.91 \\ 0.95 & 0.24 & 0.16 \end{pmatrix} \end{aligned}$$

Confirmation with MATLAB®

```
phi = 2*pi/3; u = [2;-1;2];
c = cos(phi); s = sin(phi);
u0 = u/norm(u);
R = c*eye(3) + (1-c)*u0*u0' + ...
    [0 -u0(3) u0(2); u0(3) 0 -u0(1); -u0(2) u0(1) 0];
```

8.8 Rotation with Prescribed Axis and Image of a Vector

Determine the matrix of the right-hand rotation with axis direction $(1,1,1)^{\mathrm{t}}$, which maps $(0,2,1)^{\mathrm{t}}$ to $(1,2,0)^{\mathrm{t}}$.

Resources: Rotation

Problem Variants

■ axis direction: $(2,1,2)^{\mathrm{t}}$, $(1,1,0)^{\mathrm{t}} \mapsto (0,1,1)^{\mathrm{t}}$

---

largest matrix entry: ?.??:

[ ] check

■ axis direction: $(1,1,1)^{\mathrm{t}}$, $(2,0,1)^{\mathrm{t}} \mapsto (0,2,1)^{\mathrm{t}}$

---

largest matrix entry: ?.??:

[ ] check

■ axis direction: $(1,-1,1)^{\mathrm{t}}$, $(1,1,1)^{\mathrm{t}} \mapsto (-1,-1,-1)^{\mathrm{t}}$

---

largest matrix entry: ?:

[ ] check

Solution

formula for a right-hand rotation with normalized axis direction u by an angle φ, using scalar and cross products:

$$v \mapsto Qv = \cos\varphi\, v + (1-\cos\varphi)\, uu^{\mathrm{t}}v + \sin\varphi\, u \times v \tag{1}$$

substituting $u = (1,1,1)^{\mathrm{t}}/\sqrt{3}$, $v = (0,2,1)^{\mathrm{t}}$, $Qv = (1,2,0)^{\mathrm{t}}$ with $c = \cos\varphi$, $s = \sin\varphi \quad \rightsquigarrow$

$$\begin{aligned}
\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} &= c\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} + \frac{1-c}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}}_{3} + \frac{s}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1-c \\ 1+c \\ 1 \end{pmatrix} + \frac{s}{\sqrt{3}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}
\end{aligned}$$

solving the resulting equations of the three vector components $\quad \rightsquigarrow \quad s = \sin\varphi = -\sqrt{3}/2$, $c = \cos\varphi = 1/2$ and, hence, $\varphi = -\pi/3$[2]

elements of the rotation matrix Q according to (1):

$$q_{j,k} = c\delta_{j,k} + (1-c)u_j u_k + s\sum_{\ell} \varepsilon_{j,\ell,k} u_\ell\,,$$

i.e., with $u = (1,1,1)^{\mathrm{t}}/\sqrt{3}$ and the computed values for c and s

$$\begin{aligned}
Q &= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\frac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} - \frac{\sqrt{3}}{2}\frac{1}{\sqrt{3}}\begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \\
&= \frac{1}{3}\begin{pmatrix} 2 & 2 & -1 \\ -1 & 2 & 2 \\ 2 & -1 & 2 \end{pmatrix}
\end{aligned}$$

[2]The angle would equal $+\pi/3$ if the direction of the axis is reversed ($u \to -u$).

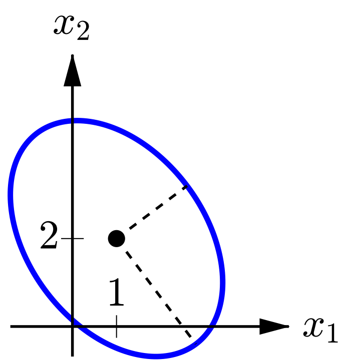
Chapter 9

Conic Sections and Quadrics

9.1 Equation of an Ellipse

Represent the depicted ellipse with midpoint $(1,2)$, directions $(3,-4)^{\mathrm{t}}$ and $(4,3)^{\mathrm{t}}$ of the principal axes with lengths 3 and 2 by an equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0\,.$$



Resources: Euclidean Normal Form of Two-Dimensional Quadrics

Problem Variants

■ focal points: $(0,2)^{\mathrm{t}}$, $(4,2)^{\mathrm{t}}$, length of the larger half axis: 3

$5x^2 + ?xy + ?y^2 - ??x - ??y + ? = 0$:

check

■ midpoint: $(-1,1)^{\mathrm{t}}$, principal directions: $(1,1)^{\mathrm{t}}$, $(-1,1)^{\mathrm{t}}$, lengths: $\sqrt{2}$, 1

$?x^2 - ?xy + ?y^2 + ?x - ?y + 4 = 0$:

check

■ focal points: $(3,0)^{\mathrm{t}}$, $(-3,-2)^{\mathrm{t}}$, point on the ellipse: $(-1,2)^{\mathrm{t}}$

$??x^2 - ?xy + ??y^2 - ?x + ??y - 181 = 0$:

check

Solution

normal form of the ellipse with lengths 3 and 2 of the principal axes:

$$\frac{\tilde{x}^2}{9} + \frac{\tilde{y}^2}{4} = 1$$

where $\tilde{x}, \tilde{y}$ are the coordinates with respect to the coordinate system with origin $p = (1,2)^{\mathrm{t}}$ and the normalized basis vectors (directions of the principal axes)

$$u = \frac{1}{5}\begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad v = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

transformation to coordinates $(x,y)^{\mathrm{t}}$ of the standard coordinate system with origin $(0,0)^{\mathrm{t}}$ and the basis vectors $(1,0)^{\mathrm{t}}$, $(0,1)^{\mathrm{t}}$:

$$\begin{pmatrix} x \\ y \end{pmatrix} = p + Q\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}, \quad Q = (u,v) = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}$$

with a translation p and a rotation matrix Q
substituting

$$\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix} = \underbrace{\frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}}_{Q^{-1}=Q^{\mathrm{t}}} \underbrace{\begin{pmatrix} x-1 \\ y-2 \end{pmatrix}}_{(x,y)^{\mathrm{t}}-p} = \begin{pmatrix} \frac{3}{5}x - \frac{4}{5}y + 1 \\ \frac{4}{5}x + \frac{3}{5}y - 2 \end{pmatrix}$$

into the normal form $\rightsquigarrow$

$$\frac{1}{9}\left(\frac{3}{5}x - \frac{4}{5}y + 1\right)^2 + \frac{1}{4}\left(\frac{4}{5}x + \frac{3}{5}y - 2\right)^2 = 1$$

simplification and multiplication with 180 $\rightsquigarrow$

$$36x^2 + 24xy + 29y2 - 120x - 140y + 20 = 0$$

matrix form of the equation

$$\begin{pmatrix} x & y \end{pmatrix} A \begin{pmatrix} x \\ y \end{pmatrix} + 2b^{\mathrm{t}}\begin{pmatrix} x \\ y \end{pmatrix} + c = 0, \quad A = \begin{pmatrix} 36 & 12 \\ 12 & 29 \end{pmatrix},\ b = \begin{pmatrix} -60 \\ -70 \end{pmatrix},\ c = 20$$

9.2 Implicit Representation of a Parabola

Determine an implicit representation

$$P:\ x^{\mathrm{t}}Ax + 2b^{\mathrm{t}}x + c = 0$$

of the parabola with focal point $F = (1, 0)$ and directrix $g:\ x_1 - 2x_2 = 3$.

Resources: Parabola

Problem Variants

■ $F = (0, 2)$, $g:\ x_1 - 3x_2 = -7$

$A = [?, ?; ?, ?]$, $b = [-?; ?]$, $c = -9$ (MATLAB® notation):

check

■ $F = (-2, -3)$, $g:\ x_1 - x_2 = 5$

$A = [?, ?; ?, ?]$, $b = [?; ?]$, $c = 1$ (MATLAB® notation):

check

■ $F = (2, 3)$, $g:\ x_2 = 4$

$A = [-?, ?; ?, ?]$, $b = [?; -?]$, $c = 3$ (MATLAB® notation):

check

Solution

definition of a parabola with focal point F and directrix g:

$$P:\ |(x_1, x_2) - F| = \operatorname{dist}((x_1, x_2), g)$$

Hesse normal form (normalized normal vector $(1, -2)^{\mathrm{t}} \to (1, -2)^{\mathrm{t}}/\sqrt{5}$) of the straight line $g:\ x_1 - 2x_2 = 3$:

$$g:\ (x_1 - 2x_2)/\sqrt{5} = 3/\sqrt{5}$$

$\rightsquigarrow$ simple computation of the distance:

$$\operatorname{dist}((x_1, x_2), g) = |(x_1 - 2x_2)/\sqrt{5} - 3/\sqrt{5}|$$

equating this expression with the distance $|(x_1 - 1, x_2)^{\mathrm{t}}|$ of (x_1, x_2) to the focal point $(1, 0)$ and squaring $\rightsquigarrow$

$$P:\ (x_1 - 1)^2 + x_2^2 = (x_1 - 2x_2 - 3)^2/5$$

expanding and simplifying $\rightsquigarrow$

$$P:\ x_1^2 + x_1 x_2 + x_2^2/4 - x_1 - 3x_2 - 1 = 0$$

matrix form of this equation

$$\begin{pmatrix} x_1 & x_2 \end{pmatrix} \underbrace{\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/4 \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + 2 \underbrace{\begin{pmatrix} -1/2 & -3/2 \end{pmatrix}}_{b} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{(-1)}_{c} = 0$$

⚠ Note that $a_{1,2} = a_{2,1} = 1/2$, since the terms $x_1 a_{1,2} x_2$ and $x_2 a_{2,1} x_1$ are identical.

9.3 Equation of a Hyperbola

Represent the hyperbola H with focal points $F_1 = (0,6)$, $F_2 = (8,0)$ and $P = (0,0) \in H$ by an equation $q(x_1, x_2) = 0$.

Resources: Hyperbola

Problem Variants

■ focal points: $(0,0)$, $(4,2)$, point on the hyperbola: $(4,3)$

$?x_1x_2 - ?x_2^2 - ?x_1 - ?x_2 + 1 = 0$:

[ ] check

■ midpoint: $(-1,-1)$, direction of the principal axis containing the focal points: $(1,1)^{\mathrm{t}}$, length: $\sqrt{2}$, focal length: $\sqrt{5}$

$x_1^2 + ??x_1x_2 + ?x_2^2 + ??x_1 + ??x_2 = 0$:

[ ] check

■ midpoint: $(2,1)$, direction of the principal axis not containing the focal points: $(-3,1)^{\mathrm{t}}$, corresponding length: $\sqrt{3}$, length of the other half axis: 1

$-?x_1^2 + ??x_1x_2 + ??x_2^2 + ?x_1 - ??x_2 + 10 = 0$:

[ ] check

Solution

Parameter of the hyperbola H

focal points $F_1 = (0,6)$, $F_2 = (8,0)$, $P = (0,0) \in H \quad \rightsquigarrow$

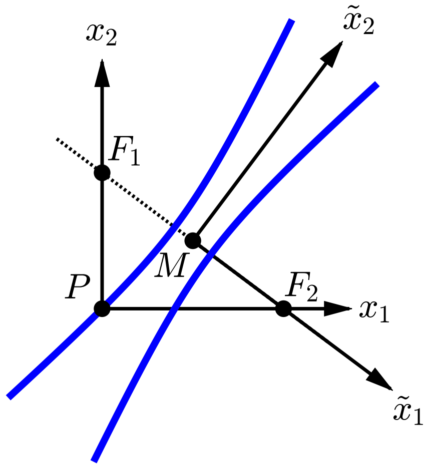
- midpoint: $M = (F_1 + F_2)/2 = (4,3)$
- focal length: $f = |\overrightarrow{F_1F_2}|/2 = |(8,-6)^{\mathrm{t}}|/2 = \sqrt{64+36}/2 = 5$
- lengths of the half axes: $a = ||\overrightarrow{F_1P}| - |\overrightarrow{F_2P}||/2 = |6-8|/2 = 1$, $b = \sqrt{f^2 - a^2} = \sqrt{24}$

Normal form

canonical representation in the coordinate system aligned with the principal axes

normalized directions of the axes:
$\tilde{e}_1 = \overrightarrow{F_1F_2}^{\circ} = (8,-6)^{\mathrm{t}}/10 = (4,-3)^{\mathrm{t}}/5$,
$\tilde{e}_1 \perp \tilde{e}_2 = (3,4)^{\mathrm{t}}/5$
origin $M \quad \rightsquigarrow \quad$ normal form of the hyperbola

$$H:\ 1 = \frac{\tilde{x}_1^2}{a^2} - \frac{\tilde{x}_2^2}{b^2} = \tilde{x}_1^2 - \frac{\tilde{x}_2^2}{24}$$



Transformation of the coordinates ($\tilde{x} \to x$)

formula for the coefficients with respect to an orthonormal basis $\quad \Longrightarrow$

$$\begin{aligned}
\tilde{x}_1 &= (x-M)\tilde{e}_1 = (x_1-4, x_2-3)(4,-3)^{\mathrm{t}}/5 = (4x_1 - 3x_2 - 7)/5 \\
\tilde{x}_2 &= (x-M)\tilde{e}_2 = (x_1-4, x_2-3)(3,4)^{\mathrm{t}}/5 = (3x_1 + 4x_2 - 24)/5
\end{aligned}$$

substitution into the equation for the hyperbola $\quad \rightsquigarrow$

$$H:\ ((4x_1 - 3x_2 - 7)/5)^2 - \frac{((3x_1 + 4x_2 - 24)/5)^2}{24} = 1$$

and, after simplification,

$$H:\ \frac{5}{8}x_1^2 - x_1x_2 + \frac{1}{3}x_2^2 - 2x_1 + 2x_2 = 0$$

9.4 Intersection of a Double Cone with the xy Plane

Determine the implicit representation

$$C: \; ax^2 + bxy + cy^2 + dx + ey + f = 0$$

of the intersection of the double cone with tip $p = (0, 1, 2)$, opening angle $\varphi = \pi/2$, and symmetry axis parallel to $v = (1, 1, 1)^{\mathrm{t}}$ with the xy plane.

Resources: Conic Section, Scalar Product

Problem Variants

■ $p = (1, -1, 2)$, $\varphi = \pi/2$, $v = (2, -1, 1)^{\mathrm{t}}$

$a =?$, $b = -?$, $c = -?$, $d = -??$, $e = -?$, $f = 7$:

check

■ $p = (0, 0, 1)$, $\varphi = \pi/3$, $v = (1, 2, 3)^{\mathrm{t}}$

$a =??$, $b = -?$, $c =??$, $d =??$, $e =??$, $f = 3$:

check

■ $p = (1, 0, 2)$, $\varphi = 2\pi/3$, $v = (2, 1, 0)^{\mathrm{t}}$

$a = -??$, $b = -??$, $c =?$, $d =??$, $e =??$, $f = 9$:

check

Solution

For a point (x,y,z) of the double cone D with tip p, the vector $(x,y,z)^{\mathrm{t}} - p$ forms a constant angle of $\varphi/2$ with the direction vector v, i.e.

$$D: \frac{((x,y,z)-p)(\pm v)}{|(x,y,z)^{\mathrm{t}} - p^{\mathrm{t}}||v|} = \cos(\varphi/2)\,,$$

where the formula $\cos\sphericalangle(u,v) = (u^{\mathrm{t}}v)/(|u|\,|v|)$ was used.
substituting $p = (0,1,2)^{\mathrm{t}}$, $v = (1,1,1)^{\mathrm{t}}$, $\varphi = \pi/2$, and multiplying with the denominator $\quad\rightsquigarrow$

$$\pm \begin{pmatrix} x & y-1 & z-2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \cos(\pi/4)\,|\begin{pmatrix} x & y-1 & z-2 \end{pmatrix}^{\mathrm{t}}|\,\left|\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right|$$

squaring $\quad\rightsquigarrow$

$$(x+y-1+z-2)^2 = \underbrace{\cos^2(\pi/4)}_{1/2}(x^2+(y-1)^2+(z-2)^2)(\sqrt{3})^2$$

setting $z=0$ and simplifying $\quad\rightsquigarrow$

$$C: x^2 - 4xy + y^2 + 12x + 6y - 3 = 0$$

9.5 Focal Points and Principal Axes of an Ellipse

Determine the focal points $F_\pm$ and the lengths of the principal axes of the ellipse

$$E:\ 3x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1 + 6x_2 = 1\,.$$

Resources: Ellipse, Transformation to Normal Form

Problem Variants

■ $E:\ 5x_1^2 + 4x_1x_2 + 2x_2^2 - 10x_1 - 4x_2 = 1$

---

focal points $(?,?)^{\mathrm{t}} \pm (?,-?)^{\mathrm{t}}$:

check

■ $E:\ 2x_1^2 + 2x_1x_2 + 2x_2^2 - 6x_1 - 6x_2 = -3$

---

focal points $(?,?)^{\mathrm{t}} \pm (?,-?)^{\mathrm{t}}$:

check

■ $E:\ 2x_1^2 + 6x_1x_2 + 10x_2^2 = 11$

---

focal points $(?,?)^{\mathrm{t}} \pm (?,-?)^{\mathrm{t}}$:

check

Solution

Transformation to normal Form

matrix form of the equation $3x_1^2 - 2x_1x_2 + 3x_2^2 - 2x_1 + 6x_2 = 1$, representing the ellipse:

$$E: \, x^{\mathrm{t}} \underbrace{\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}}_{A} x + 2 \underbrace{(-1, 3)^{\mathrm{t}}}_{b^{\mathrm{t}}} x = 1$$

eigenvalues λ_k and normalized eigenvectors v_k (directions chosen, so that $\det(v_1, v_2) = 1$) of A

$$\lambda_1 = 2, \, v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \lambda_2 = 4, \, v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

transformation to diagonal form with the rotation $Q = (v_1, v_2)$ and the substitution $x = Qy \quad \rightsquigarrow$

$$\begin{aligned} x^{\mathrm{t}} A x &= y^{\mathrm{t}} Q^{\mathrm{t}} A Q y = y^{\mathrm{t}} \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} y \,, \\ b^{\mathrm{t}} x &= \begin{pmatrix} -1 & 3 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} y = \begin{pmatrix} \sqrt{2} & 2\sqrt{2} \end{pmatrix} y \,, \end{aligned}$$

i.e.

$$E: \, (y^{\mathrm{t}} Q^{\mathrm{t}}) A (Q y) + 2 b^{\mathrm{t}} Q y = 2y_1^2 + 4y_2^2 + 2\sqrt{2} y_1 + 4\sqrt{2} y_2 = 1$$

completing the square $\quad \rightsquigarrow$

$$2(y_1 + \sqrt{2}/2)^2 + 4(y_2 + \sqrt{2}/2)^2 = 4$$

substituting $y = z - (\sqrt{2}/2, \sqrt{2}/2)^{\mathrm{t}}$ and dividing by 4 $\quad \rightsquigarrow \quad$ normal form

$$\frac{z_1^2}{2} + \frac{z_2^2}{1} = 1$$

Lengths of the princial axes and focal points

denominators in the normal form $\quad \rightsquigarrow \quad$ lengths of the principal axes,

$$a = \sqrt{2}, \quad b = \sqrt{1} = 1$$

and transformed focal points

$$(\pm f, 0), \quad f = \sqrt{a^2 - b^2} = 1$$

undoing the transformation $x \to y \to z \quad \rightsquigarrow$

$$x = Qy = Q(z - (\sqrt{2}/2, \sqrt{2}/2)^{\mathrm{t}})$$

substituting $z = (\pm f, 0) \quad \rightsquigarrow$

$$F_\pm = Q\begin{pmatrix} \pm 1 \\ 0 \end{pmatrix} - Q\begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix} = \pm\begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

9.6 Transformation of a Quadric to Normal Form

Determine the type, the principal directions, the lengths of the half axes, and the midpoint of the quadric, represented by the equation

$$x_1^2 + 6x_1x_2 - 7x_2^2 - 4x_1 + 4x_2 = 8\,,$$

and sketch the quadratic curve.

Resources: Transformation to Normal Form, Euclidean Normal Form of Two-Dimensional Quadrics

Problem Variants

■ $-5x_1^2 + 12x_1x_2 - 12x_1 = 1$

---

type (e/p/h): ?:

[ ] check

■ $17x_1^2 - 12x_1x_2 + 8x_2^2 - 22x_1 - 4x_2 = 7$

---

type (e/p/h): ?:

[ ] check

■ $9x_1^2 + 24x_1x_2 + 16x_2^2 + 40x_1 - 30x_2 = 0$

---

type (e/p/h): ?:

[ ] check

Solution

Matrix form of the quadric

$x_1^2 + 6x_1x_2 - 7x_2^2 - 4x_1 + 4x_2 = 8 \quad \Longleftrightarrow$

$$x^{\mathrm{t}} \underbrace{\begin{pmatrix} 1 & 3 \\ 3 & -7 \end{pmatrix}}_{A} x \,+\, 2\underbrace{(-2,2)}_{b^{\mathrm{t}}}\, x = 8$$

⚠ Note that $a_{1,2} = a_{2,1} = 3$, since the two terms $x_1 a_{1,2} x_2 + x_2 a_{2,1} x_1$ combine to $6x_1x_2$.

Diagonalization of A

- eigenvalues λ_k:
  characteristic polynomial:

$$\det(A - \lambda E) = \begin{vmatrix} 1-\lambda & 3 \\ 3 & -7-\lambda \end{vmatrix} = \lambda^2 + 6\lambda - 16$$

  zeros $\quad\rightsquigarrow\quad \lambda_1 = 2,\ \lambda_2 = -8$
  Since $\lambda_1\lambda_2 < 0$, the quadric is a hyperbola.

- eigenvectors:
  eigenvector u to $\lambda_1 = 2$:

$$\begin{pmatrix} 1-2 & 3 \\ 3 & -7-2 \end{pmatrix} u = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad u \parallel \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

  eigenvector v to $\lambda_2 = -8$:

$$\text{symmetry of } A \quad \Longrightarrow \quad v \perp u, \quad \text{i.e. } v \parallel \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

  eigenvectors $\quad\rightsquigarrow\quad$ normalized principal directions

$$u^\circ = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v^\circ = \frac{1}{\sqrt{10}} \begin{pmatrix} -1 \\ 3 \end{pmatrix}$$

  The orientation (signs of the directions) has been chosen, so that the matrix of normalized eigenvectors,

$$Q = (u^\circ, v^\circ) = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix},$$

  is a rotation, i.e. $\det(Q) = 1$.

substitution $x = Qy$ (rotation of the coordinate system) $\rightsquigarrow$ diagonal form

$$y^{\mathrm{t}} \underbrace{Q^{\mathrm{t}}AQ}_{\mathrm{diag}(\lambda_1,\lambda_2)} y + 2b^{\mathrm{t}}Qy = y^t \begin{pmatrix} 2 & 0 \\ 0 & -8 \end{pmatrix} y + \frac{1}{\sqrt{10}}(-8, 16)y$$

$$= \left[2y_1^2 - \frac{8}{\sqrt{10}}y_1\right] - \left\{8y_2^2 + \frac{16}{\sqrt{10}}y_2\right\} = 8$$

<u>Transformation to normal form</u>

elimination of the linear terms by completing the squares:

$$[\ldots] = 2\left(y_1^2 - (4/\sqrt{10})y_1\right) = 2\left(y_1 - 2/\sqrt{10}\right)^2 - 8/10$$

$$\{\ldots\} = 8\left(y_2 - 1/\sqrt{10}\right)^2 - 8/10$$

translating the coordinate system by substituting $z_1 = y_1 - 2/\sqrt{10}$, $z_2 = y_2 - 1/\sqrt{10}$ $\rightsquigarrow$

$$2z_1^2 - \frac{8}{10} - 8z_2^2 + \frac{8}{10} = 8$$

scaling (normalizing the right-hand side to 1) $\rightsquigarrow$

$$\left(\frac{z_1}{2}\right)^2 - \left(\frac{z_2}{1}\right)^2 = 1$$

normal form of a hyperbola with lengths 2 and 1 of the half axes

<u>Midpoint</u>

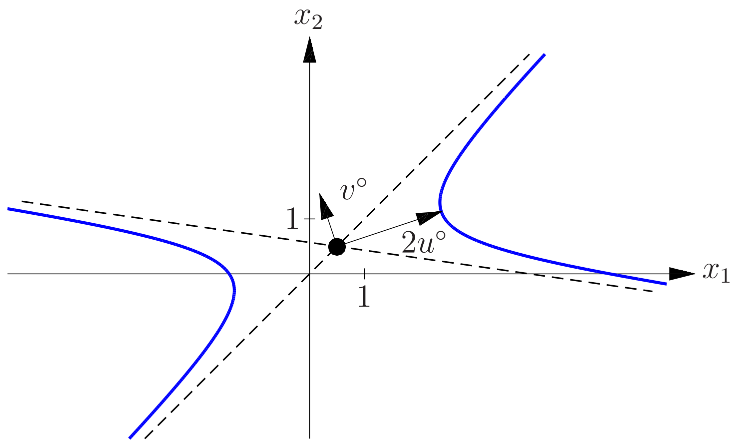
combining the coordinate transformations (rotation and translation) $\rightsquigarrow$

$$x = Qy = Q(z + p), \quad p = (2, 1)^{\mathrm{t}}/\sqrt{10}$$

$\rightsquigarrow$ midpoint of the quadric (image of $z = (0, 0)^{\mathrm{t}}$)

$$m = Qp = \frac{1}{\sqrt{10}} \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$$

Sketch of the quadric



9.7 Principal Axes and Midpoint of a Quadric

Determine the directions of the principal axes, the midpoint, and the type of the quadric

$$Q:\ 6x_1x_2 + 8x_2x_3 + 2x_2 = 1\,.$$

Resources: Transformation to Normal Form, Euclidean Normal Form of Three-Dimensional Quadrics

Problem Variants

■ $Q:\ -x_1^2 + 4x_1x_2 - 8x_1x_3 + 2x_2^2 + 4x_2x_3 - x_3^3 + 2x_1 - 4x_2 + 8x_3 = 8$

---

midpoint: $(?, ?, ?)$:

[ ] check

■ $Q:\ 5x_1^2 - 3x_1x_3 + x_2^2 + 5x_3^3 - 2x_1 + 2x_2 - 2x_3 = -2$

---

midpoint: $(?, -?, ?)$:

[ ] check

■ $Q:\ 2x_1^2 - 2x_1x_3 + x_2^2 + 5x_3^3 + 2x_1 - 2x_2 - 10x_3 = 12$

---

midpoint: $(?, ?, ?)$:

[ ] check

Solution

Transformation to normal form

matrix form of the equation $6x_1x_2 + 8x_2x_3 + 2x_2 = 1$:

$$x^{\mathrm{t}}Ax + 2b^{\mathrm{t}}x = 1, \quad A = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix}, \, b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

eigenvalues λ_k and orthogonal matrix $Q = (v_1, v_2, v_3)$ of normalized eigenvectors (directions of the principal axes) with directions chosen so that $\det Q = 1$

$$\lambda: \; 0, -5, 5, \quad Q = \frac{1}{5}\begin{pmatrix} 4 & -3/\sqrt{2} & 3/\sqrt{2} \\ 0 & 5/\sqrt{2} & 5/\sqrt{2} \\ -3 & -4/\sqrt{2} & 4/\sqrt{2} \end{pmatrix}$$

transformation to diagonal form with the substitution $x = Qy \quad \rightsquigarrow$

$$-5y_2^2 + 5y_3^2 + \sqrt{2}y_2 + \sqrt{2}y_3 = 1$$

completing the squares $\quad \rightsquigarrow$

$$-5(y_2 - 1/(5\sqrt{2}))^2 + 5(y_3 + 1/(5\sqrt{2}))^2 = 1$$

translation $y = z + p$ with $p = (0, 1, -1)^{\mathrm{t}}/(5\sqrt{2}) \quad \rightsquigarrow \quad$ normal form

$$-\frac{z_2^2}{1/5} + \frac{z_3^2}{1/5} = 1$$

Geometric characteristics

- type: hyperbolic cylinder since $\lambda_1 = 0$, $\lambda_2\lambda_3 < 0$
- lengths a_k of the principal axes: denominators in the normal form $\quad \rightsquigarrow$ $a_2 = a_3 = \sqrt{1/5}$
- midpoint: back transformation of the midpoint $z = (0, 0, 0)^{\mathrm{t}}$, corresponding to the normal form $\quad \rightsquigarrow$

$$Q(z+p) = Qp = \frac{1}{5}\begin{pmatrix} 4 & -3/\sqrt{2} & 3/\sqrt{2} \\ 0 & 5/\sqrt{2} & 5/\sqrt{2} \\ -3 & -4/\sqrt{2} & 4/\sqrt{2} \end{pmatrix} \frac{1}{5\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 0 \\ -3 \\ -4 \end{pmatrix}$$

9.8 Equation, Describing a Cylinder

Describe the surface of a (infinite) cylinder C with symmetry axis

$$g:\ (1,1,0)^{\mathrm{t}} + t(0,1,1)^{\mathrm{t}}$$

and radius $r=1$ by an equation $f(x_1,x_2,x_3)=0$.

Resources: Euclidean Normal Form of Three-Dimensional Quadrics

Problem Variants

■ $g:\ (1,2,1)^{\mathrm{t}} + t(2,-1,2)^{\mathrm{t}}$, $r=2$

$f(1,1,1)/f(0,0,0) = -?$:

[ ] check

■ $C:\ \begin{pmatrix}0\\1\\0\end{pmatrix} + s\begin{pmatrix}2\\-1\\2\end{pmatrix} + \cos t\begin{pmatrix}-1\\2\\2\end{pmatrix} + \sin t\begin{pmatrix}2\\2\\-1\end{pmatrix}$, $s\in[0,2\pi)$, $t\in\mathbb{R}$

$f(1,1,1)/f(0,0,0) = ?.??$:

[ ] check

■ surface of an elliptic cylinder with cross section (perpendicular to the symmetry axis) bounded by $S:\ 2\cos t\begin{pmatrix}1\\0\\1\end{pmatrix} + 3\sin t\begin{pmatrix}1\\0\\-1\end{pmatrix}$, $t\in[0,2\pi)$

$f(1,1,1)/f(0,0,0) = ?.??$:

[ ] check

Solution

The points x of the cylinder C have constant distance r from the symmetry axis g:

$$M:\ r = \operatorname{dist}(x, g)\,.$$

applying the formula for the distance of a point x from a straight line $g: t \mapsto p + tu$,

$$\operatorname{dist}(x, g) = \frac{|u \times (x-p)|}{|u|}\,,$$

and substituting the given data $\rightsquigarrow$

$$\begin{aligned} 1 = r \quad &= \quad \frac{|(0,1,1)^{\mathrm{t}} \times (x_1-1, x_2-1, x_3)|}{|(0,1,1)^{\mathrm{t}}|} = \frac{|(x_3-x_2+1, x_1-1, 1-x_1)^{\mathrm{t}}|}{\sqrt{2}} \\ &= \quad \frac{\sqrt{(x_3-x_2+1)^2+2(x_1-1)^2}}{\sqrt{2}} \end{aligned}$$

squaring and simplifying $\rightsquigarrow$

$$f(x_1, x_2, x_3) = 2x_1^2 + x_2^2 - 2x_2x_3 + x_3^2 - 4x_1 - 2x_2 + 2x_3 + 1 = 0$$

9.9 Visualizing Quadrics with MATLAB®

Visualize the quadrics

$$Q_\pm : x^2 + (y/2)^2 = (z/3)^2 \pm 1$$

for $|z| \leq 10$.

Resources: Visualization of Surfaces with MATLAB®

Problem Variants

In the following variants the MATLAB® functions (`surf` or `fimplicit3`) to be used are intentionally prescribed, to make the problem slightly more difficult by requiring to change the representation of the quadric (parametric $\leftrightarrow$ implicit).

- paraboloid: $x^2 + 4y^2 = z$, `surf`
- ellipsoid: $(\vartheta, \varphi) \mapsto (2\sin\vartheta\cos\varphi, 3\sin\vartheta\sin\varphi, 4\cos\vartheta)$, $0 \leq \vartheta \leq \pi$, $0 \leq \varphi \leq 2\pi$, `fimplicit3`
- hyperbolic cylinder: $x^2 - y^2 = 1$, `surf`

Solution

One-sheeted Hyperboloid

$$Q_+ : x^2 + (y/2)^2 = (z/3)^2 + 1$$

representation as parametrized surface $Q_+ : (z, \varphi) \mapsto (x, y, z)$ using cylindrical coordinates

```
[z,phi] = meshgrid([-10:1/2:10],[-pi:pi/20:pi]);
r = sqrt((z/3).^2+1);
x = r.*cos(phi); y = 2*r.*sin(phi);
surf(x,y,z)
```

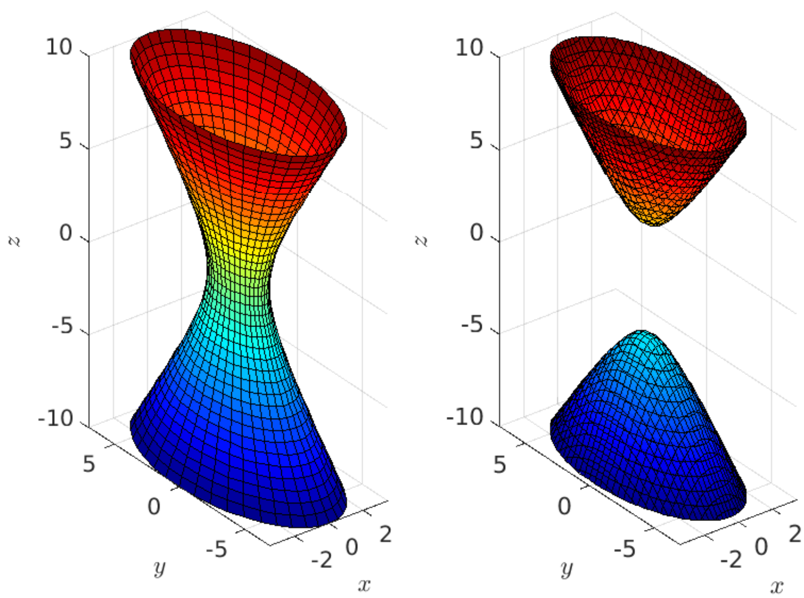
Two-sheeted hyperboloid

$$Q_- : x^2 + (y/2)^2 = (z/3)^2 - 1$$

implicit representation $Q_- : f(x, y, z) = 0$

```
interval = [-10 10 -10 10 -10 10];
f = @(x,y,z) -x.^2-(y/2).^2+(z/3).^2-1;
fimplicit3(f,interval)
```

Parametric and implicit surface plots



9.10 Equation of a Quadric

Determine an implicit representation

$$Q : x^{\mathrm{t}}Ax + 2b^{\mathrm{t}}x + c = 0$$

of the one-sheeted hyperboloid with midpoint $(-1, 1, -1)$, symmetry axis parallel to the vector $(1, 0, 1)^{\mathrm{t}}$, and lengths of the half axes $a_1 = 1$ (corresponding to the symmetry axis), $a_2 = a_3 = 2$.

Resources: Quadric, Euclidean Normal Form of Three-Dimensional Quadrics

Problem Variants

■ double cone, midpoint $(1, 1, 1)$, symmetry axis parallel to $(2, 2, -1)^{\mathrm{t}}$, all lengths of the half axis equal to 1

$?x_1^2 - ??x_1x_2 + ?x_1x_3 + ?x_2^2 + ?x_2x_3 + ?x_3^2 + ?x_1 + ?x_2 - ??x_3 = -9$:

[ ] check

■ elliptic paraboloid, midpoint $(0, 1, 2)$, symmetry axis parallel to $(3, 4, 0)^{\mathrm{t}}$, lengths of the half axes orthogonal to the symmetry axis equal to 2

$??x_1^2 - ??x_1x_2 + ?x_1x_3 + ?x_2^2 + ?x_2x_3 + ??x_3^2 - ??x_1 - ???x_2 - ???x_3 = -269$:

[ ] check

■ two-sheeted hyperboloid, midpoint $(0, 1, 0)$, symmetry axis parallel to $(2, 0, 3)^{\mathrm{t}}$, all lengths of the half axis equal to 1

$-?x_1^2 + ?x_1x_2 - ??x_1x_3 - ??x_2^2 + ?x_2x_3 + ?x_3^2 + ?x_1 + ??x_2 + ?x_3 = 26$:

[ ] check

Solution

Normalized directions v_k of the principal axes

symmetry axis $\parallel (1,0,1)^{\mathrm{t}} \quad \Longrightarrow$

$$v_1 = (1,0,1)^{\mathrm{t}}/\sqrt{2}$$

Since $a_2 = a_3$, the directions v_2 and v_3 can be any unit vectors which extend v_1 to an orthonormal basis; e.g.

$$v_2 = (0,1,0)^{\mathrm{t}}, \quad v_3 = (-1,0,1)^{\mathrm{t}}/\sqrt{2}\,.$$

Derivation of the implicit equation

representation of the one-sheeted hyperboloid in the coordinate system with the midpoint of the quadric, $p = (-1,1,-1)$, as origin and the directions v_k as basis vectors (normal form):

$$-\frac{z_1^2}{a_1^2} + \frac{z_2^2}{a_2^2} + \frac{z_3^2}{a_3^2} = 1, \quad z_k = v_k^{\mathrm{t}}(x - p^{\mathrm{t}}), \quad a_1 = 1,\, a_2 = a_3 = 2$$

The coordinates z_k are scalar products with the orthonormal basis vectors v_k, taking the translation by p, $x \to (x_1+1, x_2-1, x_3+1)^{\mathrm{t}}$, into account.

substituting the concrete values and assembling the normal form with Maple™ :

```
with(LinearAlgebra)
p := Vector([-1,1,-1])
s := sqrt(2)
v[1] := Vector([1,0,1])/s
v[2] := Vector([0,1,0])
v[1] := Vector([-1,0,1])/s
for k from 1 to 3 do
   s[k] := DotProduct(v[k],Vector(3,symbol=x)-p)
od
Q := expand(-s[1]^2+s[2]^2/4+s[3]^2/4=1)
```

multiplying the resulting equation by the common denominator ($=8$) of the summands $\quad \rightsquigarrow$

$$Q:\ -x_1^2 + 2x_2^2 - 3x_3^2 - 10x_1x_3 - 16x_1 - 4x_2 - 16x_3 = 22$$

or, in matrix form,

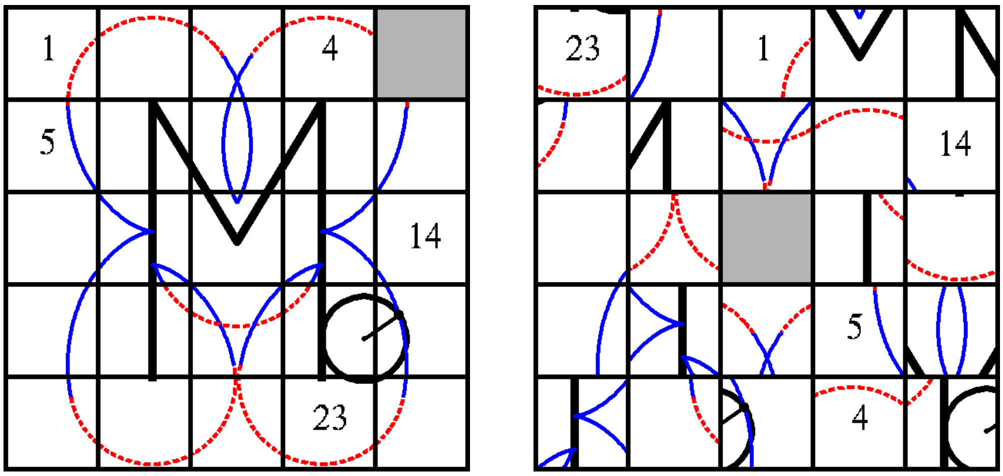
$$Q:\ x^{\mathrm{t}}Ax+2b^{\mathrm{t}}x+c=0 \quad \text{with} A=\begin{pmatrix} -3 & 0 & -5 \\ 0 & 2 & 0 \\ -5 & 0 & -3 \end{pmatrix},\ b=\begin{pmatrix} -8 \\ -2 \\ -8 \end{pmatrix},\ c=-22$$

Chapter 10

Calculus Highlights

10.1 Mathematics-Online Puzzle

The depicted puzzle consists of 24 (square) pieces on 5×5 fields, one of which is left blank. Adjacent pieces can be pushed vertically or horizontally onto the empty field.



In the target configuration (left)[1], the pieces of the puzzle are numbered for better identification, beginning with 1 in the top left corner, proceeding from left to right and from the top to the bottom row. The last piece in the bottom right corner is assigned the number 24. Hence, the start configuration (right) corresponds to the following permutation of the puzzle pieces:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 23 & & 1 & & & & & & & 14 & & & & & & & & 5 & & & & & 4 & \end{pmatrix}$$

Complete the representation of π, and determine the cycles and the sign of this permutation. Identify the puzzle operations (horizontal and vertical shifts) with special permutations and decide whether the puzzle is solvable, i.e., if the target configuration can be reached with a sequence of shift operations from the configuration shown in the right figure[2].

Resources: Permutation

---

[1] Mathematics-Online Logo: https://mo.mathematik.uni-stuttgart.de

[2] This beautiful problem was contributed by Dr. Joachim Wipper.

Solution

Completion of the permutation

numbering of the puzzle pieces in the target configuration, depicted on the left:

$$\text{row}\,1:\ 1-4,\quad \text{row}\,2:\ 5-9,\quad \text{row}\,3:\ 10-14,\quad \ldots$$

identifying the puzzle pieces in the start configuration (right) $\rightsquigarrow$

$$\text{row}\,1:\ 23,9,1,12,6,\quad \text{row}\,2:\ 24,8,17,2,14,\quad \text{row}\,3:\ 10,22,16,21,\quad \ldots$$

resulting permutation:

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\ 23 & 9 & 1 & 12 & 6 & 24 & 8 & 17 & 2 & 14 & 10 & 22 & 16 & 21 & 15 & 11 & 3 & 5 & 7 & 13 & 20 & 19 & 4 & 18 \end{pmatrix}$$

Cycles

$1 \mapsto 23 \mapsto 4 \mapsto 12 \mapsto 22 \mapsto 19 \mapsto 7 \mapsto 8 \mapsto 17 \mapsto 3 \mapsto 1, \quad 2 \mapsto 9 \mapsto 2, \quad \ldots \quad \Longrightarrow$

$$\pi = \underbrace{(1\,23\,4\,12\,22\,19\,7\,8\,17\,3)}_{\text{cycle of length 10}}(2\,9)\,(5\,6\,24\,18)\,(10\,14\,21\,20\,13\,16\,11)\,(15)$$

Sign

applying the formula

$$\sigma(\pi) = \prod_k (-1)^{m_k - 1}$$

with m_k the lengths of the cycles of π $\Longrightarrow$

$$\begin{aligned} \sigma(\pi) &= (-1)^{10-1} \cdot (-1)^{2-1} \cdot (-1)^{4-1} \cdot (-1)^{7-1} \cdot (-1)^{1-1} \\ &= (-1)^9 \cdot (-1)^1 \cdot (-1)^3 \cdot (-1)^6 \cdot (-1)^0 = (-1)^{19} = -1 \end{aligned}$$

Permutations of the shift operations

- horizontal shift:
  The order of the puzzle pieces remains unchanged, i.e. the corresponding permutation is the identity.

- vertical shift:
  cyclic change of 5 positions
  For example, if the piece 17 is pushed down onto the empty field in the start configuration, the corresponding permutation is

$$\ldots 8\,17\,2\,14\,10\,22\,16 \ldots \quad \to \quad \ldots 8\,2\,14\,10\,22\,17\,16 \ldots,$$

  i.e., $17 \mapsto 2 \mapsto 14 \mapsto 10 \mapsto 22 \mapsto 17 \;\widehat{=}\;$ 5-cycle $(17\,2\,14\,10\,22)$.
  Analogously, exchanging the piece 3 with the empty field yields the 5-cycle $(3\,11\,15\,21\,16)$.

Solvability of the puzzle

$\sigma(\tau) = (-1)^{5-1} = 1$ for a 5-cycle τ $\quad\Longrightarrow\quad$ no sign changes for the puzzle operations
$\sigma(\pi) = -1$ for the start configuration and $\sigma(\pi_\star) = 1$ for the target configuration $\pi_\star = (1)\,(2)\,\ldots\,(24)$ $\quad\Longrightarrow\quad$ π cannot be obtained from $\pi_\star$ with the shift operations.

10.2 Tournament Tables and Galois Fields

Design a pairing table for a tournament with 9 teams; each team playing all other teams (72 matches). On each of 4 dates, the teams are split into 3 groups, playing in different cities against each other (6 games in every city).

Resources: Field

Solution

The teams are identified with the points in the finite plane $E = \mathbb{Z}_3 \times \mathbb{Z}_3 = \{0,1,2\} \times \{0,1,2\}$:

$$\{\underbrace{0,1,\ldots,8}_{\text{team numbers}}\} \ni m \quad \underset{m=j+3k}{\longleftrightarrow} \quad (j,k) \in E,\ 0 \le j,k < 3\,.$$

The 3 groups for each date correspond to the points of three parallel straight lines in E.

- date 1 (horizontal lines): The teams with numbers $m \widehat{=} (j,n)$, $j = 0,1,2$, play in city n, i.e. the groups are

$$\{(0,0),(1,0),(2,0)\},\ \{(0,1),(1,1),(2,1)\},\ \{(0,2),(1,2),(2,2)\}\,.$$

- date 2 (vertical lines): (n,k), $k = 0,1,2 \to$ city n
- date 3 (lines with slope $\mathbf{1}$): $(j, \mathbf{1}\cdot j + n \bmod 3)$, $j = 0,1,2$
- date 4 (lines with slope $\mathbf{2}$): $(j, \mathbf{2}\cdot j + n \bmod 3)$, $j = 0,1,2$

Since parallel lines do not intersect and two non-parallel lines do have a unique point in common (arithmetic modulo 3 in the field $\mathbb{Z}_3$), regardles of the date, two groups can have at most one team in common. This ensures that all 72 games are played; there is no duplication of pairings[3].

Converting the points $(j,k) \in E$ to the team numbers $m \in \{0,1,\ldots,8\}$ yields the following pairing table:

	date 1	date 2	date 3	date 4
city 0	0 1 2	0 3 6	0 4 8	0 7 5
city 1	3 4 5	1 4 7	3 7 2	3 1 8
city 2	6 7 8	2 5 8	6 1 5	6 4 2

For example, the team numbers 6 1 5 for city 2 and date 3 are given by

$$m = j + 3(\underbrace{\mathbf{1}\cdot j + 2}_{k} \bmod 3), \quad j = 0,1,2\,.$$

[3]cf. K. Höllig: *Approximationszahlen von Sobolev-Einbettungen*, Dissertation, Bonn 1979, Lemma 2, for the following more general problem: Construct subsets A_i of $\{1,\ldots,n\}$ with m elements and $|A_j \cap A_k| < \ell\ \forall j,k$ ($n = 9$, $m = 3$, $\ell = 2$ in the problem).

10.3 Identities in Modulo Arithmetic

Prove the identities

a) $z^p = z \bmod p$ b)[4] $(x+y)^p = x^p + y^p \bmod p$

for any prime number p and $x, y, z \in \mathbb{N}_0$.

Resources: Field

[4] Of course, this identity, sometimes referred to as *Freshman's Dream*, is false in general (p not a prime number, standard arithmetic).

Solution

a) $z^p = z \bmod p$

proof by induction:

- induction start
  $z = 0$: $0^p = 0$ ✓, $z = 1$: $1^p = 1$ ✓
- induction step $z \to z + 1$
  binomial formula and induction hypothesis ($z^p = z \bmod p$) $\Longrightarrow$

  $$(z+1)^p = z^p + \left[\sum_{k=1}^{p-1} \binom{p}{k} z^{p-k}\, 1^k\right] + 1^p = z + [\ldots] + 1 \bmod p$$

  $[\ldots] = 0 \bmod p$, since

  $$\mathbb{N} \ni \binom{p}{k} = \frac{p \cdot (p-1) \cdots 1}{k \cdot (k-1) \cdots 1}$$

  can be divided by p (Since $k < p$, the prime factor decomposition of the denominator contains no factor p), and, therefore, $\binom{p}{k} = 0 \bmod p$

b) $(x + y)^p = x^p + y^p \bmod p$

proof by applying twice the identity a):

$$(x+y)^p \underset{z=x+y}{=} x + y \bmod p \underset{z=x,z=y}{=} x^p + y^p \bmod p$$

Alternative Solution
binomial formula $\Longrightarrow$

$$(x+y)^p = x^p + \left[\sum_{k=1}^{p-1} \binom{p}{k} x^{p-k}\, y^k\right] + y^p$$

and $[\ldots] = 0 \bmod p$, analogously to a), since $\binom{p}{k} = 0 \bmod p$ $\Longrightarrow$ $\binom{p}{k} q = 0 \bmod p$ for each factor q

10.4 Solution of Three Congruences

Determine the smallest solution $x \in \mathbb{N}_0$ of

$$x = 1 \bmod 2, \quad x = 4 \bmod 5, \quad x = 3 \bmod 7\,.$$

Resources: Field, Chinese Remainder Theorem

Solution

$2, 5, 7$ pairwise prime, Chinese remainder theorem $\implies$
$\exists!\, x \in \{0, \dots 2 \cdot 5 \cdot 7 - 1\}$ with the representation

$$x = \left(1 \cdot \frac{70}{2}\left(\frac{1}{35} \bmod 2\right) + 4 \cdot \frac{70}{5}\left(\frac{1}{14} \bmod 5\right) + 3 \cdot \frac{70}{7}\left(\frac{1}{10} \bmod 7\right)\right) \bmod 70$$

computation of the reciprocals

$$\begin{aligned} 1/35 &= 1/1 = 1 \mod 2 \\ 1/14 &= 1/4 = 4 \mod 5 \\ 1/10 &= 1/3 = 5 \mod 7 \end{aligned}$$

($1 \cdot 1 = 1 \bmod 2$, $4 \cdot 4 = 1 \bmod 5$, $3 \cdot 5 = 1 \bmod 7$)

inserting into the formula for x $\rightsquigarrow$

$$\begin{aligned} x &= 1 \cdot 35 \cdot 1 + 4 \cdot 14 \cdot 4 + 3 \cdot 10 \cdot 5 \\ &= 35 + 224 + 150 = 35 + 14 + 10 = 59 \mod 70 \end{aligned}$$

<u>Alternative solution</u>

solving the congruences

$$x = 1 + 2r, \quad x = 4 + 5s, \quad x = 3 + 7t$$

with $r, s, t \in \mathbb{Z}$ by successive substitution

equating the first two congruences $\rightsquigarrow$

$$2r = (4-1) + 5s \quad \iff \quad r = \frac{3}{2} + \frac{5s}{2} \underset{(\star)}{=} \left[\frac{3}{2} + \frac{5}{2}\right] + 5\,\frac{s-1}{2} = 4 + 5u$$

with $u \in \mathbb{Z}$, since $5(s-1)/2 \in \mathbb{Z}$ $\implies$ $2\,\text{divides}\,(s-1)$ (prime factor decomposition)

$(\star)$ The fraction $5/2$ has been added and subtracted, so that $[\dots] \in \mathbb{Z}$.

substituting the expression for r into the first congruence and comparing the resulting congruence with the third congruence $\rightsquigarrow$

$$1 + 2(4 + 5u) = 3 + 7t \quad \iff \quad u = \frac{-6}{10} + \frac{7t}{10} = \left[\frac{-6}{10} + \frac{56}{10}\right] + 7\,\frac{t-8}{10} = 5 + 7v$$

with $v \in \mathbb{Z}$ since $u \in \mathbb{Z}$

substituting into the first congruence $\rightsquigarrow$ general solution

$$x = 1 + 2(4 + 5(5 + 7v)) = 59 + 70v, \quad v \in \mathbb{Z}$$

10.5 Gram-Schmidt Orthogonalization with Maple™

Construct an orthonormal basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors

$$u_j : u_{j,k} = k^{j-1},\ k = 1, \ldots, 4, \quad j = 1, 2, 3\,.$$

Moreover, determine the matrices of the projections onto V and onto the orthogonal complement $V^\perp$.

Resources: Matrix Operations with Maple™

Solution

```
with(LinearAlgebra):   # loading relevant functions
U := Matrix(3,4,(j,k)->k^(j-1)):   # Matrix of row vectors
```

Orthogonalization with the Gram-Schmidt method

```
# Orthogonalization with normalization
V := GramSchmidt([U[1,1..4],U[2,1..4],U[3,1..4]],normalized);
```

$$V := \left[\left[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right],\ \left[-\frac{3\sqrt{5}}{10},-\frac{\sqrt{5}}{10},\frac{\sqrt{5}}{10},\frac{3\sqrt{5}}{10}\right],\ \left[\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right]\right]$$

Projection matrix P for V

v_j (= `V[j]`) orthonormal (row vectors) $\quad\Longrightarrow\quad P = \sum_j v_j^{\mathrm{t}} v_j$

```
for j from 1 to 3 do
   p[j] := MatrixMatrixMultiply(Transpose(V[j]),V[j]):
end do
P := add(p[j],j=1..3):
```

$$P := \frac{1}{20}\begin{bmatrix} 19 & 3 & -3 & 1 \\ 3 & 11 & 9 & -3 \\ -3 & 9 & 11 & 3 \\ 1 & -3 & 3 & 19 \end{bmatrix}$$

Projection matrix Q for the orthogonal complement W

```
W := NullSpace(P):   # dimension 1, spanned by W[1]
w := ScalarMultiply(W[1],1/Norm(W[1],2)):   # column vector
Q := MatrixMatrixMultiply(w,Transpose(w));
```

$$Q := \frac{1}{20}\begin{bmatrix} 1 & -3 & 3 & -1 \\ -3 & 9 & -9 & 3 \\ 3 & -9 & 9 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix}$$

To check the result compute `E := P+Q` which should be the unit matrix.

10.6 Approximation by Polynomials with MATLAB®

Approximate the exponential function $f(x) = \mathrm{e}^x$ on the interval $[0, 1]$ with a polynomial p of degree ≤ 4 by minimizing the error $e = \left(\int_0^1 (f(x) - p(x))^2 \,\mathrm{d}x\right)^{1/2}$. Compare the accuracy with the Taylor approximation.

Resources: Orthogonal Projection, Matrix Operations with MATLAB®

Solution

Characterization of the best approximation

A polynomial $p = \sum_{k=1}^{n} c_k x^{k-1}$ is the best approximation to a function f with respect to the scalar product norm $\langle g, h\rangle = \int_0^1 g(x)h(x)\,\mathrm{d}x$ if $\langle p_j, f-p\rangle = 0$ with the monomials $p_j(x) = x^{j-1}$, $j = 1,\ldots,n$ (basis functions $p_j \perp$ error $f-p$), i.e.

$$\sum_{k=1}^{n} \underbrace{\int_0^1 x^{j-1}x^{k-1}\,\mathrm{d}x}_{a_{j,k}}\; c_k = \underbrace{\int_0^1 x^{j-1}f(x)\,\mathrm{d}x}_{b_j},\; j = 1,\ldots,n \quad \Longleftrightarrow \quad Ac = b\,.$$

Assembly of the linear system

- Gram matrix (Hilbert matrix for the scalar product of monomials):

$$a_{j,k} = \int_0^1 x^{j-1}x^{k-1}\,\mathrm{d}x = \left[\frac{1}{j+k-1}x^{j+k-1}\right]_{x=0}^{x=1} = \frac{1}{j+k-1}$$

- right-hand side: recursive computation with integration by parts $\leadsto$

$$\begin{aligned} b_1 &= \int_0^1 \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}-1 \\ b_{j+1} &= \int_0^1 x^j\mathrm{e}^x\,\mathrm{d}x = \left[x^j\mathrm{e}^x\right]_{x=0}^{x=1} - \int_0^1 jx^{j-1}\mathrm{e}^x\,\mathrm{d}x = \mathrm{e} - jb_j \end{aligned}$$

Computation of the error $|f-p|$

substituting $p = \sum_k c_k x^{k-1}$ $\leadsto$

$$\begin{aligned} |f-p|^2 &= \langle f-p, f-p\rangle = \langle f,f\rangle - 2\langle p,f\rangle + \langle p,p\rangle = \\ &= \int_0^1 \mathrm{e}^{2x}\,\mathrm{d}x - 2\sum_{k=1}^{n} c_k \underbrace{\int_0^1 x^{k-1}\mathrm{e}^x\,\mathrm{d}x}_{b_k} + \sum_{j,k=1}^{n} c_j \underbrace{\int_0^1 x^{j-1}x^{k-1}\,\mathrm{d}x}_{a_{j,k}}\; c_k \\ &= \frac{\mathrm{e}^2-1}{2} - 2c^{\mathrm{t}}b + c^{\mathrm{t}}Ac \end{aligned}$$

Matlab® script

```
% recursive computation of the scalar products with x^j
b = [exp(1)-1; 1];
for j=2:4;
    b(j+1) = exp(1)-j*b(j);
end

% coefficients of the poynomial approximation
A = hilb(5);   % Gram matrix A, a_j,k = 1/(j+k-1)

% solution of the linear system
c = A\b
c =
   1.0001 0.9984 0.5106 0.1397 0.0695

% computation of the error sqrt<f-p,f-p>
error_p = sqrt((exp(2)-1)/2 - 2*c'*b + c'*A*c)
error_p =
   1.6612e-05

% comparison with the Taylor polynomial
c_taylor = [1; 1; 1/2; 1/6; 1/24];   % coefficients 1/j!
c_taylor =
   1.0000 1.0000 0.5000 0.1667 0.0417
error_taylor = sqrt((exp(2)-1)/2 - ...
   2*c_taylor'*b + c_taylor'*A*c_taylor)
error_taylor =
   0.0030
```

The reason for the significantly larger error of the Taylor approximation is the loss of accuracy for a large distance from the expansion point. As a consequence, the error is not equally distributed over the interval $[0,1]$.

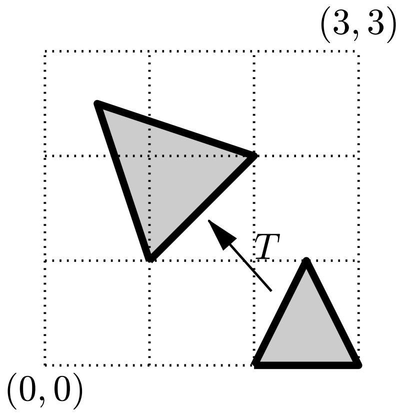
10.7 Affine Maps in Homogeneous Coordinates

Algorithms in computer graphics use homogeneous coordinates,

$$(x_1,\, x_2)^{\mathrm{t}} = (u_1/\lambda,\, u_2/\lambda)^{\mathrm{t}} \,\widehat{=}\, (u|\lambda)^{\mathrm{t}}\,,$$

since this allows to represent affine maps $x \mapsto Ax + b$ as matrix multiplications:

$$\begin{pmatrix} u \\ \hline \lambda \end{pmatrix} \mapsto \underbrace{\left(\begin{array}{c|c} A & b \\ \hline (0,\, 0) & 1 \end{array}\right)}_{T} \begin{pmatrix} u \\ \hline \lambda \end{pmatrix}\,.$$



Determine the transformation matrix T for the depicted transformation of a triangle.

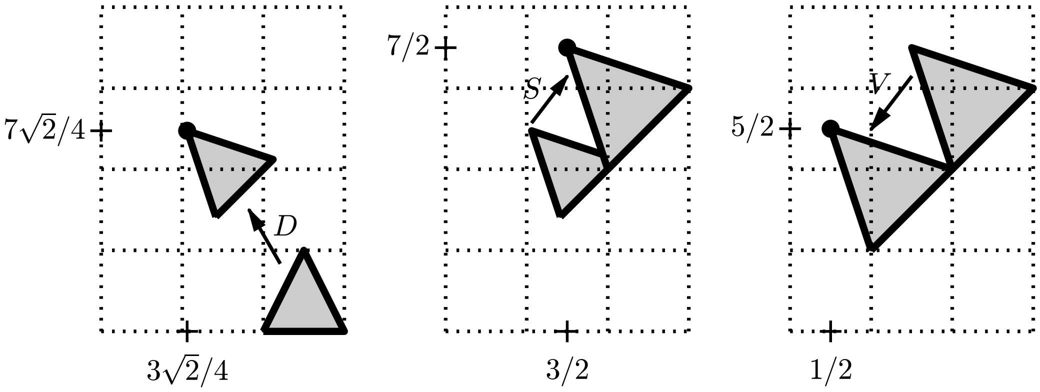
Resources: Affine Map, Matrix Multiplication

Solution

Representation of the transformation matrix in terms of elementary maps

representation of T as composition of a rotation, a scaling, and a translation:

$$T = VSD$$



(marked points: images of the vertex $(5/2, 1)$)

Rotation by $\pi/4$

$(1,\, 0)^{\mathrm{t}} \mapsto (1/\sqrt{2},\, 1/\sqrt{2})^{\mathrm{t}}$, $(0,\, 1)^{\mathrm{t}} \mapsto (-1/\sqrt{2},\, 1/\sqrt{2})^{\mathrm{t}}$, i.e.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto Dx = \underbrace{\begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}}_{A} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{b}$$

$\rightsquigarrow$ matrix

$$D = \left(\begin{array}{c|c} A & b \\ \hline (0,\, 0) & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Scaling by $\sqrt{2}$

$$x \mapsto Sx = \sqrt{2}\, x \quad \Longleftrightarrow \quad A = \sqrt{2}\, E,\ b = (0,\, 0)^{\mathrm{t}}$$

with E the 2×2 unit matrix

$\rightsquigarrow$ matrix

$$S = \left(\begin{array}{cc|c} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Translation by $(-1,\,-1)^{\mathrm{t}}$

$$x \mapsto Vx = x + (-1,\,-1)^{\mathrm{t}} \quad \Longleftrightarrow \quad A = E,\, b = (-1,\,-1)^{\mathrm{t}}$$

$\rightsquigarrow$ matrix

$$V = \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right)$$

Multiplication of the matrices

$\rightsquigarrow$ transformation matrix

$$\begin{aligned} T = VSD &= \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ \hline 0 & 0 & 1 \end{array}\right) \\ &= \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right) \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{cc|c} 1 & -1 & -1 \\ 1 & 1 & -1 \\ \hline 0 & 0 & 1 \end{array}\right) \end{aligned}$$

Alternative solution

determination of the transformation matrix using the images of three points, e.g. the vertices of the triangle:

$$\begin{array}{ccccccc} (2,\,0)^{\mathrm{t}} & \widehat{=} & (2,\,0|1)^{\mathrm{t}} & \mapsto & (1,\,1|1)^{\mathrm{t}} & \widehat{=} & (1,\,1)^{\mathrm{t}} \\ (3,\,0)^{\mathrm{t}} & \widehat{=} & (3,\,0|1)^{\mathrm{t}} & \mapsto & (2,\,2|1)^{\mathrm{t}} & \widehat{=} & (2,\,2)^{\mathrm{t}} \\ (5/2,\,1)^{\mathrm{t}} & \widehat{=} & (5/2,\,1|1)^{\mathrm{t}} & \mapsto & (1/2,\,5/2|1)^{\mathrm{t}} & \widehat{=} & (1/2,\,5/2)^{\mathrm{t}} \end{array}$$

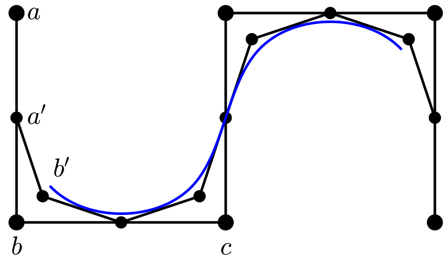
simultaneous multiplication with the transformation matrix T $\rightsquigarrow$

$$\left(\begin{array}{cc|c} 1 & 2 & 1/2 \\ 1 & 2 & 5/2 \\ \hline 1 & 1 & 1 \end{array}\right) = \underbrace{\left(\begin{array}{cc|c} a_{1,1} & a_{1,2} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \\ \hline 0 & 0 & 1 \end{array}\right)}_{T} \underbrace{\left(\begin{array}{cc|c} 2 & 3 & 5/2 \\ 0 & 0 & 1 \\ \hline 1 & 1 & 1 \end{array}\right)}_{P}$$

solution of this matrix equation by inverting P or via Gauß elimination after transposition

10.8 Cubic Spline Curves with MATLAB®

Starting with a control polygon $a, b, c, \ldots$, a cubic spline curve can be plotted by replacing the polygon by a more accurate polygon $a', b', \ldots$, and iterate this procedure until the distance of adjacent points is less than a given tolerance.



$$a' = \frac{a+b}{2},\; b' = \frac{a+6b+c}{8}$$

Write a MATLAB® Script, which plots the cubic spline curve associated with the depicted control polygon ($a = (0,1)$, $b = (0,0)$, $c = (1,0)$, $\ldots$), using this subdivision algorithm[5]. Give several other examples.

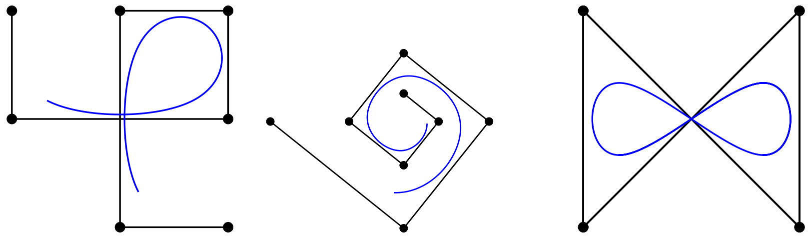
Resources: Matrix Operations with MATLAB®

[5]Spline curves of arbitrary degree can be plotted in a similar fashion; cf. K. Höllig, J. Hörner: *Approximation and Modeling with B-Splines*, SIAM, Other Titles in Applied Mathematics 132, 2013, chapter 6

Solution

```
P = [0 1; 0 0; 1 0; 1 1; 2 1; 2 0]; tol = 0.02;
while max(max(abs(diff(P)))) > tol
   % lengths of the old and new control polygon
   n = size(P,1); N = 2*n-3;
   % new points with odd and even indices
   Q(1:2:N,:) = (P(1:n-1,:)+P(2:n,:))/2;
   Q(2:2:N-1,:) = (P(1:n-2,:)+6*P(2:n-1,:)+P(3:n,:))/8;
   P = Q;
end
plot(P(:,1),P(:,2),'-b');
```

Further examples:



periodic control polygon

10.9 Algorithm of De Casteljau with Maple™

Write a Maple™ procedure which evaluates a Bézier curve[6]

$$t \mapsto p(C|t) = \sum_{k=0}^{n} \binom{n}{k} (1-t)^{n-k} t^k \, c_k, \quad C = \begin{pmatrix} c_0 \\ \vdots \\ c_n \end{pmatrix}$$

with the recursion

$$p(c_0, \ldots, c_n|t) = (1-t)p(c_0, \ldots, c_{n-1}|t) + tp(c_1, \ldots, c_n|t) \, .$$

Illustrate the algorithm by modeling the letter „B“ with three Bézier curves.

Resources: Matrix Operations with Maple

[6]Bézier and B-Spline curves have become the standard in Computer Aided Design in view of their very intuitive properties.

Solution

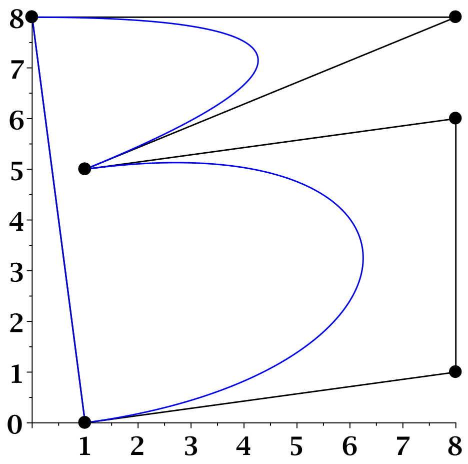
Recursive Maple™ procedure

```
with(LinearAlgebra):
p := proc(C,t)
  local n:
  # C: matrix with control points c_k as rows
  # t: curve parameter
  # n: polynomial degree = number of control points -1
  # return value: point p(C|t) on the curve

  n := RowDimension(C)-1:
  if n=0 then
    return C
    # p(C|t) = C for a single control point (degree = 0)
  else
    # degree of p > 0 -> recursive call of the procedure
    # without the first and without the second control point
    # deletion of one row of C in both cases
    return (1-t)*bezier(DeleteRow(C,1),t)
       + t*bezier(DeleteRow(C,n+1),t):
  end if
end proc
```

Plotting of Bézier curves

```
with(LinearAlgebra): with(plots):
# matrices of control points for curves of degree n=1,2,3
C1:=<0,1|8,0>: C2:=<0,8,1|8,8,5>: C3:=<1,8,8,1|5,6,1,0>

# plot of the control points and the control polygon
Cpoints := plot({C1,C2,C3},color=black,symbol=solidcircle)
Cpolygons := plot({C1,C2,C3},color=black)

# plot of the Bezier curves (parameter interval [0,1])
curves := plot({
     [p(C1,t)[1,1],p(C1,t)[1,2],t=0..1],
     [p(C2,t)[1,1],p(C2,t)[1,2],t=0..1],
     [p(C3,t)[1,1],p(C3,t)[1,2],t=0..1]
```

```
    },color=blue)

# merging the three plots
display(Cpoints,Cpolygons,curves)
```



Remark The input of the Maple™ function `plot` is a list $\{d1, d2, \ldots\}$ of plot data d of the same type:

- matrices with rows containing the vertices of the polygons in the first two calls;
- two functions, which compute the two coordinates of the points on the curves, in the third call. Since `p` returns the point $p(t|C)$ on the curve as 1×2 matrix, the Maple™ procedure has to be split into its two components `p(C,t)[1,1]` and `p(C,t)[1,2]`.

10.10 Determinant of a 5 × 5 Matrix

Compute

$$\begin{vmatrix} 99 & 0 & 95 & 10 & 0 \\ 98 & 1 & 95 & 10 & 1 \\ 97 & 0 & 97 & 10 & -1 \\ 96 & 1 & 98 & 10 & 1 \\ 95 & 0 & 99 & 10 & 0 \end{vmatrix}.$$

Resources: Determinant, Properties of Determinants, Expansion of Determinants

Solution

manipulating determinants with Gauß transformations and expansion rules:

- no change if a multiple of a row/column is added to another row/column
- change of sign if rows/columns are interchanged
- linear dependence on rows/columns

application to the given matrix $A \quad \rightsquigarrow$

$$\det A \overset{(1)}{=} 10 \cdot \begin{vmatrix} 99 & 0 & 95 & 1 & 0 \\ 98 & 1 & 95 & 1 & 1 \\ 97 & 0 & 97 & 1 & -1 \\ 96 & 1 & 98 & 1 & 1 \\ 95 & 0 & 99 & 1 & 0 \end{vmatrix} \overset{(2)}{=} 10 \cdot \begin{vmatrix} 2 & 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & 1 & 1 \\ -2 & 0 & 2 & 1 & 0 \end{vmatrix}$$

(1) division of column 4 by 10

(2) subtraction of column 4 $\times$ 97 from columns 1 and 3

subtracting column 2 from column 5 and expanding with respect to column 5 $\quad \rightsquigarrow$

$$10 \cdot \begin{vmatrix} 2 & 0 & -2 & 1 & 0 \\ 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & 1 & 1 & 1 & 0 \\ -2 & 0 & 2 & 1 & 0 \end{vmatrix} = 10 \cdot (-1)^{3+5} \cdot (-1) \cdot \begin{vmatrix} 2 & 0 & -2 & 1 \\ 1 & 1 & -2 & 1 \\ -1 & 1 & 1 & 1 \\ -2 & 0 & 2 & 1 \end{vmatrix}$$

adding the last to the first row and expanding with respect to the first row $\rightsquigarrow$

$$-10 \cdot \begin{vmatrix} 0 & 0 & 0 & 2 \\ 1 & 1 & -2 & 1 \\ -1 & 1 & 1 & 1 \\ -2 & 0 & 2 & 1 \end{vmatrix} = -10 \cdot (-1)^{1+4} \cdot 2 \cdot \begin{vmatrix} 1 & 1 & -2 \\ -1 & 1 & 1 \\ -2 & 0 & 2 \end{vmatrix}$$

Sarrus rule $\quad \rightsquigarrow$

$$\det A = 20\,(2 + (-2) + 0 - 4 - 0 - (-2)) = -40$$

10.11 Recursion for the Determinant of a Tridiagonal Matrix

Derive a recursion for the determinant d_n of the tridiagonal $n \times n$ matrix T with

$$t_{k,k-1} = a, \quad t_{k,k} = b, \quad t_{k,k+1} = c\,.$$

Which values a, b, c yield the Fibonacci numbers?

Resources: Determinant, Expansion of Determinants

Solution

First three determinants

$$d_1 = b, \quad d_2 = \begin{vmatrix} b & c \\ a & b \end{vmatrix} = b^2 - ac, \quad \begin{vmatrix} b & c & 0 \\ a & b & c \\ 0 & a & b \end{vmatrix} = b^3 - bca - cab = b(b^2 - 2ac)$$

Illustration of the recursion for $n = 5$

expanding with respect to the first row $\rightsquigarrow$

$$d_5 = \begin{vmatrix} b & c & 0 & 0 & 0 \\ a & b & c & 0 & 0 \\ 0 & a & b & c & 0 \\ 0 & 0 & a & b & c \\ 0 & 0 & 0 & a & b \end{vmatrix} = b \underbrace{\begin{vmatrix} b & c & 0 & 0 \\ a & b & c & 0 \\ 0 & a & b & c \\ 0 & 0 & a & b \end{vmatrix}}_{d_4} - c \begin{vmatrix} a & c & 0 & 0 \\ 0 & b & c & 0 \\ 0 & a & b & c \\ 0 & 0 & a & b \end{vmatrix}$$

expanding the last determinant with respect to the first column $\rightsquigarrow$

$$d_5 = bd_4 - ca \underbrace{\begin{vmatrix} b & c & 0 \\ a & b & c \\ 0 & a & b \end{vmatrix}}_{d_3} = bd_4 - acd_3$$

proceeding analogously in general $\rightsquigarrow$ recursion

$$d_n = bd_{n-1} - acd_{n-2}$$

Fibonacci numbers

$b = 1$ and $ac = -1$, e.g. $a = 1$, $c = -1$ $\rightsquigarrow$ $d_n = d_{n-1} + d_{n-2}$ with the sequence

$$d_1 = 1,\ d_2 = 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ \ldots$$

10.12 Determinant of an $n \times n$ Matrix

Compute the determinant of the matrix A_n with $a_{k,k} = 2$ and $a_{j,k} = 1$ for $j \neq k$.

Resources: Determinant, Properties of Determinants

Solution

Small n

$$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = 3, \qquad \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} \underset{\text{Sarrus}}{=} 8 + 1 + 1 - 2 - 2 - 2 = 4$$

$|A_4| = 5$, $|A_5| = 6$, ... $\rightsquigarrow$ conjecture $|A_n| = n + 1$

General n

The kth column is the sum of a vector u of n ones and the kth unit vector e_k (diagonal element $1 + 1 = 2$), i.e.

$$|A_n| = \det(u + e_1, u + e_2, \ldots, u + e_n)$$

multilinearity of the determinant $\rightsquigarrow$ sum of 2^n determinants (analogously to expanding expressions with brackets)

$$\begin{aligned} |A_n| &= \det(e_1, \ldots, e_n) + [\det(u, e_2 \ldots, e_n) + \cdots + \det(e_1, \ldots, e_{n-1}, u)] \\ &\quad + \{\text{determinants with at least 2 columns } = u\} \end{aligned}$$

- $\{\ldots\} = 0$ in view of the antisymmetry of determinants
- $u = e_1 + \ldots + e_n \implies$

  $$\begin{aligned} \det(e_1, \ldots, u, \ldots, e_n) &= \det(e_1, \ldots, \underbrace{\sum_\ell e_\ell}_{\text{column } k}, \ldots, e_n) \\ &= \det(e_1, \ldots, e_k, \ldots, e_n) \end{aligned}$$

  again in view of the antisymmetry
- $\det(e_1, \ldots, e_n) = 1$

combining these observations $\rightsquigarrow$

$$|A_n| = 1 + [1 + \cdots + 1] = 1 + n$$

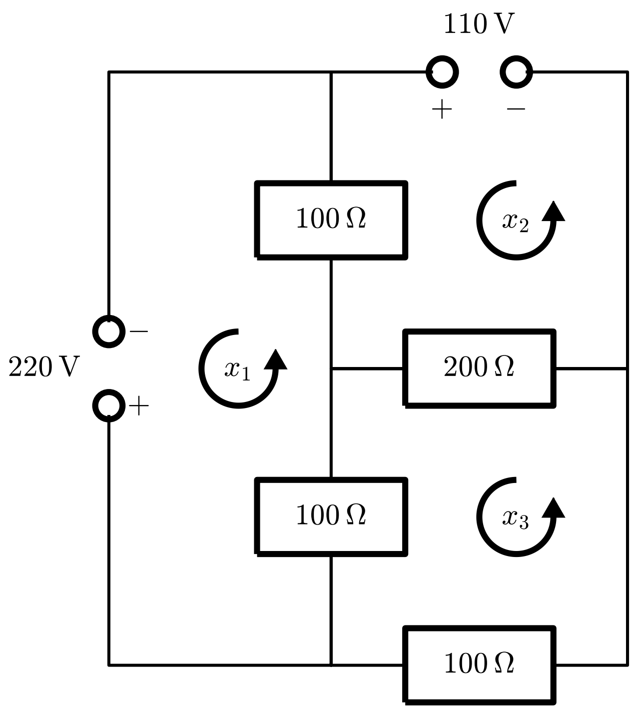
in agreement with the conjecture

10.13 Electrical Circuit

The laws of Ohm and Kirchhoff imply

$$R_j x_j + \sum_{k \neq j} R_{j,k}(x_j - x_k) = U_j \,,$$

where R_j is the resistance of a resistor belonging only to the j-th loop, $R_{j,k}$ is the resistance of a resistor shared by the j-th and k-th loop, x_j and $(x_j - x_k)$ are the currents in these resistors, and U_j is the voltage applied to the j-th loop ($U_j = 0$ if the j-th loop contains no voltage source).



Compute the maximal current for the given data.

Resources: Linear System, Gauß Elimination

Solution

Assembling the linear system

$R_1 = 0$ (no resistor belonging only to loop 1), $R_{1,2} = 100$, $R_{1,3} = 100$, $U_1 = 220 \quad \Longrightarrow$

$$100(x_1 - x_2) + 100(x_1 - x_3) = 220 \quad \longleftrightarrow \quad 200x_1 - 100x_2 - 100x_3 = 220$$

analogously: $R_2 = 0$, $R_{1,2} = 100$, $R_{2,3} = 200$, $U_2 = 110$ and $R_3 = 100$, $R_{1,3} = 100$, $R_{2,3} = 200$, $U_3 = 0 \quad \Longrightarrow$

$$\begin{aligned} 100(x_2 - x_1) + 200(x_2 - x_3) &= 110 \\ 100x_3 + 100(x_3 - x_1) + 200(x_3 - x_2) &= 0 \end{aligned}$$

$\rightsquigarrow$ matrix A and right-hand side b of the linear system $Ax = b$:

$$\left(\; A \mid b \; \right) = \left(\begin{array}{ccc|c} 200 & -100 & -100 & 220 \\ -100 & 300 & -200 & 110 \\ -100 & -200 & 400 & 0 \end{array} \right) \qquad \begin{array}{c} \text{(r1)} \\ \text{(r2)} \\ \text{(r3)} \end{array}$$

Gauß elimination

row operations (1): r2 $\leftarrow 2 \cdot$ r2 + r1, r3 $\leftarrow 2 \cdot$ r3 + r1 and (2): r3 $\leftarrow$ r2 + r3

$$\underset{(1)}{\rightarrow} \left(\begin{array}{ccc|c} 200 & -100 & -100 & 220 \\ 0 & 500 & -500 & 440 \\ 0 & -500 & 700 & 220 \end{array} \right) \underset{(2)}{\rightarrow} \left(\begin{array}{ccc|c} 200 & -100 & -100 & 220 \\ 0 & 500 & -500 & 440 \\ 0 & 0 & 200 & 660 \end{array} \right)$$

Backward substitution

$\rightsquigarrow$ currents

$$\begin{aligned} x_3 &= 660/200 = 3.3, \quad x_2 = (440 + 500 \cdot 3.3)/500 = 4.18 \\ x_1 &= (220 + 100 \cdot 3.3 + 100 \cdot 4.18)/200 = 4.84 \end{aligned}$$

$\max_{j,k}\{|x_j|, |x_j - x_k|\} = 4.84\,$Ampere

10.14 Farmer Bill

Every Wednesday at half past seven, farmer Bill delivers potatoes, onions, and tomatoes to three greengrocers in parc avenue. This week the quantities (in kg) are as follows:

	potatoes	onions	tomatoes
dealer 1	200	100	120
dealer 2	150	50	80
dealer 3	280	150	120

With Bill granting a 10 % discount for a total purchase of over 350 kg, the first dealer pays 738 dollars, the second 530 dollars, and the third 900 dollars.

Determine the prices of 1 kg potatoes, onions, and tomatoes.

Resources: Linear System, Gauß Elimination

Solution

Assembly of the linear system

x_1, x_2, x_3: prices for 1 kg potatoes, onions, and tomatoes

adjusting the payments, taking the discounts for dealers 1 and 3 into account

$$738 \longrightarrow 738 \cdot \frac{10}{9} = 820, \quad 900 \longrightarrow 900 \cdot \frac{10}{9} = 1000$$

total price, sum of the (quantities in kg) × (price per kg) $\rightsquigarrow$ linear system:

$$\begin{array}{rcrcrcrl} 200x_1 & + & 100x_2 & + & 120x_3 & = & 820 & (1) \\ 150x_1 & + & 50x_2 & + & 80x_3 & = & 530 & (2) \\ 280x_1 & + & 150x_2 & + & 120x_3 & = & 1000 & (3) \end{array}$$

Gauß elimination

$(1) - 2 \cdot (2)$ and $(3) - 3 \cdot (2) \quad \rightsquigarrow$

$$-100x_1 - 40x_3 = -240 \qquad (4)$$
$$-170x_1 - 120x_3 = -590 \qquad (5)$$

for example: $-40 = 120 - 2 \cdot 80$, $-170 = 280 - 3 \cdot 150$

$(5) - 3 \cdot (4) \quad \rightsquigarrow$

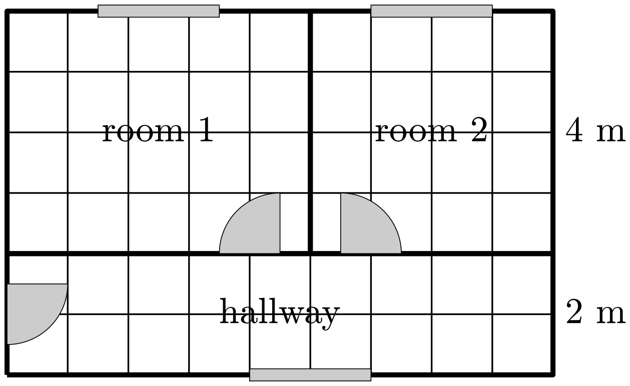
$$130x_1 = 130 \quad \Longrightarrow \quad x_1 = 1$$

substituting into (4) $\quad \Longrightarrow \quad x_3 = (-240 + 100 \cdot 1)/(-40) = 7/2$

substituting into (2) $\quad \Longrightarrow \quad x_2 = (530 - 150 \cdot 1 - 80 \cdot 7/2)/50 = 2$

10.15 Invoice of a Painter

After having renovated an appartment, a painter charges for the first room 790 dollars, for the second room 690 dollars, and for the hallway 870 dollars.



height of the rooms: 2.5 m

In order to check the invoice, determine the different costs per square meter for painting the ceiling, the wall (openings for windows and doors regarded as wall), and a window.

Resources: Linear System, Gauß Elimination

Solution

Assembly of the linear system

areas (m^2):

	room 1	room 2	hallway
ceiling	$4 \cdot 5 = 20$	16	18
wall	$(4+4+5+5) \cdot 2.5 = 45$	40	55

$\rightsquigarrow$ linear system for the costs: x dollars per m^2 ceiling, y dollars per m^2 wall, and z dollars per window

$$\begin{array}{rcrcrcl} 20x & + & 45y & + & z & = & 790 \\ 16x & + & 40y & + & z & = & 690 \\ 18x & + & 55y & + & z & = & 870 \end{array}$$

Gauß elimination

processing the linear system in tableau form $\rightsquigarrow$

$$\left(\begin{array}{rrr|r} 20 & 45 & 1 & 790 \\ 16 & 40 & 1 & 690 \\ 18 & 55 & 1 & 870 \end{array}\right) \underset{(1)}{\rightarrow} \left(\begin{array}{rrr|r} 20 & 45 & 1 & 790 \\ -4 & -5 & 0 & -100 \\ -2 & 10 & 0 & 80 \end{array}\right) \underset{(2)}{\rightarrow} \left(\begin{array}{rrr|r} 20 & 45 & 1 & 790 \\ -4 & -5 & 0 & -100 \\ -10 & 0 & 0 & -120 \end{array}\right)$$

(1) row 2 $\leftarrow$ row 2 - row 1, row 3 $\leftarrow$ row 3 - row 1

(2) row 3 $\leftarrow$ row 3 + 2·row 2

backward substitution $\rightsquigarrow$

$$\begin{aligned} \text{third equation}: \; x &= \frac{-120}{-10} = 12 \\ \text{second equation}: \; y &= \frac{-100 + 4 \cdot 12}{-5} = 10.40 \\ \text{first equation}: \; z &= 790 - 20 \cdot 12 - 45 \cdot 10.40 = 82 \end{aligned}$$

10.16 Construction of an Affine Transformation with Matlab®

Determine the affine transformation $x \mapsto y = Ax + b$ from the following images of three points:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \mapsto \begin{pmatrix} 4 \\ -3 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 3 \end{pmatrix} \mapsto \begin{pmatrix} 5 \\ -4 \end{pmatrix}.$$

Resources: Affine Map, Matrix Operations with Matlab

Solution

Matrix formulation

2×3 matrices of the given points and their images

$$X = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \quad Y = \begin{pmatrix} 3 & 4 & 5 \\ -2 & -3 & -4 \end{pmatrix}$$

affine map $x \mapsto Ax + b \quad \rightsquigarrow$

$$Y = AX + \begin{pmatrix} b & b & b \end{pmatrix} = AX + b \underbrace{\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}}_{e^{\mathrm{t}}}$$

$\iff$ linear system for A and b

$$Y = \underbrace{\begin{pmatrix} A \mid b \end{pmatrix}}_{C} \underbrace{\begin{pmatrix} X \\ \hline e^{\mathrm{t}} \end{pmatrix}}_{Z} \quad \iff \quad Y = CZ$$

substituting the given data $\quad \rightsquigarrow$

$$\underbrace{\left(\begin{array}{cc|c} a_{1,1} & a_{1,1} & b_1 \\ a_{2,1} & a_{2,2} & b_2 \end{array}\right)}_{C} = YZ^{-1} = \begin{pmatrix} 3 & 4 & 5 \\ -2 & -3 & -4 \end{pmatrix} \left(\begin{array}{ccc} 0 & 1 & 2 \\ 1 & 2 & 3 \\ \hline 1 & 1 & 1 \end{array}\right)^{-1}$$

MATLAB® script

```
X = [0 1 2; 1 2 3]; Y = [3 4 5; -2 -3 -4];
e = [1;1;1];
Z = [X; e'];
C = Y/Z;
A = C(:,1:2), b = C(:,3)
```

$$A = \begin{pmatrix} 0 & 1 \\ 2 & -3 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

10.17 Interpolation by Radial Functions with MATLAB®

Interpolate the data $z_k = f(x_k, y_k)$ with a linear combination $\sum_k c_k f_k$ of radial functions,

$$f_k(x,y) = \exp(-s((x-x_k)^2 + (y-y_k)^2)),\ s = 0.2\,,$$

associated with the locations of the data.
Visualize the interpolant in different ways [7].

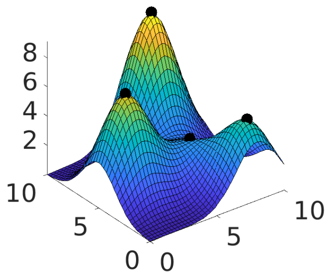
x_k	2	8	7	6
y_k	5	1	9	4
z_k	7	5	9	3

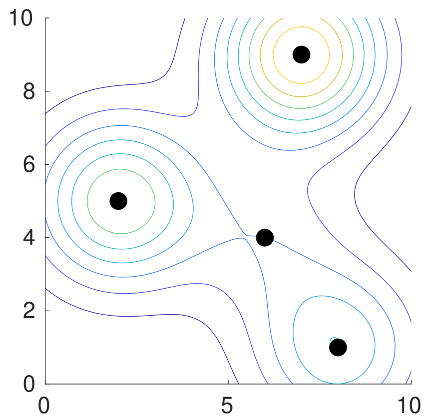
Resources: Visualization of Surfaces with MATLAB® , Matrix Operations with MATLAB®

[7]The interpolation method, described in this problem, is particularly useful for irregularly distributed data, e.g. for fitting height measurements.

Solution

```
% data and radial functions
x = [2 8 7 6]; y = [5 1 9 4]; z = [7 5 9 3];
s = 0.2; f = @(x,y) exp(-s*(x.^2+y.^2));

% interpolation matrix a_{j,k} = f(x_j-x_k,y_j-y_k)
u = ones(size(x)); dx = x'*u-u'*x; dy = y'*u-u'*y;
A = f(dx,dy);

% alternatively (simpler but contrary to the Matlab spirit!):
% computation of a_{j,k} with a double loop

% solution of the linear system Ac = z
c = A\z';

% evaluation grid for visualizing the interpolant
dxy = 0.25; [X,Y] = meshgrid([0:d:10]);

% evaluation of the linear combination sum_k c_k f_k
Z = zeros(size(X));
for k=1:length(x), Z = Z+c(k)*f(X-x(k),Y-y(k)); end

% graph of the interpolant and contour plot
surf(X,Y,Z), contour(X,Y,Z,[0:10])
```





10.18 Jacobi Iteration for Poisson's Equation with MATLAB®

A difference approximation with grid width $1/n$ of Poisson's equation

$$-\Delta u = f$$

on the unit square $[0,1]^2$ with Dirichlet boundary conditions leads to the linear system

$$\begin{aligned} &4u_{j,k} - u_{j-1,k} - u_{j+1,k} - u_{j,k-1} - u_{j,k+1} = f_{j,k}/n^2, \quad j,k = 2,\ldots,n \\ &u_{1,k} = u_{n+1,k} = u_{j,1} = u_{j,n+1} = 0\,, \end{aligned}$$

with $u_{j,k}$ and $f_{j,k}$ the values of the functions u and f at the grid points $(x,y) = ((j-1)/n, (k-1)/n)$.

Write a MATLAB® function `u = jacobi(f,tol)` which solves the linear system with the Jacobi iteration.

Resources: Jacobi Iteration

Solution

Jacobi iteration

solving

$$4u_{j,k} - u_{j-1,k} - u_{j+1,k} - u_{j,k-1} - u_{j,k+1} = f_{j,k}/n^2$$

for the diagonal term $\rightsquigarrow$ iteration step $u \to v$ of the Jacobi iteration:

$$\underbrace{v_{j,k}}_{\text{replacing } u_{j,k}} = (u_{j-1,k} + u_{j+1,k} + u_{j,k-1} + u_{j,k+1} + f_{j,k}/n^2)/4$$

The grid values can be updated simultaneously, which leads to a very simple[8] MATLAB® implementation.

MATLAB® function

```
function u = jacobi(f,tol)
% f(j,k), u(j,k): values at ((j-1)/n,(k-1)/n), j,k=1:n+1
n = size(f,1)-1;
ind = [2:n];   % indices of the (n-1)^2 interior grid points
% start values, incorporating the zero boundary values
u = zeros(n+1,n+1); v = u;

error = inf;
while error > tol
   % In view of the convergence of the Jacobi iteration
   % no bound on the number of iterations is necessary
   % for tolerances significantly larger than eps.

   % Jacobi step
   v(ind,ind) = (u(ind-1,ind)+u(ind+1,ind)+ ...
      u(ind,ind-1)+u(ind,ind+1)+f/n^2)/4;

   % maximum norm of difference -> estimate of the error
   error = norm(u(:)-v(:),inf);
   u = v;
end
```

[8]Simple, but slow! Multigrid methods yield the fastest convergence.

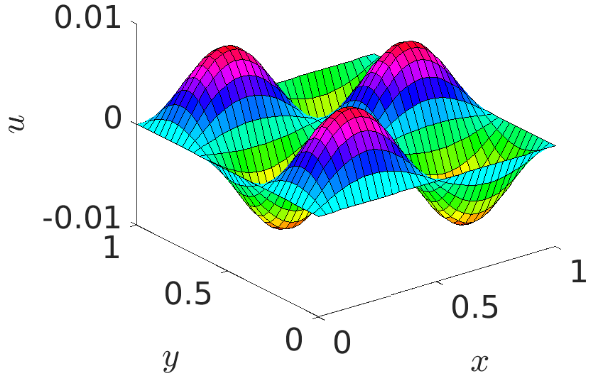
Sample problem

As a test case, consider

$$f(x,y) = \sin(2\pi x)\sin(3\pi y)$$

with the exact solution $u(x,y) = f(x,y)/((2\pi)^2 + (3\pi)^2)$ (eigenfunction of the negative Laplace operator).

```
% generate the grid values of f
n = 32;
[x,y] = meshgrid([0:n]/n);
f = sin(2*pi*x).*sin(3*pi*y);

% Jacobi iteration and plot of the solution
u = jacobi(f,1.0e-5);
surf(x,y,u);
view(3); shading interp; colormap(hsv)
```



10.19 Rank 1 Update of an Inverse Matrix

Determine, for an invertible matrix A and a rank one matrix uv^{t}, vectors x and y, so that

$$(A + uv^{\mathrm{t}})^{-1} = A^{-1} + xy^{\mathrm{t}}\,.$$

Which restriction on the vectors u and v is necessary? Apply the formula to invert the matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Resources: Matrix Multiplication

Solution

Determination of x and y

$(A+uv^{\mathrm{t}})(A^{-1}+xy^{\mathrm{t}})=E$ with E the unit matrix $\quad\Longrightarrow$

$$E+Axy^{\mathrm{t}}+uv^{\mathrm{t}}A^{-1}+uv^{\mathrm{t}}xy^{\mathrm{t}}=E \quad\Longleftrightarrow\quad (Ax+uv^{\mathrm{t}}x)y^{\mathrm{t}}=-u(v^{\mathrm{t}}A^{-1})$$

comparison of the columns and rows of the rank one matrices on both sides $\rightsquigarrow$

$$Ax \parallel u, \quad y^{\mathrm{t}} \parallel v^{\mathrm{t}}A^{-1}$$

choosing $y^{\mathrm{t}}=v^{\mathrm{t}}A^{-1}$, $x=sA^{-1}u \quad \rightsquigarrow$

$$(su+uv^{\mathrm{t}}sA^{-1}u)v^{\mathrm{t}}A^{-1}=-uv^{\mathrm{t}}A^{-1} \quad\Longleftrightarrow\quad s+s(v^{\mathrm{t}}A^{-1}u)=-1$$

i.e. $s=-(1+v^{\mathrm{t}}A^{-1}u)^{-1}$ if $v^{\mathrm{t}}A^{-1}u\neq -1$, and

$$(A+uv^{\mathrm{t}})^{-1}=A^{-1}-\underbrace{\frac{1}{1+v^{\mathrm{t}}A^{-1}u}A^{-1}u}_{-x}\,\underbrace{v^{\mathrm{t}}A^{-1}}_{y^{\mathrm{t}}}$$

Application to the example

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix} = E+uv^{\mathrm{t}}, \quad u=\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v^{\mathrm{t}}=\begin{pmatrix} 0 & 1 & 0 \end{pmatrix},$$

application of the formula with $A^{-1}=A=E \quad \rightsquigarrow$

$$A^{-1}=E-\frac{1}{1+v^{\mathrm{t}}u}uv^{\mathrm{t}}=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}-\frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}=\begin{pmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix}$$

10.20 Gerschgorin Inequality

Prove that the eigenvalues λ of a matrix A lie in the union of the discs

$$\{\lambda : |\lambda - a_{j,j}| \le \sum_{k \neq j} |a_{j,k}|\} .$$

As an example, state the resulting bound for the symmetric matrix

$$A = \begin{pmatrix} 8 & 0 & 1 \\ 0 & -3 & -2 \\ 1 & -2 & -3 \end{pmatrix} .$$

Resources: Eigenvalues and Eigenvectors

Solution

Proof

$Av = \lambda v \quad \iff$

$$\lambda v_j - a_{j,j} v_j = \sum_{k \neq j} a_{j,k} v_k, \quad j = 1, 2, \ldots$$

For the index j with maximal absolute value $|v_j|$, one obtains after division by v_j the estimate

$$|\lambda - a_{j,j}| \leq \left| \sum_{k \neq j} a_{j,k} (v_k / v_j) \right| \leq \left(\sum_{k \neq j} |a_{j,k}| \right) \underbrace{\max_{k \neq j} |v_k / v_j|}_{\leq 1} .$$

Example

$$A = \begin{pmatrix} 8 & 0 & 1 \\ 0 & -3 & -2 \\ 1 & -2 & -3 \end{pmatrix}$$

$A = A^{\mathrm{t}} \quad \Longrightarrow \quad$ All eigenvalues λ are real and the Gerschgorin discs can be replaced by intervals:

$$\begin{aligned} \lambda &\in [8-1, 8+1] \cup [-3-2, -3+2] \cup [-3-(1+2), -3+(1+2)] \\ &= [7,9] \cup [-5,-1] \cup [-6,0] = [7,9] \cup [-6,0] \subset [-6,9] \end{aligned}$$

Note that no eigenvalues lie in the interval $(0, 7)$.

To illustrate the sharpness of Gerschgorin's theorem, the eigenvalues are computed with MATLAB® .

```
A = [8 0 1; 0 -3 -2; 1 -2 -3];
lambda = eig(A)
lambda =
   -5.0387
   -1.0545
    8.0931
```

10.21 Eigenvalues of Permutation Matrices

Determine the eigenvalues of all 3×3 permutation matrices without resorting to the characteristic polynomials.

Resources: Eigenvalues and Eigenvectors, Sum and Product of Eigenvalues

Solution

General considerations

matrix, associated with a permutation $p:\ k \mapsto p(k)$:

$$A:\ a_{k,p(k)} = 1, \quad a_{k,\ell} = 0,\ \ell \neq p(k)\,,$$

i.e. $x \mapsto Ax = (x_{p(1)}, x_{p(2)}, \ldots)^{\mathrm{t}}$

- $\sum_k a_{j,k} = 1 \quad\Longrightarrow\quad (1,1,\ldots)^{\mathrm{t}}$ is an eigenvector with eigenvalue $\lambda = 1$
- $|Ax| = |x| \quad\Longrightarrow\quad |\lambda| = 1$ for all eigenvalues
  $a_{j,k} \in \{0,1\} \subset \mathbb{R} \quad\Longrightarrow\quad \lambda = \pm 1 \vee \lambda = \cos\varphi \pm \mathrm{i}\sin\varphi$ (pair of complex conjugate eigenvalues)

In view of these properties, there exist 4 possibilities for the eigenvalues of a 3×3 permutation matrix:

$$\underbrace{\{1,1,1\},\quad \{1,1,-1\},\quad \{1,-1,-1\}}_{\text{real eigenvalues}},\quad \underbrace{\{1, \cos\varphi + \mathrm{i}\sin\varphi, \cos\varphi - \mathrm{i}\sin\varphi\}}_{\text{eigenvalue 1 and a complex conjugate pair}}\,.$$

The different cases can be distinguished, using that

$$\operatorname{trace} A = a_{1,1} + a_{2,2} + a_{3,3} = \sum_{k=1}^{3} \lambda_k\,.$$

Eigenvalues of the six 3×3 permutation matrices

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$: $\operatorname{trace} A = 3 \rightsquigarrow \{1,1,1\}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$: $\operatorname{trace} A = 1 \rightsquigarrow \{1,1,-1\}$
$A^{-1} = A^{\mathrm{t}}$, $\det A = -1 \implies A$ is a reflection

$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$: $\operatorname{trace} A = 0 \rightsquigarrow \{1, \cos\varphi + \mathrm{i}\sin\varphi, \cos\varphi - \mathrm{i}\sin\varphi\}$
$\sum_{k=1}^{3} \lambda_k = 1 + 2\cos\varphi = 0 \implies \varphi = \arccos(-1/2) = \pm 2\pi/3$ and $\cos\varphi \pm$

$\mathrm{i}\sin\varphi = (-1 \pm \sqrt{3}\mathrm{i})/2$
$A^{-1} = A^{\mathrm{t}}$, $\det A = 1 \quad \Longrightarrow \quad A$ is a rotation by an angle $2\pi/3$ with axis $(1,1,1)^{\mathrm{t}}$.

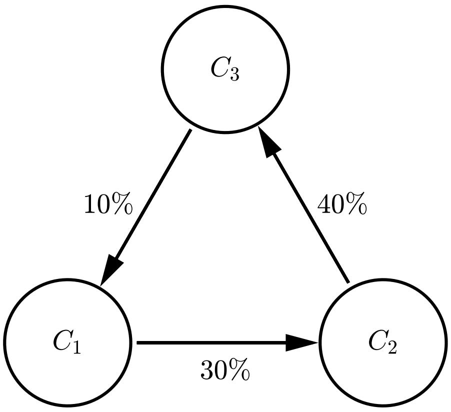
The eigenvalues $\{1, -1, -1\}$ are not possible for a 3×3 permutation matrix.

10.22 Market Shares of Competing Companies

The figure shows the annual change of the market shares x_k ($\sum_k x_k = 1$) of three companies C_k. Set up the transition matrix A, which decribes the update

$$x^{\mathrm{alt}} \to x^{\mathrm{neu}} = Ax^{\mathrm{alt}},$$

and determine the asymptotic distribution of the market shares, $x^{\infty} = \lim_{n\to\infty} A^n x$, resulting from almost all initial distributions x.



Resources: Basis of Eigenvectors, Convergence of Matrix Powers

Solution

annual change of the market shares x_k

$$\begin{array}{lclcl} x_1 & \to & (1-0.1)x_1 & + & 0.4x_3 \\ x_2 & \to & (1-0.3)x_2 & + & 0.1x_1 \\ x_3 & \to & (1-0.4)x_3 & + & 0.3x_2 \end{array} \quad \Longleftrightarrow \quad x^{\text{new}} = \underbrace{\begin{pmatrix} 0.9 & 0 & 0.4 \\ 0.1 & 0.7 & 0 \\ 0 & 0.3 & 0.6 \end{pmatrix}}_{A} x^{\text{old}}$$

Since the columns of A sum to 1, $\lambda_1 = 1$ is an eigenvalue of A^{t} ($A^{\text{t}}(1,1,1)^{\text{t}} = (1,1,1)^{\text{t}}$) and, therefore, also of A.

$\det A = 0.39 = \lambda_1\lambda_2\lambda_3$, $\operatorname{trace} A = 2.2 = \underbrace{\lambda_1}_{=1} + \lambda_2 + \lambda_3 \quad \Longrightarrow$

$$(\lambda - \lambda_2)(\lambda - \lambda_3) = \lambda^2 - 1.2\lambda + 0.39, \quad \lambda_{2,3} = 0.6 \pm \sqrt{0.03}\,\mathrm{i}$$

computation of the limit $x^\infty = \lim\limits_{n\to\infty} A^n x$ by expressing x as linear combinations of eigenvectors u, v, w, corresponding to the eigenvalues λ_1, λ_2, λ_3:

$$x = \alpha u + \beta v + \gamma w \quad \Longrightarrow \quad A^n x \underset{\lambda_1 = 1}{=} \alpha u + \lambda_2^n \beta v + \lambda_3^n \gamma w$$

$|\lambda_2| = |\lambda_3| < \lambda_1 = 1 \quad \rightsquigarrow \quad$ limit $x^\infty = \alpha u$ if $\alpha \neq 0$ (satisfied for almost all x)

asymptotic ratios of the market shares $\widehat{=}$ ratios of the components of u

computation of this dominant eigenvector

$$(A - E)u = 0 \quad \Leftrightarrow \quad \begin{pmatrix} -0.1 & 0 & 0.4 \\ 0.1 & -0.3 & 0 \\ 0 & 0.3 & -0.4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

choosing $u_2 = 4 \quad \rightsquigarrow \quad u_3 = 3$, $u_1 = 12$, i.e.

$$u = (12, 4, 3)^{\text{t}} \quad \Longrightarrow \quad x_1^\infty : x_2^\infty : x_3^\infty = 12 : 4 : 3$$

normalization (market shares sum to 1) $\quad \Longrightarrow \quad x^\infty = \frac{1}{13}(12, 4, 3)^{\text{t}}$

10.23 Straight Line of Minimal Distance

Determine the straight line

$$g: x_1u_1 + x_2u_1 = c \geq 0, \quad |u| = 1\,,$$

which minimizes the sum of the squared (geometric) distances to the points $(1,\,1)$, $(2,\,2)$, $(4,\,3)$, $(5,\,2)$.

⚠ Note the difference to the least squares line, where the deviation from the data is measured in the x_2 direction and, hence, is not equal to the geometric distance.

Resources: Eigenvalues and Eigenvectors, Rayleigh Quotient

Solution

Sum of the squared distances

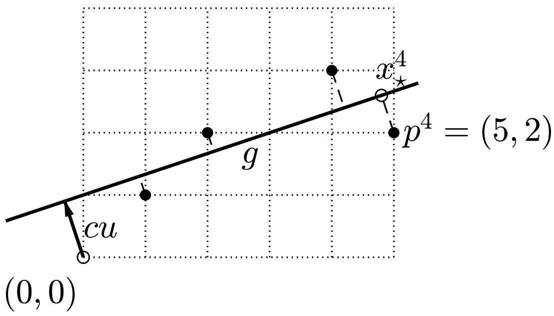
Hesse normal form of a straight line g in the x_1/x_2 plane:

$$g:\ x_1u_1 + x_2u_1 = c \geq 0, \quad |u| = 1$$

with normal $u = (u_1, u_2)^{\mathrm{t}}$ and c the distance from the origin

distance d of a point $p = (p_1,\, p_2)$ from g:
$x_\star \in g$ closest point to p
$\implies \quad (p - x_\star) \parallel u$ and

$$\begin{aligned} d &= |p - x_\star| \\ &\underset{|u|=1}{=} |(p - x_\star)u| = |pu - c| \end{aligned}$$



sum of the squared distances d_k to the points $p^k = (p_1^k,\, p_2^k)$

$$s = \sum_k d_k^2 = \sum_k (p^k u - c)^2 = \sum_k \left((p^k u)^2 - 2\,c\,p^k u + c^2\right)$$

or, since $(p^k u)^2 = (u^{\mathrm{t}}(p^k)^{\mathrm{t}})\,(p^k u) = u^{\mathrm{t}}\,((p^k)^{\mathrm{t}} p^k)\,u$,

$$s = u^{\mathrm{t}} Q u - 2n\,c\,\bar{p}u + n\,c^2$$

with n the number of points p^k and

$$Q = \sum_{k=1}^{n} (p^k)^{\mathrm{t}} p^k, \quad \bar{p} = \frac{1}{n}\sum_{k=1}^{n} p^k$$

Optimal straight line

optimal constant c:

$$0 \overset{!}{=} \frac{\partial s}{\partial c} = -2n\,\bar{p}u + 2n\,c$$

$\implies \quad c = \bar{p}u$

substituting this value $\quad\rightsquigarrow\quad$ simplification

$$s = u^{\mathrm{t}} Q u - 2n\,\underbrace{(\bar{p}u)(\bar{p}u)}_{=u^{\mathrm{t}}(\bar{p}^{\mathrm{t}}\bar{p})u} + n\,(\bar{p}u)^2 = u^{\mathrm{t}}\,\underbrace{\left(Q - n\bar{p}^{\mathrm{t}}\bar{p}\right)}_{=:A}\,u = r_A(u)$$

property of the Rayleigh quotient $r_A \quad \Longrightarrow \quad$ minimal s for the normalized eigenvector u ($u^{\mathrm{t}}u = 1$) corresponding to the smallest eigenvalue $\lambda_{\min}$ of A and

$$s_{\min} = u^{\mathrm{t}}(Au) = u^{\mathrm{t}}(\lambda_{\min}u) = \lambda_{\min}$$

<u>Computation for the given data</u>

$$\begin{aligned}
Q &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}(1,\, 1) + \begin{pmatrix} 2 \\ 2 \end{pmatrix}(2,\, 2) + \begin{pmatrix} 4 \\ 3 \end{pmatrix}(4,\, 3) + \begin{pmatrix} 5 \\ 2 \end{pmatrix}(5,\, 2) \\
&= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 4 & 4 \\ 4 & 4 \end{pmatrix} + \begin{pmatrix} 16 & 12 \\ 12 & 9 \end{pmatrix} + \begin{pmatrix} 25 & 10 \\ 10 & 4 \end{pmatrix} = \begin{pmatrix} 46 & 27 \\ 27 & 18 \end{pmatrix} \\
\bar{p} &= \frac{1}{4}((1,\, 1) + (2,\, 2) + (4,\, 3) + (5,\, 2)) = (3,\, 2) \\
A &= \begin{pmatrix} 46 & 27 \\ 27 & 18 \end{pmatrix} - 4\begin{pmatrix} 3 \\ 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \end{pmatrix} = \begin{pmatrix} 46 & 27 \\ 27 & 18 \end{pmatrix} - \begin{pmatrix} 36 & 24 \\ 24 & 16 \end{pmatrix} \\
&= \begin{pmatrix} 10 & 3 \\ 3 & 2 \end{pmatrix}
\end{aligned}$$

eigenvalues of A: $1, 11 \quad \Longrightarrow \quad \lambda_{\min} = 1$ with normalized eigenvector $u = (-1,\, 3)^{\mathrm{t}}/\sqrt{10}$

$c = \bar{p}u = (3,\, 2)\begin{pmatrix} -1/\sqrt{10} \\ 3/\sqrt{10} \end{pmatrix} = 3/\sqrt{10}$

$\rightsquigarrow$ optimal straight line (multiplying the Hesse normal form by $\sqrt{10}$)

$$g:\ -x_1 + 3x_2 = 3$$

10.24 Approximation with a Linear Map Using MATLAB®

Determine a 2×2 matrix A which minimizes the sum of the squared errors $e(A) = \sum_k |f(x_k) - Ax_k|^2$ for the data

$$X = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \; Y = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad x_k = X(:,k),\; f(x_k) = y_k = Y(:,k)\,.$$

Resources: Singular Value Decomposition

Solution

Simplification

matrix formulation:

$$\sum_k |Y(:,k) - AX(:,k)|^2 = |Y - AX|_F^2$$

with the Frobenius norm $|Z|_F = \sqrt{\sum_{j,k} z_{j,k}^2}$
replacing the matrix X by its singular value decomposition USV^{t} and using the invariance of $|\ |_F$ under orthogonal transformations $\rightsquigarrow$

$$|Y - AX|_F^2 = |Y - AUSV^{\mathrm{t}}|_F \underset{V^{\mathrm{t}}V=E}{=} |YV - (AU)S|_F =: |Z - BS|_F$$

Since

$$B \underbrace{\begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{pmatrix}}_{S} = \begin{pmatrix} s_1 b_{1,1} & s_2 b_{1,2} & 0 \\ s_1 b_{2,1} & s_2 b_{2,2} & 0 \end{pmatrix}$$

the optimal entries of the matrix B are obtained by annihilating the corresponding entries of Z:

$$b_{1,1} = z_{1,1}/s_1,\ b_{1,2} = z_{1,2}/s_2,\ \ldots \iff B = Z(:,1:2)S(:,1:2)^{-1} = ZS^+$$

with S^+ the pseudoinverse of S, i.e. a 2×3 matrix with diagonal entries $1/s_1$ and $1/s_2$
summarizing:

$$X = USV^{\mathrm{t}},\ Z = YV,\ B = ZS^+,\ A = BU^{\mathrm{t}}$$

squared error

$$e(A) = |Y - AX|_F^2 = |Z(:,3)|^2$$

Implementation

```
% data
X = [1 2 0; 0 2 1]; Y = [0 1 1; 1 1 0];
% singular value decomposition
[U,S,V] = svd(X)
% computation of the approximation
Z = Y*V
```

```
B = Z*pinv(S)   % multiplication with the pseudoinverse
A = B*U'
    -0.2222  0.7778
     0.7778 -0.2222
% error in the Frobenius norm (squared)
e_A = norm(Y-A*X,'fro')^2
    0.2222
```

10.25 Recursive Least Squares

If an overdetermined linear system is augmented by an additional equation,

$$Ax \overset{!}{=} b \quad \rightarrow \quad \begin{array}{c} A\tilde{x} \overset{!}{=} b \\ a^{\mathrm{t}}\tilde{x} \overset{!}{=} \beta \end{array},$$

then

$$\tilde{x} = x + \frac{\beta - a^{\mathrm{t}}x}{1 + a^{\mathrm{t}}Pa}Pa$$

with $P = (A^{\mathrm{t}}A)^{-1}$. Prove this recursion, which is particularly useful for problems with a large amount of successive measurement data.

Resources: Normal Equation, Matrix Multiplication

Solution

It has to be shown that $\tilde{x}$ is a solution of the normal equation for the augmented least squares problem $\begin{pmatrix} A \\ a^{\mathrm{t}} \end{pmatrix} \tilde{x} \stackrel{!}{=} \begin{pmatrix} b \\ \beta \end{pmatrix}$, i.e. that

$$\left(\begin{pmatrix} A^{\mathrm{t}} & a \end{pmatrix} \begin{pmatrix} A \\ a^{\mathrm{t}} \end{pmatrix} \right) \tilde{x} = \begin{pmatrix} A^{\mathrm{t}} & a \end{pmatrix} \begin{pmatrix} b \\ \beta \end{pmatrix}$$

multiplying the block matrices and substituting $\tilde{x} = x + \dfrac{\beta - a^{\mathrm{t}} x}{1 + a^{\mathrm{t}} P a} P a$ and $A^{\mathrm{t}} A = P^{-1} \quad \rightsquigarrow$

$$\left(P^{-1} + a a^{\mathrm{t}} \right) \left(x + \frac{\beta - a^{\mathrm{t}} x}{1 + a^{\mathrm{t}} P a} P a \right) \stackrel{!}{=} A^{\mathrm{t}} b + \beta a$$

simplifying the left side with

$$\left(P^{-1} + a a^{\mathrm{t}} \right) P a = (1 + a^{\mathrm{t}} P a) a$$

and the right side with the normal equation $P^{-1} x = A^{\mathrm{t}} A x = A^{\mathrm{t}} b \quad \rightsquigarrow$

$$\left(P^{-1} + a a^{\mathrm{t}} \right) x + \left(\beta - a^{\mathrm{t}} x \right) a \stackrel{!}{=} P^{-1} x + \beta a \quad \checkmark$$

Remark

For a repeated application of the recursion, the inverse of the matrix of the normal equation can be updated in a similar fashion:

$$\tilde{P} = \left(P^{-1} + a a^{\mathrm{t}} \right)^{-1} = P - \frac{1}{1 + a^{\mathrm{t}} P a} P a a^{\mathrm{t}} P \,.$$

10.26 Least Squares Solution Closest to a Given Point

Use the singular value decomposition

$$A = USV^{\mathrm{t}} = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 30 & 0 & 0 \\ 0 & 15 & 0 \end{pmatrix}\frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}$$

to determine the solution x of the linear system $Ax = b = (2, 11)^{\mathrm{t}}$ with minimal distance to $c = (-5, 2, 4)^{\mathrm{t}}$.

Resources: Singular Value Decomposition

Solution

Reformulation of the problem

The problem

$$\underbrace{USV^{\mathrm{t}}}_{A} x = b, \quad |x-c|^2 \to \min$$

can be simplified using the orthogonalitiy of V and V^{t}. Since $|x-c| = |V^{\mathrm{t}}(x-c)|$, an equivalent formulation is

$$SV^{\mathrm{t}}x = U^{\mathrm{t}}b, \quad |V^{\mathrm{t}}x - V^{\mathrm{t}}c|^2 \to \min\,,$$

or, with $y = V^{\mathrm{t}}x$, $d = U^{\mathrm{t}}b$, $e = V^{\mathrm{t}}c$,

$$s_1y_1 = d_1,\ s_2y_2 = d_2, \quad (y_1-e_1)^2 + (y_2-e_2)^2 + (y_3-e_3)^2 \to \min\ .$$

The minimum is attained for $y_1 = d_1/s_1$, $y_2 = d_2/s_2$, $y_3 = e_3$.

Solution for the given data

$$U = \frac{1}{5}\begin{pmatrix} 3 & -4 \\ 4 & 3 \end{pmatrix},\ V = V^{\mathrm{t}} = \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix},\ b = \begin{pmatrix} 2 \\ 11 \end{pmatrix},\ c = \begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix}$$

and $s_1 = 30$, $s_2 = 15$ $\quad\rightsquigarrow$

$$d = \frac{1}{5}\begin{pmatrix} 3 & 4 \\ -4 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 11 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$$e = \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} -5 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix}$$

$$y_1 = 10/30 = 1/3,\ y_2 = 5/15 = 1/3,\ y_3 = -3$$

backward substitution:

$$x = Vy = \frac{1}{3}\begin{pmatrix} -2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & -2 \end{pmatrix}\begin{pmatrix} 1/3 \\ 1/3 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ -5/3 \\ 7/3 \end{pmatrix}$$

10.27 Iterative Solution of Least Squares Problems

Prove that every least squares problem, $|Ax - b| \to \min$, with a matrix A of maximal rank can be solved with the iteration

$$x_{\ell+1} = x_\ell - \omega A^{\mathrm{t}} A x_\ell + \omega A^{\mathrm{t}} b, \quad x_0 = \omega A^{\mathrm{t}} b\,,$$

if the positive parameter ω is not larger than $1/(\|A^{\mathrm{t}}\|\|A\|)$ for matrix norm $\|\ \|$ corresponding to a vector norm [9].

Resources: Diagonal Form of Hermitian Matrices

[9] $\omega \le 1/\Big((\max_k \sum_j |a_{j,k}|)(\max_j \sum_k |a_{j,k}|)\Big)$ for the matrix norm corresponding to the maximum norm of vectors

Solution

representation of the start vector with an orthonormal basis of eigenvectors u_k, corresponding to the eigenvalues $\lambda_k > 0$ of the symmetric, positive definite matrix $A^{\mathrm{t}}A$:

$$x_0 = \omega A^{\mathrm{t}} b = \sum_k c_k u_k$$

repeated application of the recursion with $Q = E - \omega A^{\mathrm{t}} A$ and E the unit matrix $\quad\rightsquigarrow$

$$\begin{aligned} x_1 &= Qx_0 + x_0 \\ x_2 &= Q(Qx_0 + x_0) + x_0 = Q^2 x_0 + Qx_0 + x_0 \\ &\cdots \\ x_n &= Q^n x_0 + Q^{n-1} x_0 + \cdots + Qx_0 + x_0 \end{aligned}$$

$Qu_k = (\underbrace{1 - \omega\lambda_k}_{\varrho_k}) u_k$, $Q^\ell u_k = \varrho_k^\ell u_k$, the inequality $1 > \varrho_k \geq 1 - \omega \underbrace{\|A^{\mathrm{t}}A\|}_{\geq \lambda_k} \geq 1 - \omega \|A^{\mathrm{t}}\| \|A\| \geq 0$, and the formula for a geometric series $\quad\Longrightarrow$

$$x_n = \sum_k \frac{\varrho_k^{n+1} - 1}{\varrho_k - 1} c_k u_k \quad \underset{n \to \infty}{\longrightarrow} \quad x_\infty = \sum_k \frac{1}{1 - \varrho_k} c_k u_k = \sum_k \frac{1}{\omega \lambda_k} c_k u_k$$

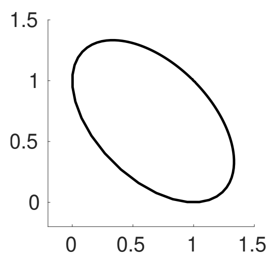
and

$$A^{\mathrm{t}} A x_\infty = \sum_k \frac{1}{\omega \lambda_k} c_k A^{\mathrm{t}} A u_k = \frac{1}{\omega} \sum_k c_k u_k = \frac{1}{\omega} x_0 = A^{\mathrm{t}} b$$

Remark

It is not necessary to assume that A has maximal rank. First, it is proved that in the representation of $\omega A^{\mathrm{t}} b$ with respect to the eigenvector basis the components of the eigenvectors to the eigenvalue 0 vanish. Then the proof proceeds analogously to the positive definite case.

10.28 Equation of a Rationally Parametrized Conic Section with Maple™



Every rational parametrization $t \mapsto \left(\frac{p(t)}{r(t)}, \frac{q(t)}{r(t)}\right)$ with quadratic polynomials p, q, r describes a conic section

$$Q: \; ax^2 + bxy + cy^2 + dx + ey + f = 0\,.$$

Determine $a, b, \ldots, f$ for the depicted ellipse with

$$p(t) = (1-t)^2,\; q(t) = t^2,\; r(t) = 1 - t + t^2\,.$$

Resources: Linear System

Solution

```
p := t->(1-t)^2: q := t->t^2: r := t->1-t+t^2:
```

substituting $x = p/r$, $y = q/r$ into the equation of the ellipse

$$Q:\ ax^2 + bxy + cy^2 + dx + ey + f = 0$$

and multiplying by r^2 $\quad\rightsquigarrow\quad$ $u(t) = 0$ with a polynomial u with degree ≤ 4,

$$u(t) = ap(t)^2 + bp(t)q(t) + \cdots + eq(t)r(t) + fr(t)^2 =: \sum_{k=0}^{4} u_k t^k$$

setting the coefficients u_k or, equivalently, the derivatives at $t = 0$ to zero $\rightsquigarrow$ 5 linear equations L_k for $a, b, \ldots, f$

```
u := t->a*p(t)^2+b*p(t)*q(t)+c*q(t)^2
             +d*p(t)*r(t)+e*q(t)*r(t)+f*r(t)^2:
L[1] := u(0)=0;
for k from 1 to 4 do
   # setting the k-th derivative at 0 to zero
   L[k+1] := D[1$k](u)(0)=0
od
```

$$\begin{aligned}
L_1 &:= a + d + f = 0\\
L_2 &:= -4a - 3d - 2f = 0\\
L_3 &:= 12a + 2b + 8d + 2e + 6f = 0\\
L_4 &:= -24a - 12b - 18d - 6e - 12f = 0\\
L_5 &:= 24a + 24b + 24c + 24d + 24e + 24f = 0
\end{aligned}$$

```
# solving the underdetermined linear system
solve([L[1],L[2],L[3],L[4],L[5]],[a,b,d,c,d,e,f]);
```

$$\rightsquigarrow \quad [[a = c,\ b = c,\ c = c,\ f = c,\ d = -2c,\ e = -2c]]$$

one-dimensional solution space, parametrized by c
choosing $c = 1$ $\quad\rightsquigarrow\quad$ $Q:\ x^2 + xy + y^2 - 2x - 2y + 1 = 0$

10.29 Normal Form, Type and Lenghts of the Principal Axes of a Quadric

Transform the quadric

$Q:\ 4x_1^2 - 2x_1x_2 + 4x_1x_3 + x_2^2 + 2x_2x_3 + 4x_3^2 - \sqrt{6}x_1 - 2\sqrt{6}x_2 + \sqrt{6}x_3 = -3$

to normal form and determine the type as well as the lengths of the principal axes.

Resources: Euclidean Normal Form of Three-Dimensional Quadrics

Solution

Matrix form of the quadric

rewrite the equation

$$4x_1^2 - 2x_1x_2 + 4x_1x_3 + x_2^2 + 2x_2x_3 + 4x_3^2 - \sqrt{6}x_1 - 2\sqrt{6}x_2 + \sqrt{6}x_3 = -3$$

in the form $x^{\mathrm{t}}Ax + 2b^{\mathrm{t}}x + c = 0$ with

$$A = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 1 & 1 \\ 2 & 1 & 4 \end{pmatrix}, \quad b = \sqrt{6}\begin{pmatrix} -1/2 \\ -1 \\ 1/2 \end{pmatrix}, \quad c = 3$$

Eigenvalues

characteristic polynomial

$$\det(A - \lambda E) = \begin{vmatrix} 4-\lambda & -1 & 2 \\ -1 & 1-\lambda & 1 \\ 2 & 1 & 4-\lambda \end{vmatrix}$$
$$\underset{\text{Sarrus}}{=} (4-\lambda)(1-\lambda)(4-\lambda) - 2 - 2 - (4-\lambda) - (4-\lambda) - 4(1-\lambda)$$
$$= -\lambda^3 + 9\lambda^2 - 18\lambda$$

zeros $\quad\rightsquigarrow\quad$ eigenvalues $\lambda_1 = 6,\ \lambda_2 = 3,\ \lambda_3 = 0$

Principal axes

Since the algebraic multiplicity of all eigenvalues λ equals 1, $\operatorname{Rang}(A-\lambda E) = 2$ and, hence, every eigenvector is parallel to the cross product of two linearly independent rows of

$$A - \lambda E = \begin{pmatrix} 4-\lambda & -1 & 2 \\ -1 & 1-\lambda & 1 \\ 2 & 1-\lambda & 4 \end{pmatrix}$$

(cross product $\perp$ rows, i.e. $\in \ker(A - \lambda E)$ $\quad\rightsquigarrow\quad$ directions of the principal axes

$$\begin{array}{rclcrcl} \lambda_1 & = & 6: & u & = & (-2,-1,2)^{\mathrm{t}} \times (-1,-5,1)^{\mathrm{t}} & = & (9,0,9)^{\mathrm{t}} \\ \lambda_2 & = & 3: & v & = & (1,-1,2)^{\mathrm{t}} \times (-1,-2,1)^{\mathrm{t}} & = & (3,-3,-3)^{\mathrm{t}} \\ \lambda_3 & = & 0: & w & = & (4,-1,2)^{\mathrm{t}} \times (-1,1,1)^{\mathrm{t}} & = & (-3,-6,3)^{\mathrm{t}} \end{array}$$

Transformation

normalizing u, v, w $\rightsquigarrow$ transformation matrix

$$Q = \left(\frac{u}{|u|}, \frac{v}{|v|}, \frac{-w}{|w|}\right) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix}$$

correction of the orientation of w to ensure that $\det Q = 1$ (rotation, orientation preserving)

coordinate transformation $x = Qy$

$$x^{\mathrm{t}}Ax + 2b^{\mathrm{t}}x + c = 0 \quad \Longleftrightarrow \quad y^{\mathrm{t}}Dy + 2d^{\mathrm{t}}y + c = 0$$

with $D = Q^{\mathrm{t}}AQ$, $d^{\mathrm{t}} = b^{\mathrm{t}}Q$, i.e.

$$\begin{aligned} D &= \operatorname{diag}(6, 3, 0) \\ d^{\mathrm{t}} &= \sqrt{6}\left(-\frac{1}{2}, -1, \frac{1}{2}\right) \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -3 \end{pmatrix} \end{aligned}$$

$\rightsquigarrow$ transformed equation

$$6y_1^2 + 3y_2^2 - 6y_3 + 3 = 0$$

translation $y = z + p$, $p = (0, 0, 1/2)^{\mathrm{t}}$, and scaling $\rightsquigarrow$

$$2z_1^2 + z_2^2 - 2z_3 = 0$$

type: two positive eigenvalues, one eigenvalue equal to 0, linear term $\rightsquigarrow$ elliptic paraboloid

transformation: $x = Qy = Qz + Qp$

lengths of the principal axes: $a_1 = 1/\sqrt{2}$, $a_2 = 1$

10.30 Normal Form of an Ellipsoid with MATLAB®

Determine the normal form of the ellipsoid, represented by the equation

$$5x_1^2 + 4x_1x_2 + 6x_2^2 - 4x_2x_3 + 7x_3^2 - 10x_1 - 4x_2 = -2\,,$$

with MATLAB® . Compute, in particular, the directions and lengths of the principal axes and the midpoint.

Resources: Transformation to Normal Form, Euclidean Normal Form of Three-Dimensional Quadrics

Solution

```
% matrix form x^t A x + 2 b^t x = c of the equation
% 5x_1^2+4x_1x_2+6x_2^2-4x_2x_3+7x_3^2-10x_1-4x_2=-2
A = [5 2 0; 2 6 -2; 0 -2 7]; b = [-5;-2;0]; c = -2;

% eigenvectors (principal directions) and eigenvalues of A
[Q,D] = eig(A)

Q =                          D =
    -0.6667 0.6667 -0.3333       3.000   0      0
     0.6667 0.3333 -0.6667       0       6.0000 0
     0.3333 0.6667  0.6667       0     0        9.000

% diagonalisation with the substitution x = Qy
% -> y'Dy + 2d'y + c = 0, D = Q'*A*Q;
d = Q'*b

d =
     2.0000
    -4.0000
     3.0000

% completing the square with the substituion y = z+p
% D(1,1) y_1^2 + ... -> D(1,1) (y_1-d_1/D(1,1))^2 + ...
% -> sum_k a_k z_k^2 = e
lambda = diag(D); p = -d./lambda, e = c + sum(d.^2./lambda)

p =           e =
    -0.6667       3.0000
     0.6667
    -0.3333

% scaling -> sum_k z_k^2/a_k^2 = 1
% lambda_k z_k^2 -> lambda_k z_k^2/e =: z_k^2/a_k^2
a = sqrt(e./lambda)   % lengths

a = 1.0000
    0.7071
    0.5774
```

```
% midpoint
m = Q*p

m =
    1.0000
    0.0000
    0.0000
```

10.31 Normal Form and Type of a parametrized Quadric

Determine the normal form and the type of the quadric

$$Q: x^{\mathrm{t}} \begin{pmatrix} 1 & 0 & t \\ 0 & t & 1 \\ t & 1 & 0 \end{pmatrix} x = t\,,$$

depending on the parameter $t \in \mathbb{R}$.

Resources: Euclidean Normal Form of Three-Dimensional Quadrics

Solution

Eigenvalues λ_k

$$Q:\; x^{\mathrm{t}}Ax = t, \quad A = \begin{pmatrix} 1 & 0 & t \\ 0 & t & 1 \\ t & 1 & 0 \end{pmatrix}$$

$\sum_k a_{j,k} = 1 + t \;\forall j \quad \Longrightarrow \quad \lambda_1 = 1 + t$ with eigenvector $(1,1,1)^{\mathrm{t}}$

$\lambda_1 + \lambda_2 + \lambda_3 = \operatorname{trace} A = 1 + t \quad \Longrightarrow \quad \lambda_3 = -\lambda_2$

$\lambda_1\lambda_2\lambda_3 = \det A = -1 - t^3 \quad \Longrightarrow$

$$\lambda_2^2 = \lambda_2(-\lambda_3) = -\frac{\det A}{\lambda_1} = \frac{1+t^3}{1+t} = t^2 - t + 1 = \left[(t-1/2)^2 + 3/4\right]$$

$\Longrightarrow \quad \forall t\colon\; \lambda_2 = \varrho = \sqrt{[\ldots]},\; \lambda_3 = -\varrho$ with $\varrho \geq \sqrt{3}/2$

Normal form and type

transformation to diagonal form ($x = Vy$ with V the matrix of eigenvectors) $\rightsquigarrow$

$$(1+t)y_1^2 + \varrho y_2^2 - \varrho y_3^2 = t \tag{1}$$

- type for $t = 0$:
  double cone (3 eigenvalues $\neq 0$, not all of the same sign)

- type for $t \neq 0$:
  division of (1) by $t \quad \rightsquigarrow \quad$ normal form

  $$(1/t + 1)y_1^2 + (\varrho/t)y_2^2 - (\varrho/t)y_3^2 = 1$$

  type determined by the signs of the coefficients $1/t + 1$, ϱ/t, $-\varrho/t$:

$t < -1$:	$+ - +$	one-sheeted hyperboloid
$t = -1$:	$0 - +$	hyperbolic cylinder
$-1 < t < 0$:	$- - +$	two-sheeted hyperboloid
$0 < t$:	$+ + -$	one-sheeted hyperboloid

Chapter 11

Lexicon

11.1 Groups and Fields

Group

$(G, \diamond)$, neutral element e

$$(a \diamond b) \diamond c = a \diamond (b \diamond c), \quad a \diamond e = e \diamond a = a, \quad a \diamond a^{-1} = e$$

commutative if $a \diamond b = b \diamond a$

group table: matrix A with $a_{j,k} = g_j \diamond g_k$

Subgroup: $U \subseteq G$ with $u_1 \diamond u_2 \in U$ and $u^{-1} \in U$ for any elements of U

$|U|$ divides $|G|$

Permutation

bijective map $\{1, \ldots, n\} \to \{1, \ldots, n\}$, e.g.

$$p : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 5 & 4 & 1 & 2 \end{pmatrix}, \quad 1 \mapsto 3,\ 2 \mapsto 5,\ \ldots$$

representation via composition of transpositions or cycles

$$p = \underbrace{(1\,3) \circ (3\,4) \circ (2\,5)}_{m=3\,\text{transpositions}} = \underbrace{(1\,3\,4)}_{3-\text{cycle}}\ \underbrace{(2\,5)}_{2-\text{cycle}}$$

signum: $\sigma(p) = (-1)^m$ or, alternatively, $m = \sum_k (m_k - 1)$ where m_k are the lengths of the cycles ($m_1 = 3$, $m_2 = 2$ in the example)

Field

- $(F, +)$: (additive) commutative group with zero element 0
- $(F \backslash \{0\}, \cdot)$: (multiplicative) commutative group with neutral element 1
- distributive law: $a \cdot (b + c) = a \cdot b + a \cdot c$

Prime Field: $\mathbb{Z}_p = \{0, 1, \ldots, p-1\}$ with addition and multiplication modulo a prime number p

Galois Field $\quad |F| < \infty \implies |F| = p^m$

Euclid's Algorithm

computes the greatest common divisor n_K of two positive integers $n_1 > n_2$ by successive division:

$$n_{k-1} = q_k n_k + \underbrace{n_{k+1}}_{<n_k}, \quad k = 2, \ldots, K,\ n_{K+1} = 0\,,$$

i.e. n_{k+1} is the remainder when n_{k-1} is divided by n_k

MATLAB® / Maple™ `gcd(n_1,n_2)`

Chinese Remainder Theorem

solution x of $x = a_k \bmod p_k$ for pairwise prime natural numbers p_k:

$$\{0, \ldots, P-1\} \ni x = \sum_k a_k Q_k P_k \bmod P$$

with $P = \prod p_k$, $P_k = P/p_k$ and $Q_k P_k = 1 \bmod p_k$

11.2 Vector Spaces, Scalar Products, and Bases

Vector Space
commutative group $(V,+)$ of vectors v which can be multiplied with scalars s from a field F, according to the rules

$$\begin{array}{rclrcl} (s_1+s_2)v &=& s_1v+s_2v, & s(v_1+v_2) &=& sv_1+sv_2 \\ (s_1s_2)v &=& s_1(s_2v), & 1v &=& v \end{array}$$

$\mathbb{R}^n$ ($\mathbb{C}^n$): vector space of real (complex) n-tupels $v=(v_1,\ldots,v_n)^{\mathrm{t}}$

Subspace: subset $U\subset V$ with

$$u,v\in U \implies u+v\in U, \quad s\in K, u\in U \implies su\in U$$

Linear Combination
sum of scalar multiples of vectors:

$$v=s_1v_1+\cdots+s_mv_m$$

Linear Span: subspace $\operatorname{span}(U)$ of all linear combinations of elements from U

Convex Combination
linear combination with nonnegative scalars which sum to 1:

$$v=s_1v_1+\cdots+s_mv_m, \quad s_k\geq 0, \sum_k s_k=1$$

Convex Hull: smallest convex set $\operatorname{conv}(U)$ containing U

Linear Independence
Vectors $v_1,\ldots,v_m$ of a vector space V are

- **linearly independent**, if

$$s_1v_1+\cdots+s_mv_m=0_V \quad \implies \quad s_1=\cdots=s_m=0$$

- **linearly dependent**, if 0_V can be represented as nontrivial linear combination (not all scalars equal to 0) of the vectors v_k.

Basis

Vectors $b_1, \ldots, b_n$ form a basis of a vector space V, if every vector $v \in V$ can be uniquely represented as linear combination of the vectors b_k.
Dimension: $n = \dim V$

Scalar Product

$(u, v) \mapsto \langle u, v\rangle \in F$ with $F = \mathbb{R}$ $(\mathbb{C})$ for a real (complex) vector space
properties:

$$\langle v, v\rangle > 0 \quad \text{for } v \neq 0_V, \quad \langle u, sv + tw\rangle = s\langle u, v\rangle + t\langle u, w\rangle$$

- real: symmetric, i.e. $\langle u, v\rangle = \langle v, u\rangle$
  ($\implies$ linear also with respect to the first argument)
- complex: skew symmetric, i.e. $\langle u, v\rangle = \overline{\langle v, u\rangle}$
  ($\implies$ $\langle su + tv, w\rangle = \bar{s}\langle u, w\rangle + \bar{t}\langle v, w\rangle$)

Euklidean Scalar Product for the vector spaces $\mathbb{R}^n$ and $\mathbb{C}^n$:

$$\langle x, y\rangle = x^* y = \bar{x}^{\mathrm{t}} y = \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n$$

no complex conjugation for a real Euclidean scalar product ($x^* = x^{\mathrm{t}}$, $\bar{x}_k = x_k$)

Angle: defined for a real scalar product

$$\cos \sphericalangle(u, v) = \frac{\langle u, v\rangle}{|u|\,|v|}$$

Norm

$V \ni v \mapsto \|v\| \in \mathbb{R}$

$$\|v\| > 0 \quad \text{for } v \neq 0_V, \quad \|sv\| = |s|\|v\|, \quad \|u + v\| \leq \|u\| + \|v\|$$

Scalar Product Norm: $|v| = \sqrt{\langle v, v\rangle}$
special norms for $\mathbb{R}^n$ and $\mathbb{C}^n$:

$$|z| = \|z\|_2 = \sqrt{|z_1|^2 + \cdots + |z_n|^2}, \quad \|z\|_\infty = \max_k |z_k|, \quad \|z\|_1 = \sum_k |z_k|$$

MATLAB® `norm(z), norm(z,inf), norm(z,1)`

Cauchy-Schwarz Inequality
$|\langle u, v\rangle| \le |u||v|$ with equality if $u \parallel v$

Orthogonal Basis
$\langle u_j, u_k\rangle = 0,\ j \ne k$

$$v = \sum_k \underbrace{\frac{\langle u_k, v\rangle}{|u_k|^2}}_{c_k} u_k, \quad |v|^2 = \sum_k |c_k|^2 |u_k|^2$$

Orthonormal Basis: $|u_k| = 1 \quad \rightsquigarrow \quad$ simplified formulas

Orthogonal Projection
$V \ni v \mapsto P_U(v) \in U$

$$P_U(v) = \sum_{k=1}^{m} \frac{\langle u_k, v\rangle}{\langle u_k, u_k\rangle} u_k, \quad v \in V\,,$$

for an orthogonal basis $\{u_1, \ldots, u_m\}$ of a subspace $U \subset V$

Gram-Schmidt Algorithm
basis $\{b_1, \ldots, b_n\} \to$ orthogonal basis $\{u_1, \ldots, u_n\}$

$$u_j = b_j - \sum_{k<j} \frac{\langle u_k, b_j\rangle}{\langle u_k, u_k\rangle} u_k, \quad j = 1, \ldots, n$$

11.3 Linear Maps and Matrices

Linear Map

$L: V \to W$

$$L(u+v) = L(u) + L(v), \quad L(sv) = sL(v)$$

uniquely determined by the images $L(b_k) \in W$ of a basis $\{b_1, b_2, \ldots\}$ of V

Matrix of a Linear Map

$$V \ni v \mapsto Lv = w \in W \quad \Longleftrightarrow \quad w_j = \sum_k a_{j,k} v_k \quad \text{i.e.} \quad w = Av$$

with $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ the coordinates of v and w with respect to bases $\{e_1, \ldots, e_n\}$ and $\{f_1, \ldots, f_m\}$ of V and W

k-th column of A: coordinates of $L(e_k)$, i.e. $L(e_k) = a_{1,k} f_1 + \cdots + a_{m,k} f_m$

Affine Map

linear map with translation: $v \mapsto L(v) + w$ or $x \mapsto Ax + b$ if the matrix representation of L is used

Change of Basis

$v = \sum_{k=1}^n x_k e_k \to v = \sum_{j=1}^n y_j f_j \quad \Longleftrightarrow$

$$y = Ax \quad \text{i.e.} \quad y_j = \sum_{k=1}^n a_{j,k} x_k \quad \text{with} \quad e_k = \sum_{j=1}^n a_{j,k} f_j\,,$$

i.e. $a_{1,k}, \ldots, a_{n,k}$ are the coordinates of the basis vectors e_k with respect to the basis $\{f_1, \ldots, f_n\}$

$e_k, f_j \in K^n \quad \rightsquigarrow \quad$ formula for the transformation matrix A in terms of the matrices with the basis vectors as columns:

$$A = (f_1, \ldots, f_n)^{-1} (e_1, \ldots, e_n)$$

Range and Kernel

$L: V \to W$ linear

$$v \in \underbrace{\ker L}_{\text{kernel}} \subseteq V \iff Lv = 0_W, \quad w \in \underbrace{\operatorname{ran} L}_{\text{range}} \subseteq W \iff \exists v: Lv = w$$

$\dim V < \infty$: $\dim V = \dim \ker L + \dim \operatorname{ran} L$

Inverse of a Linear Map

injectivity $L : V \to W \iff$ existence of $L^{-1} : \operatorname{ran} L \to V \iff$

$$Lv = 0_W \implies v = 0_V, \quad \text{i.e. } \ker L = 0_V$$

Matrix Multiplication

$$\underbrace{C}_{\ell\times n} = \underbrace{A}_{\ell\times m}\underbrace{B}_{m\times n}, \quad c_{i,k} = \sum_{j=1}^{m} a_{i,j}b_{j,k} \quad \text{„row } i \cdot \text{column } k\text{“}$$

The number of columns of A must coincide with the number of rows of B.

Inverse Matrix

$$\det A \neq 0 \quad \implies \quad \exists A^{-1} : AA^{-1} = A^{-1}A = E \quad \text{(unit matrix)}$$

$(AB)^{-1} = B^{-1}A^{-1}$, $(A^{\mathrm{t}})^{-1} = (A^{-1})^{\mathrm{t}}$, $(A^*)^{-1} = (A^{-1})^*$

Transposition and Conjugation of a Matrix

$$B = A^{\mathrm{t}},\ b_{j,k} = a_{k,j}, \quad C = A^* = \bar{A}^{\mathrm{t}},\ c_{j,k} = \bar{a}_{k,j}$$

$(PQ)^\diamond = Q^\diamond P^\diamond$, $(A^\diamond)^{-1} = (A^{-1})^\diamond$ with $\diamond = \mathrm{t}, *$
symmetric: $A = A^{\mathrm{t}}$, **hermitian**: $A = A^*$

Trace

A: $n \times n$ matrix with eigenvalues λ_k

$$\operatorname{trace} A = \sum_{k=1}^{n} a_{k,k} = \sum_{k=1}^{n} \lambda_k$$

$\operatorname{trace}(AB) = \operatorname{trace}(BA), \quad \operatorname{trace}(Q^{-1}AQ) = \operatorname{trace} A$

Rank

number of pivots of the row echelon form

$$\operatorname{rank} A = \dim \underbrace{\operatorname{span}(u_1, \ldots, u_m)}_{U} = \dim \underbrace{\operatorname{span}(v_1, \ldots, v_n)}_{V},$$

where u_j^{t} denote the rows and v_k the columns of the $m \times n$ matrix A

$U = (\ker A)^\perp$, $V = \operatorname{ran} A$, $n = \dim U^\perp + \dim V$

Norm of a Matrix

norm associated with a vector norm $\| \ \|$:

$$\|A\| = \sup_{\|x\|\neq 0} \frac{\|Ax\|}{\|x\|} = \max_{\|x\|=1} \|Ax\|$$

submultiplicative, i.e. $\|AB\| \leq \|A\|\|B\|$
Maximum Norm: $\|A\|_\infty = \max_j \sum_k |a_{j,k}|$
Euclidean Norm: $\|A\|_2 = \max\{\sqrt{\lambda} : \lambda \text{ is an eigenvalue of } A^*A\}$ (largest singular value)

Frobenius Norm: $\|A\|_F = \left(\sum_{j,k} |a_{j,k}|^2\right)^{1/2}$ (not submultiplicative)

MATLAB® `norm(A,inf), norm(A), norm(A,'fro')`

Unitary Matrix

$$A^{-1} = \overline{A}^{\mathrm{t}} = A^* \quad \Longleftrightarrow \quad |Av| = |v| \quad \forall v \in \mathbb{C}^n$$

$A^{-1} = A^{\mathrm{t}}$ for real matrives ($\rightarrow$ **orthogonal matrix**)
The columns of A (as well as the rows) form an orthonormal basis.

Normal Matrix

$$AA^* = A^*A \,(\text{complex}), \quad AA^{\mathrm{t}} = A^{\mathrm{t}}A \,(\text{real})\,,$$

e.g., unitary (real: orthogonal) and hermitian (real: symmetric) matrices

Cyclic Matrix

$C : c_{j,k} = a_{j-k \bmod n}$ (successive translation of the first column) $(a_0, a_1, \ldots, a_{n-1})^{\mathrm{t}}$)
compatible with transposition, multiplication, and inversion

Positive Definite Matrix

$$v^*Av > 0 \quad \forall v \neq 0_n, \quad v^* = v^{\mathrm{t}} \text{ for real vectors}$$

positive diagonal elements and eigenvalues

Matrix Operations with MATLAB®

```
% definition of vectors and matrices
c = [1; 2; 3], r = [1 2 3], r = c'   % column/row
A = [11 12 13; 21 22 23]

% indexing
A(j,k), A(:,k), A(ind_j,ind_k)

% matrix multiplication
A*B

% pointwise operations
C = A.*B, C = A.^B, C = A./B   % c_j,k = a_j,k*b_j,k, etc.

% matrix functions
det, diag, inv, rank, trace, tril, triu
```

Matrix Operations with Maple™

```
with(LinearAlgebra)

# definition of vectors and matrices
c := Vector([1,2,3]); r = Vector[row]([1,2])
A := Matrix([[11 12 13],[21 22 23]])

% multiplication of vectors and matrices
MatrixVectorMultiply(A,c)
VectorMatrixMultiply(r,A)
MatrixMatrixMultiply(A,Transpose(A))

% some matrix functions
Determinant, MatrixAdd, MatrixInverse, MatrixNorm,
MatrixScalarMultiply, Rank, VectorAdd, VectorNorm,
VectorScalarMultiply
```

11.4 Determinants

Determinant

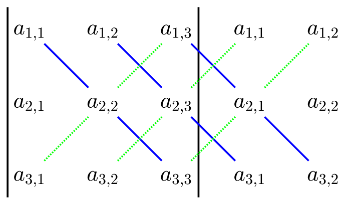
$$|A| = \det A = \det(a_1, \ldots, a_n) = \sum_{\text{permutations}\, p} \sigma(p)\, a_{p(1),1} \cdots a_{p(n),n}$$

$|\det A|$: volume of the parallelepiped $A[0,1]^n$,
spanned by the columns $a_1, \ldots, a_n$ of A (parallelogram for $n = 2$)
- multilinear: $\det(\ldots, sa_k + tb_k, \ldots) = s\det(\ldots, a_k, \ldots) + t\det(\ldots, b_k, \ldots)$
- antisymmetric: $\det(\ldots, u, \ldots, v, \ldots) = -\det(\ldots, v, \ldots, u, \ldots)$,
  $= 0$ if $u = sv$
- unit matrix: $\det E = \det(e_1, \ldots, e_n) = 1$

$\det A = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$ for a 2×2 matrix

Sarrus Scheme for a 3×3 determinant:
products of the blue minus products of the green diagonals, i.e.

$$\det A = a_{1,1}a_{2,2}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2}$$
$$-a_{1,3}a_{2,2}a_{3,1} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3}$$



MATLAB® `det(A)`

Maple™ `Determinant(A)`

Properties of Determinants
adding multiples of rows and/or columns of A ($\det A$ unchanged) and permutations ($\det A \cdot (-1)$) $\rightsquigarrow$ triangular matrix D and

$$\det(a_1, \ldots, a_n) = (-1)^\ell \det D = (-1)^\ell d_{1,1} d_{2,2} \cdots$$

with ℓ the number of row and column permutations

- $\{\overbrace{a_1, \ldots, a_n}^{\text{columns}}\}$ basis $\iff$ $\det A \neq 0$
- $\det(AB) = (\det A)(\det B)$
- $\det A = \det A^{\mathrm{t}}$, $\det(A^{-1}) = (\det A)^{-1}$, $\det(sA) = s^n \det A$

Expansion of Determinants

$$\begin{aligned} \det A &= \sum_{k=1}^{n} (-1)^{j+k} a_{j,k} \det \tilde{A}_{j,k} \quad \text{(expansion with respect to row } j\text{)} \\ &= \sum_{j=1}^{n} (-1)^{j+k} a_{j,k} \det \tilde{A}_{j,k} \quad \text{(expansion with respect to column } k\text{)} \end{aligned}$$

with $\tilde{A}_{j,k}$ the matrix after deleting row j and column k of A

11.5 Linear Systems

Linear System

$$\begin{array}{ccccccc} a_{1,1}x_1 & + & \cdots & + & a_{1,n}x_n & = & b_1 \\ & & & & & \cdots & \\ a_{m,1}x_1 & + & \cdots & + & a_{m,n}x_n & = & b_m \end{array} \quad \Longleftrightarrow \quad Ax = b$$

$U = \ker A$: subspace of the solutions of the homogeneous system $Ax = 0_m = (0, \ldots, 0)^{\mathrm{t}}$, $\dim U = n - \operatorname{rank} A$, $\operatorname{rank} A = \dim \operatorname{range} A$

- unique solution: $b \in \operatorname{range} A$ and $U = 0_n$
- unique solution for all b: A quadratic ($m = n$) and $\det A \neq 0$
  $\Longrightarrow \quad x = A^{-1}b$
- no solution: $b \notin \operatorname{range} A$
- infinitely many solutions: $b \in \operatorname{range} A$, $\dim U \geq 1$
  $\Longrightarrow \quad x \in v + U$ with v a particular solution

MATLAB® `x = A\b` (least squares solution)

Maple™ `x := LinearSolve(A,b)`

Cramer's Rule

- solution of $Ax = b$, $A = (a_1, \ldots, a_n)$:

$$x_j = \det \underbrace{(a_1, \ldots, a_{j-1}, b, a_{j+1}, \ldots, a_n)}_{\text{column } a_j \text{ of } A \text{ replaced by } b} \,/\, \det A$$

- inverse matrix $C = A^{-1}$:

$$c_{j,k} = \underbrace{(-1)^{j+k} \det \tilde{A}_{k,j}}_{\text{cofactor}} \,/\, \det A$$

with $\tilde{A}_{k,j}$ obtained from A by deleting row k and column j[1]

[1]Note the permutation of indices: $c_{j,k} \leftrightarrow \tilde{A}_{k,j}$.

2 × 2 matrix:

$$x_1 = \frac{b_1 a_{2,2} - b_2 a_{1,2}}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}}, \quad x_2 = \frac{b_2 a_{1,1} - b_1 a_{2,1}}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}}$$

and

$$\begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} = \frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}$$

Gauß Elimination

algorithm for solving $Ax = b$

- permuting and adding multiples of rows in the tableau $A|b$
  $\rightsquigarrow$ triangular form $D|c$ $(d_{j,k} = 0,\ j > k)$
  typical elimination step:
  subtracting a pivot row $(0, \ldots, 0, a_{k,k}, \ldots, a_{k,n}|b_k)$ multiplied with $(a_{j,k}/a_{k,k})$
  from a row $(0, \ldots, 0, a_{j,k}, \ldots, a_{j,n}|b_j)$ $\rightsquigarrow$ 0 in position (j, k)
  Gauß-Jordan Algorithm generating zeros in the elimination step
  also above the diagonal $(j < k)$ $\rightsquigarrow$ diagonal matrix D

- backward substitution $\rightsquigarrow$ $x_n, x_{n-1}, \ldots$
  typical step:
  solving the equation $d_{k,k}x_k + \cdots + d_{k,n}x_n = c_k$ for x_k and substituting
  the previously computed values $x_{k+1}, \ldots, x_n$

Echelon Form

generalized triangular form

$$Ax = b \quad \overset{\text{Gauß operations}}{\Longrightarrow} \quad Dx = c$$

$d_{j,k} = 0$ for $j > r = \operatorname{rank} A$, strictly increasing number of leading zeros for the nontrivial rows, i.e.

$$D(k, :) = (0, \ldots, 0, \underbrace{d_{k,i_k}}_{\text{pivots} \neq 0}, \ldots), \quad i_1 < i_2 < \cdots < i_r$$

existence of solutions $\iff$ $0 = c_{r+1} = c_{r+2} = \cdots,$ arbitrary choice of the unknowns x_ℓ, $\ell \neq i_k$

Reduced Echelon Form $d_{k,i_k} = 1$ and $d_{j,i_k} = 0$ also for $j < i_k$ (the pivot is the only nonzero entry of the column i_k)

MATLAB® `rref(A)`

Maple™ `ReducedRowEchelonForm(A)`

Linear Iteration

iterative solution of a linear system $Ax = b$:

$$x^{\text{new}} = (\underbrace{E - CA}_{Q})x^{\text{old}} + Cb$$

with E the unit matrix and C a suitable approximation of A^{-1}
convergence $\iff$ spectral radius (maximal absolute value of the eigenvalues) of Q less than 1

Jacobi Iteration

iterative solution of a linear system $Ax = b$ with the iteration

$$x_j^{\text{new}} = (b_j - \sum_{k \neq j} a_{j,k} x_k^{\text{old}})/a_{j,j}$$

convergence for diagonally dominant matrices, i.e. if $|a_{j,j}| > \sum_{k \neq j} |a_{j,k}|$

Gauß-Seidel Iteration

iterative solution of a linear system $Ax = b$ with the iteration

$$x_j^{\text{new}} = (b_j - \sum_{k<j} a_{j,k} x_k^{\text{new}} - \sum_{k>j} a_{j,k} x_k^{\text{old}})/a_{j,j}, \quad j = 1, 2, \ldots$$

convergence for symmetric, positive definite matrices ($\rightarrow$ application to least squares problems)

Relaxation (SOR: successive-over-relaxation)

$$x^{\text{new}} = x^{\text{old}} + \omega(x^{\text{new}} - x^{\text{old}})$$

with $\omega > 1$ suitably chosen to accelerate convergence

11.6 Eigenvalues and Normal Forms

Eigenvalues and Eigenvectors

$$Av = \lambda v, \quad v \in \text{Eigenspace}\, V_\lambda \quad \Longleftrightarrow \quad (A - \lambda E)v = (0, \ldots, 0)^{\mathrm{t}}$$

Spectral Radius $\varrho(\mathbf{A})$: largest absolute value of the eigenvalues

MATLAB® `[V,D] = eig(A)` → eigenvectors `V(:,k)` and eigenvalues `D(k,k`, i.e. $AV = VD$

Maple™ `Lambda := Eigenvalues(A)` → vector of eigenvalues `Lambda,V := Eigenvectors(A)` → vector of eigenvalues `Lambda(k)` and eigenvectors `Column(V,k)`

Similarity Transformation

$$A \to B = Q^{-1}AQ$$

invariant: eigenvalues, determinant, trace
v eigenvector of A $\Longleftrightarrow$ $Q^{-1}v$ eigenvector of B

Characteristic Polynomial

$$p_A(\lambda) = \det(A - \lambda E) = \begin{vmatrix} a_{1,1} - \lambda & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} - \lambda \end{vmatrix}$$

zeros = eigenvalues

Algebraic and Geometric Multiplicity
m_λ: multiplicity of λ as zero of $p_A(\lambda) = \det(A - \lambda E)$
d_λ: dimension of the eigenspace V_λ

$$d_\lambda = n - \text{Rang}(\underbrace{A}_{n\times n} - \lambda E) \le m_\lambda, \quad \sum_\lambda m_\lambda = n$$

Sum and Product of Eigenvalues

$$\sum_{k=1}^{n} \lambda_k = \text{trace}\, \underbrace{A}_{n\times n}, \quad \prod_{k=1}^{n} \lambda_k = \det A$$

$n = 2$: $\lambda_1 + \lambda_2 = a_{1,1} + a_{2,2}$, $\lambda_1\lambda_2 = a_{1,1}a_{2,2} - a_{1,2}a_{2,1}$

Basis of Eigenvectors

$\rightsquigarrow$ transformation to diagonal form

$$V^{-1}AV = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \quad V = (v_1, \ldots, v_n)\,,$$

with $\{v_1, \ldots, v_n\}$ a basis of eigenvectors of A and λ_k, $k = 1, \ldots, n$, the corresponding eigenvectors

Diagonal Form of Cyclic Matrices

$A = (a_{j-k \bmod n})_{j,k=0,\ldots,n-1}$

eigenvalues: $\lambda_\ell = \sum_{k=0}^{n-1} a_k w^{-k\ell}$, $\ell = 0, \ldots, n-1$, $\quad w = \exp(2\pi \mathrm{i}/n)$

eigenvectors: $(1, w^\ell, w^{2\ell}, \ldots, w^{(n-1)\ell})^{\mathrm{t}}$ (columns of the Fourier matrix)

Unitary Diagonalization

$A^*A = AA^*$ (normal matrix) $\iff$

$$U^*AU = \operatorname{diag}(\lambda_1, \ldots, \lambda_n), \quad U = (u_1, \ldots, u_n),\ U^* = U^{-1} \text{ unitary}$$

with $\{u_1, \ldots, u_n\}$ an orthonormal basis of eigenvectors of A corresponding to the eigenvalues λ_k

Diagonal Form of Hermitian Matrices

$A = A^* = \overline{A}^{\mathrm{t}}$ ($A = A^{\mathrm{t}}$ for real matrices) $\iff$

$$A = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U^*, \quad U = (u_1, \ldots, u_n)$$

with real eigenvalues λ_k and orthonormal eigenvectors u_k

Rayleigh Quotient

$S^* = S$, $x^*Sx \geq 0$ (Hermitian, positive semidefinite) $\implies$ The maximal and minimal eigenvalue are the extremal values of

$$r_S(x) = \frac{x^*Sx}{x^*x}, \quad x \neq (0, \ldots, 0)^{\mathrm{t}}\,.$$

Triangular Form

$\longleftarrow$ unitary similarity transformation

$$U^*AU = \begin{pmatrix} \lambda_1 & \cdots & r_{1,n} \\ & \ddots & \vdots \\ 0 & & \lambda_n \end{pmatrix}, \quad U^* = \overline{U}^{\mathrm{t}} = U^{-1}$$

$U^{-1} = U^{\mathrm{t}}$ (orthogonal) for a real $n \times n$ matrix A

Jordan Form

$\longleftarrow$ similarity transformation with eigenvectors and generalized eigenvectors as columns of the transformation matrix Q

$$J = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix} = Q^{-1}AQ, \quad J_\ell = \underbrace{\begin{pmatrix} \lambda_\ell & 1 & & 0 \\ & \ddots & \ddots & \\ & & \lambda_\ell & 1 \\ 0 & & & \lambda_\ell \end{pmatrix}}_{\text{Jordan block}}$$

MATLAB® `[V,J] = jordan(A)` $\rightarrow$ eigenvectors and principal vectors `V(:,k)` and jordan block matrix `J`, i.e. $AV = VJ$

Maple™ `J,V := JordanForm(A,output=['J','Q'])`

Convergence of Matrix Powers

basis $\{v_1, v_2, \ldots\}$ of eigenvectors to the eigenvalues λ_k of A with $|\lambda_1| > |\lambda_2| > \cdots$ (dominant largest eigenvalue) $\Longrightarrow$

$$A^n(\underbrace{c_1}_{\neq 0} v_1 + c_2 v_2 + \cdots) = \lambda_1^n (c_1 v_1 + o(1)) \quad \text{for}\, n \to \infty$$

$|\lambda_k| < 1 \,\forall\, k \quad \Longrightarrow \quad A^n \underset{n\to\infty}{\longrightarrow}$ zero matrix

11.7 Least Squares and Singular Value Decomposition

Least Squares Line

$\sum_{k=1}^{n} (\underbrace{u + vt_k}_{\text{straight line}} - f_k)^2 \to \min$

$$u = \frac{(\sum t_k^2)(\sum f_k) - (\sum t_k)(\sum t_k f_k)}{n(\sum t_k^2) - (\sum t_k)^2}, \quad v = \frac{n(\sum t_k f_k) - (\sum t_k)(\sum f_k)}{n(\sum t_k^2) - (\sum t_k)^2}$$

Normal Equation

$$\underbrace{A^{\mathrm{t}} A}_{n\times n} x = A^{\mathrm{t}} b \quad \Longleftrightarrow \quad |\underbrace{A}_{m\times n} x - b| \to \min$$

unique solution x if rank $A = n$

MATLAB® `x = A\b` $\Longleftrightarrow$ `x = inv(A'*A)*A'*b`

Singular Value Decomposition

$$\underbrace{A}_{m\times n} = \underbrace{U}_{m\times m} \underbrace{S}_{m\times n} \underbrace{V^*}_{n\times n}, \quad S = \operatorname{diag}(s_1, \ldots, s_r, 0, \ldots, 0)$$

with unitary matrices U, V, $r = \operatorname{rank} A$ and $s_1 \geq \cdots \geq s_r > 0$ the singular values of A (eigenvalues of A^*A and AA^*)

$$Ax = \sum_{k=1}^{r} u_k s_k v_k^* x$$

with u_k (v_k) the columns of U (V)
$\|A\|_2 = s_1$, $\|A\|_F^2 = \sum_{j,k} |a_{j,k}|^2 = s_1^2 + \cdots + s_r^2$

MATLAB® `[U,S,V] = svd(A)`

Maple™ `U,S,V_transpose := SingularValues(A,ouput=['U','S','Vt'])`

Pseudoinverse

best Euclidean approximation A^+ of an inverse for arbitrary matrices A

$$\overbrace{\underbrace{A}_{m\times n} = U \underbrace{S}_{m\times n} V^*}^{\text{singular value decomposition}} \quad \Longrightarrow \quad A^+ = V \underbrace{S^+}_{n\times m} U^*$$

with $S^+ = \operatorname{diag}(1/s_1, \ldots, 1/s_r, 0, \ldots, 0)$ and $s_1 \geq s_2 \geq \cdots$ the singular values of A ($r = \operatorname{rank} A$)
$A^+ = (A^*A)^{-1}A^*$ if $r = n \leq m$
Minimum Norm Solution of the Least Squares Problem $|\mathbf{Ax} - \mathbf{b}| \to \min$:

$$x = A^+b = \sum_{k=1}^{r} v_k s_k^{-1} u_k^* x$$

MATLAB® `pinv(A)`

Maple™ `MatrixInverse(A,method = pseudo)`

11.8 Reflections and Rotations

Reflection

$x \mapsto Qx$, $Q = Q^{\mathrm{t}} = Q^{-1}$, $\det Q = -1$

$$Q = E - 2dd^{\mathrm{t}}, \quad |d| = 1$$

with d a normal vector of the reflecting plane

Rotation

$x \mapsto Qx$, $Q^{-1} = Q^{\mathrm{t}}$, $\det Q = 1$

- $x \in \mathbb{R}^2$, right-hand rotation (counterclockwise), angle φ:

$$Q = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

- $x \in \mathbb{R}^3$, right-hand rotation (oriented like a right-hand screw) with axis direction u, $|u| = 1$, and angle φ,

$$\begin{aligned} q_{j,k} &= \cos\varphi\,\delta_{j,k} \;+\; (1-\cos\varphi)\,u_j u_k \;+\; \sin\varphi \sum_{\ell} \varepsilon_{j,\ell,k} u_\ell \\ \iff \qquad Qx &= \cos\varphi\, x \;+\; (1-\cos\varphi)\, uu^{\mathrm{t}}x \;+\; \sin\varphi\, u \times x \end{aligned}$$

$$\cos\varphi = \frac{1}{2}(\operatorname{trace} Q - 1)$$

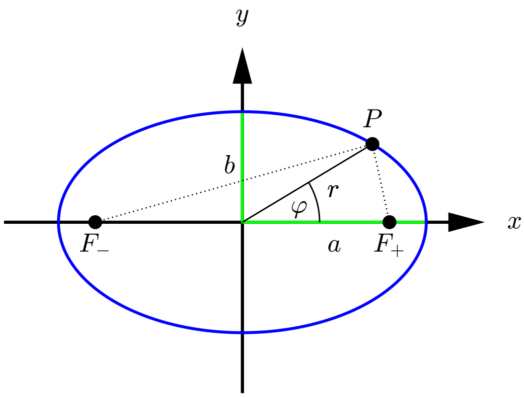
11.9 Conic Sections and Quadrics

Ellipse

constant sum of the distances of points P to two focal points $F_\pm = (\pm f, 0)$:

$$|\overrightarrow{PF_-}| + |\overrightarrow{PF_+}| = 2a$$

$$\iff \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad b^2 = a^2 - f^2$$



polar coordinates

$$r^2 = b^2/(1 - (f/a)^2 \cos^2 \varphi)$$

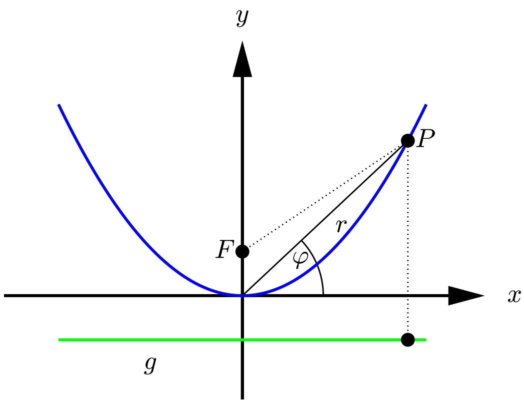
parametrization

$$x = a \cos t, \quad y = b \sin t$$

Parabola

equal difference of points P to a focal point $F = (0, f)$ and a straight line $g:\ y = -f$

$$4f\, y = x^2$$



polar coordinates

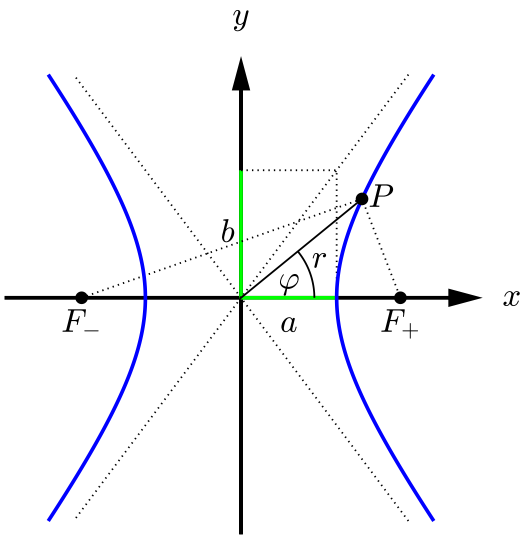
$$r = 4f \sin \varphi / \cos^2 \varphi$$

Hyperbola

constant difference of the distance to two focal points $F_\pm = (\pm f, 0)$:

$$\left| |\overrightarrow{PF_-}| - |\overrightarrow{PF_+}| \right| = 2a$$

$$\iff \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ mit } b^2 = f^2 - a^2$$



polar coordinates

$$r^2 = b^2/((f/a)^2 \cos^2 \varphi - 1)$$

parametrization

$$x = \pm a \cosh t, \quad y = b \sinh t$$

asymptotes with slopes $\pm b/a$

Quadric

$$Q:\ x^{\mathrm{t}} A x + 2b^{\mathrm{t}} x + c = \sum_{j,k} a_{j,k} x_j x_k + 2 \sum_k b_k x_k + c = 0, \quad A = A^{\mathrm{t}}$$

Transformation to Normal Form

rotation and translation, $x = U\xi + v$, and scaling of the equation of a quadric in $\mathbb{R}^n$ $\rightsquigarrow$ normal form:

$$x^{\mathrm{t}} A x + 2b^{\mathrm{t}} x + c = 0 \quad \underset{\text{alternatives}}{\rightarrow} \quad \left\{ \begin{array}{l} \sum_{k=1}^{r} \sigma_k \dfrac{\xi_k^2}{a_k^2} = \gamma \\ \sum_{k=1}^{r} \sigma_k \dfrac{\xi_k^2}{a_k^2} = 2\xi_{r+1} \end{array} \right.$$

with $r = \operatorname{rank} A$, $\sigma_k \in \{-1, 1\}$ and $\gamma \in \{0, 1\}$

columns of the rotation matrix U: normalized eigenvectors u_k (length $= 1$) of $A = A^{\mathrm{t}}$
translation vector v: midpoint of the quadric
principal axes: $x = v + t u_k$ with lengths a_k

Conic Section

double cone with tip p ($p_3 \neq 0$), direction v and opening angle α

$$C: \ (x-p)^{\mathrm{t}} v = \pm \cos(\alpha/2)\, |x-p||v|$$

intersection with the plane $E: \ x_3 = 0 \quad \rightsquigarrow \quad$ quadric in the $x_1 x_2$ plane
type determined by $\beta = \sphericalangle(E, v)$:
ellipse for $\beta > \alpha/2$, parabola for $\beta = \alpha/2$, hyperbola for $\beta < \alpha/2$

Euclidean Normal Form of Two-Dimensional Quadrics

- Ellipse: $\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} = 1$
- Parabola: $\dfrac{x_1^2}{a_1^2} = 2x_2$
- Hyperbola: $-\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} = 1$

degenerate cases: straight line(s), point, empty set

Euclidean Normal Form of Three-Dimensional Quadrics

- (double) cone: $\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} - \dfrac{x_3^2}{a_3^2} = 0$
- elliptic paraboloid: $\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} = 2x_3$
- hyperbolic paraboloid: $-\dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} = 2x_3$
- parabolic cylinder: $\dfrac{x_1^2}{a_1^2} = 2x_2$
- two-sheeted hyperboloid: $-\dfrac{x_1^2}{a_1^2} - \dfrac{x_2^2}{a_2^2} + \dfrac{x_3^2}{a_3^2} = 1$

- one-sheeted hyperboloid: $-\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$
- ellipsoid: $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} + \frac{x_3^2}{a_3^2} = 1$
- hyperbolic cylinder: $-\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1$
- elliptic cylinder: $\frac{x_1^2}{a_1^2} + \frac{x_2^2}{a_2^2} = 1$

degenerate cases: plane(s), straight line(s), point, empty set

Visualization of Surfaces with MATLAB®

```
% parametric surface S: j,k -> (X(j,k),Y(j,k),Z(j,k)),
% defined by 2-d arrays X,Y,Z with color values C
surf(X,Y,Z,C)

% implicit surface S: V(X,Y,Z)=d,
% defined by 3-d arrays X,Y,Z,V with color values C
isosurface(X,Y,Z,V,d,C)
```

Visualization of Surfaces with Maple™

```
% parametric surface S: u,v -> x,y,z,
% defined by expressions x,y,z in the variables u,v
plot3d([x,y,z],u=u0..u1,v=v0..v1)

% implicit surface S: f=0,
% defined by an expression f in the variables x,y,z
implicitplot3d(f=0,x=x0..x1,y=y0..y1,z=z0..z1)
```