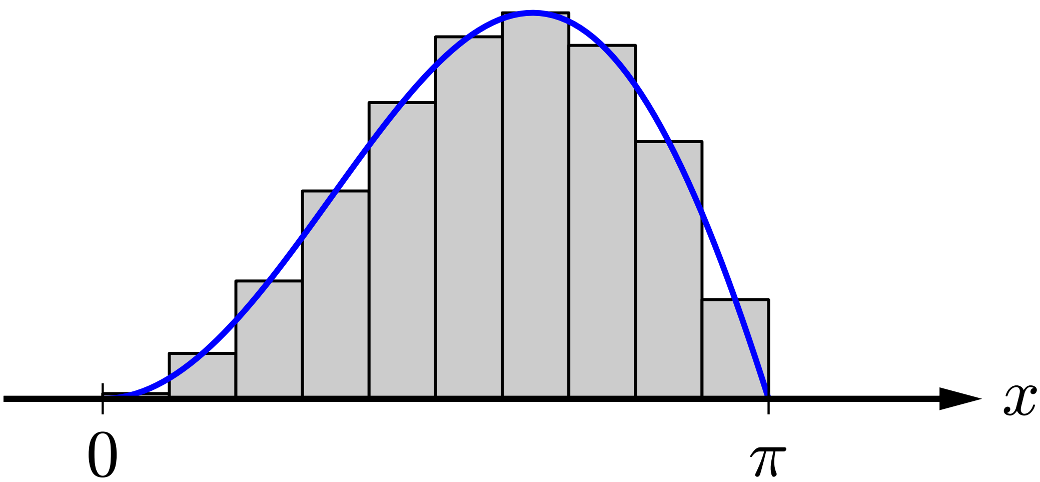
MathTraining

Integration:

Problems and Solutions



$$\int_0^\pi x \sin(x)\,\mathrm{d}x = \pi$$

Klaus Höllig

Partially translated from
Aufgaben und Lösungen zur Höheren Mathematik 1, 4. Auflage
by Klaus Höllig and Jörg Hörner

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Preface

The problem set of the book covers all basic topics of a course on *Integration*. It can be used to practice for exams, to facilitate the completion of homework assignments, and to review course material. Interactive variants to model problems with detailed solutions permit the student reader to test his comprehension of the relevant techniques. In addition to the collection of problems, a small mathematics lexicon contains brief descriptions of the relevant theorems, methods, and definitions.

There exists also a sportive aspect of mathematics - challenging problems requiring ideas beyond the standard techniques. The problems in the chapter *Calculus Highlights* are perhaps too difficult for undergraduates. They are included to initiate or strengthen fascination for mathematics. It is definitely not a mistake to practice substantially harder than necessary ...!

The book is partially translated from

Aufgaben und Lösungen zur Höheren Mathematik 1

by Jörg Hörner and the author. It supplements this textbook by providing detailed solutions to tests for the chapters on *Integration*. Moreover, the book includes additional problems, in particular problem variants for the topics of the tests.

The author wishes the readers success in their studies **and** hopes that mathematics will become one of their favorite subjects!

Klaus Höllig

Contents

Introduction

The book contains problems with detailed solutions, problem variants with interactive result verification, and a mathematics lexicon for the principal topics which are usually subject of a course on *Integration*:

- Elementary Integrals
- Rational Integrands
- Integration by Parts
- Substitution
- Solids of Revolution
- Improper Integrals

The problem set can be used to practice for exams, to facilitate the completion of homework assignments, and to deepen the comprehension of course material. How is this accomplished most effectively? Remembering his own student days, the author makes the following recommendations to a student reader.

Consider, as an example, a problem from the chapter on *Integration by Parts*:

3.3 Integrals of Products of Polynomials with Exponentials

Compute

$$\text{a)} \quad \int_0^1 x\,\mathrm{e}^x\,\mathrm{d}x \qquad \text{b)} \quad \int_1^2 (2x+1)^2\,\mathrm{e}^{2x}\,\mathrm{d}x$$

Resources: Integration by Parts, Table of Elementary Integrals

Before looking at the solution of the problem, review the relevant theory or methods (resources). Clicking on the links leads to the following brief descriptions of the relevant formulas from the *Lexicon* in chapter 7.

Integration by Parts

$$\int f'g = fg - \int fg', \quad \int_a^b f'g = [fg]_a^b - \int_a^b fg'$$

$[fg]_a^b = 0$, if fg vanishes at a and b, or, if the product is periodic with period $(b-a)$.

Typical Applications

- $g(x) = x^k$, $f(x) = \mathrm{e}^x, \cos x, \sin x$ $\rightsquigarrow$ successive reduction of the polynomial degree
- $g(x) = \ln^k x$, $f(x) = x^\ell$ $\rightsquigarrow$ elimination of the logarithm after repeated integration by parts

Table of Elementary Integrals

$f(x)$	$x^n,\ n \neq -1$	$\frac{1}{x}$	$\exp x$	$\ln x$	$\frac{1}{x^2+1}$
$F(x)$	$\frac{x^{n+1}}{n+1}$	$\ln\lvert x\rvert$	$\exp x$	$x\ln x - x$	$\arctan x$
$f(x)$	$\cos x$	$\sin x$	$\tan x$	$\cosh x$	$\sinh x$
$F(x)$	$\sin x$	$-\cos x$	$-\ln(\cos x)$	$\sinh x$	$\cosh x$

Try to solve the problem with these instructions. Then compare your computations with the solution given in the book:

Solution

For a product of a function g with a polynomial f, integrating by parts,

$$\int_a^b fg' = [fg]_a^b - \int_a^b f'g\,, \tag{1}$$

reduces the degree ($f \to f'$) and thus simplifies the integral, **provided** that a simple antiderivative of g' ($g' \to g$) exists.

a) $\int_0^1 x\mathrm{e}^x\,\mathrm{d}x$

noting that the exponential function does not change when differentiated or integrated, applying (1) with $f(x) = x$, $g'(x) = \mathrm{e}^x$ $\rightsquigarrow$

$$\begin{aligned}\int_0^1 \underbrace{x}_{f}\underbrace{\mathrm{e}^x}_{g'}\,\mathrm{d}x &= \Big[\underbrace{x}_{f}\underbrace{\mathrm{e}^x}_{g}\Big]_{x=0}^{x=1} - \int_0^1 \underbrace{1}_{f'}\underbrace{\mathrm{e}^x}_{g}\,\mathrm{d}x \\ &= (\mathrm{e}-0) - [\mathrm{e}^x]_{x=0}^{x=1} = \mathrm{e} - (\mathrm{e}-1) = 1\end{aligned}$$

b) $\int_0^1 (2x+1)^2\mathrm{e}^{2x}\,\mathrm{d}x$

noting that

$$\frac{\mathrm{d}}{\mathrm{d}x}u(px+q) = pu'(px+q), \qquad \int u(px+q)\,\mathrm{d}x = \frac{1}{p}U(px+q)$$

with $U(y)$ an antiderivative of $u(y)$, applying (1) $\rightsquigarrow$

$$\begin{aligned} s &:= \int_0^1 \underbrace{(2x+1)^2}_{f}\,\underbrace{\mathrm{e}^{2x}}_{g'}\,\mathrm{d}x \\ &= \underbrace{\left[(2x+1)^2\,\mathrm{e}^{2x}/2)\right]_{x=0}^{x=1}}_{A} - \underbrace{\int_0^1 2\cdot 2(2x+1)\,\mathrm{e}^{2x}/2\,\mathrm{d}x}_{B}\end{aligned}$$

- $A = 3^2\,\mathrm{e}^2/2 - 1/2 = 9\,\mathrm{e}^2/2 - 1/2$
- B is computed by integrating by parts a second time:

$$\begin{aligned} B &= \int_0^1 2(2x+1)\,\mathrm{e}^{2x}\,\mathrm{d}x = \left[2(2x+1)\,\mathrm{e}^{2x}/2\right]_{x=0}^{x=1} - \int_0^1 4\,\mathrm{e}^{2x}/2\,\mathrm{d}x \\ &= 3\,\mathrm{e}^2 - 1 - \left[\mathrm{e}^{2x}\right]_{x=0}^{1} = 3\,\mathrm{e}^2 - 1 - \,\mathrm{e}^2 + 1 = 2\mathrm{e}^2\end{aligned}$$

combining the results $\rightsquigarrow$

$$s = A - B = 9\,\mathrm{e}^2/2 - 1/2 - 2\,\mathrm{e}^2 = 5\,\mathrm{e}^2/2 - 1/2 \approx 17.9726$$

Remark

Noting that $a^x = \mathrm{e}^{(\ln a)x}$, integration by parts applies to integrands which are products of a^x with polynomials $p(x)$ as well.

The solutions are written in a keyword-like style, as you would employ when you comment your solutions in an exam or for homework problems. For example, the phrase

"multiplying with x $\rightsquigarrow$"

stands for "By multiplying with x we obtain". Other examples of typical phrases are "simplifying $\rightsquigarrow$ ..." or "alternative argument: ...". There is just as much detail included as is necessary for the mathematical arguments.

To gain more practice with the solution technique, it is highly recommended to solve some (preferably all ...) of the problem variants following the principal model problem for each topic. For *Integrals of Products of Polynomials with Exponentials* the variants are:

Problem Variants

■ a) $\int_0^1 (1+x)\,\mathrm{e}^{-x}\,\mathrm{d}x$, b) $\int_{-1}^0 x^2\,\mathrm{e}^x\,\mathrm{d}x$

---

a) ?.?? b) ?.??:

[ ] check

■ a) $\int_0^2 x\,\mathrm{e}^{-2x}\,\mathrm{d}x$, b) $\int_{-1}^1 (1+x)^2\,\mathrm{e}^{-x}\,\mathrm{d}x$

---

a) ?.?? b) ?.??:

[ ] check

■ a) $\int_0^1 (2+x)\,3^x\,\mathrm{d}x$, b) $\int_0^2 x^2\,2^x\,\mathrm{d}x$

---

a) ?.?? b) ?.??:

[ ] check

You can check your solution by typing your answer in the field adjacent to the check - box, replacing every question mark by a character (digit or letter). Convert your result to a decimal, truncated to the number of digits indicated. For example,

$$2/3 \to 0.6666\ldots \xrightarrow{?.??} 0.66, \qquad \text{answer}: \boxed{066}\,.$$

⚠ Note that the period is omitted; only the characters corresponding to the question marks are typed.

The solutions for the three problem variants are

(1) a) $2 - 3/\mathrm{e} \approx 0.8963$, b) $2 - 5/\mathrm{e} \approx 0.1606$

(2) a) $(1 - 5/\mathrm{e}^4)/4 \approx 0.2271$, b) $2\mathrm{e} - 10/\mathrm{e} \approx 1.7577$

(3) a) $-2/(\ln 3)^2 + 7/\ln 3 \approx 4.7146$, b) $16/\ln 2 - 16/(\ln 2)^2 + 6/(\ln 2)^3 \approx 7.7979$

Hence, the correct input is

(1) a) ?.?? b) ?.?? → 089016

(2) a) ?.?? b) ?.?? → 022175

(3) a) ?.?? b) ?.?? → 471779

As mentioned in the beginning, the problem set can also assist you in completing homework assignments. Just look for a similar problem and study its solution. Similarly, for methods and examples presented in class, practice with the relevant problems.

The above remarks pertain to the first six chapters, which exclusively discuss the solution of standard problems. Usually, such problems constitute the major portion of an exam or homework assignment. Hence, to review the basic techniques involved is of primary importance. Applying these techniques to more advanced problems is a natural next step. The chapter *Calculus Highlights* contains examples of rather challenging applications. You do not have to be disappointed if you cannot solve any of these problems; they are definitely very difficult. It is legitimate to immediately look at the solutions and learn how the methods from the previous chapters are applied in an advanced setting. Also, as mentioned in the Preface, it is not a mistake to practice substantially harder than necessary ...!

You have solved some of the problems in chapter 7 without resorting to the solutions. Then ...

... **you can take pride in your mastery of the principal techniques for solving problems concerning *Integration!***

With the previous explanations aimed at student readers, instructors could (obviously) also benefit from the interactive problem collection. The solutions of the model problems can be used as examples in class, and some of the variants assigned as homework problems. Students will welcome the possibility of checking results before submitting or presenting their solutions in the exercise sections.

Have all integration methods been described? **No: Two BEAUTIFUL techniques, Fourier Transforms and the Residue Theorem, are missing!**

Two examples,

$$\int_0^\infty \frac{\sin x}{x} \mathrm{e}^{-x}\,\mathrm{d}x = \frac{\pi}{4}, \quad \int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\,\mathrm{d}x = \frac{\pi}{\sqrt{3}},$$

provide an incentive to look at *Problems and Solutions for Fourier Analysis* and *Problems and Solutions for Complex Analysis*.

Disclaimer: Although the solutions and answers to the variants have been thoroughly checked, mistakes can always occur[1]. Please, write to the author (`Klaus.Hoellig@gmail.com`) if you find any errors.

[1] A statement by a teaching assistant to encourage students, which the author will always remember: "This year, the final exam is not too difficult - your professor could check the results without committing any errors!".

Chapter 1

Elementary Integrals

1.1 Integration of a Polynomial

Determine an antiderivative of

$$f(x) = 5x + 2x^3$$

and compute $\int_{-1}^{4} f$.

Resources: Table of Elementary Integrals, Properties of the Integral, Fundamental Theorem of Calculus

Problem Variants

■ $f(x) = 3 - 4x^2$

$\int_{-1}^{0} f =$?.??:

[ ] check

■ $f(x) = x - 3x^2 + 2x^3$

$\int_{2}^{3} f =$??:

[ ] check

■ $f(x) = (1 - 3x)^2$

$\int_{-2}^{0} f =$???:

[ ] check

Solution

Indefinite integral

linearity of the integral $\implies$

$$\int \underbrace{5x + 2x^3}_{f(x)} \,\mathrm{d}x = 5 \int x \,\mathrm{d}x + 2 \int x^3 \,\mathrm{d}x$$

applying the formula $\int x^n \,\mathrm{d}x = \frac{1}{n+1} x^{n+1} + C$ for an antiderivative of a monomial $\rightsquigarrow$

$$\int f(x) \,\mathrm{d}x = 5 \left(\frac{1}{2} x^2 \right) + 2 \left(\frac{1}{4} x^4 \right) + C = \frac{5x^2 + x^4}{2} + C =: F(x)$$

with $C \in \mathbb{R}$ an arbitrary integration constant

Definite integral

fundamental theorem of calculus,

$$\int_a^b f = [F]_a^b = F(b) - F(a)\,,$$

with F an antiderivative of f $(F' = f)$ $\implies$

$$\int_{-1}^4 5x + 2x^3 \,\mathrm{d}x = \left[\frac{5x^2 + x^4}{2} \right]_{x=-1}^{x=4} = \frac{5 \cdot 16 + 256}{2} - \frac{5+1}{2} = 165$$

1.2 Integration of a Polynomial Expression

Determine an antiderivative of

$$f(x) = (3x - 2)^4\,,$$

and compute $\int_0^1 f$.

Resources: Table of Elementary Integrals, Properties of the Integral, Fundamental Theorem of Calculus

Problem Variants

■ $f(x) = (x/4 - 4)^3$

$\int_{-4}^4 f = -???$:

[ ] check

■ $f(x) = (2 - 4x)^3$

$\int_0^1 f = ?$:

[ ] check

■ $f(x) = (1 - 2x)^3$

$\int_0^2 f = -??$:

[ ] check

Solution

Indefinite integral

$$\int g(\underbrace{px+q}_{y})\,\mathrm{d}x = \frac{1}{p}G(px+q)+C, \quad G'(y)=g(y)\,,$$

in view of the chain rule

application to $f(x)=(3x-2)^4$, i.e., with $p=3$, $q=-2$, $g(y)=y^4$ $\rightsquigarrow$ antiderivative

$$F(x)=\int(\underbrace{3x-2}_{y})^4\,\mathrm{d}x=\frac{1}{3}\underbrace{\frac{1}{5}y^5}_{G(y)}+C=\frac{1}{15}(3x-2)^5+C$$

Definite integral

fundamental theorem of calculus,

$$\int_a^b f = [F]_a^b = F(b)-F(a)$$

with F an antiderivative of f $\implies$

$$\int_0^1 (3x-2)^4\,\mathrm{d}x = \left[\frac{1}{15}(3x-2)^5\right]_{x=0}^{x=1} = \frac{1}{15}(3-2)^5-\frac{1}{15}(0-2)^5=\frac{11}{5}$$

Remark
The alternative solution method, expanding the power and integrating the resulting linear combination of monomials, is tedious, and obviously not recommended.

1.3 Integral of a Linear Combination of Powers

Compute $\int_0^3 \frac{2-x}{\sqrt{x+1}}\,\mathrm{d}x$.

Resources: Table of Elementary Integrals, Properties of the Integral, Fundamental Theorem of Calculus

Problem Variants

■ $\int_2^9 \frac{2+x}{\sqrt[3]{x-1}}\,\mathrm{d}x$

?.??:

[ ] check

■ $\int_0^4 \frac{x}{\sqrt{2x+1}}\,\mathrm{d}x$

?.??:

[ ] check

■ $\int_0^1 \frac{4x-x^2}{\sqrt[3]{x}}\,\mathrm{d}x$

?.??:

[ ] check

Solution

simplification of the integrand $f(x) = \dfrac{2-x}{\sqrt{x+1}}$ by expanding the numerator in powers of $(x+1)$:

$$2 - x = -(x+1) + 3 \quad \rightsquigarrow$$
$$f(x) = -\frac{x+1}{\sqrt{x+1}} + \frac{3}{\sqrt{x+1}} = -(x+1)^{1/2} + 3(x+1)^{-1/2}$$

applying the formula

$$\int (x - x_0)^s \mathrm{d}x = \frac{(x-x_0)^{s+1}}{s+1} + C, \quad s \neq -1\,,$$

for an antiderivative of a power $\quad\rightsquigarrow\quad$ antiderivative of f,

$$F(x) = -\frac{2}{3}(1+x)^{3/2} + 6(1+x)^{1/2} + C\,,$$

and

$$\begin{aligned}\int_0^3 f(x)\,\mathrm{d}x &= [F(x)]_{x=0}^{x=3} = \left(-\frac{2}{3}4^{3/2} + 6\cdot 4^{1/2}\right) - \left(-\frac{2}{3}1^{3/2} + 6\cdot 1^{1/2}\right) \\ &= \left(-\frac{16}{3} + 12\right) - \left(-\frac{2}{3} + 6\right) = \frac{4}{3}\end{aligned}$$

1.4 Integral of the Logarithm of a Polynomial

Compute $\displaystyle\int_2^6 \ln(x^2+2x-3)\,\mathrm{d}x$.

Resources: Fundamental Theorem of Calculus, Table of Elementary Integrals

Problem Variants

- $\int_2^3 \ln(x^3-x^2)\,\mathrm{d}x$

?.??:

[ ] check

- $\int_{-1}^1 \ln(4-x^2)\,\mathrm{d}x$

?.??:

[ ] check

- $\int_0^1 \ln(4+4x-x^2-x^3)\,\mathrm{d}x$

?.??:

[ ] check

Solution

For a polynomial p with real zeros x_k, an antiderivative F of $f(x) = \ln p(x)$ can be determined by factoring p,

$$p(x) = c(x - x_1)(x - x_2)\cdots,$$

rewriting $f(x)$ as

$$f(x) = \ln c + \ln(x - x_1) + \ln(x - x_2) + \cdots,$$

and applying the formula $\int \ln t \, \mathrm{d}t = t(\ln t - 1) + C$. A definite integral is computed with the fundamental theorem of calculus: $\int_a^b f = [F]_a^b$.

Application to $\int_2^6 \ln(\underbrace{x^2 + 2x - 3}_{p(x)}) \, \mathrm{d}x$:

<u>Factorization</u>

formula for the solutions of the quadratic equation $x^2 + 2x - 3 = 0 \quad \Longrightarrow$

$$x_{1,2} = \frac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot (-3)}}{2} = \frac{-2 \pm 4}{2},$$

i.e., $x_1 = -3$, $x_2 = 1$, and

$$p(x) = (x - 1)(x + 3)$$

<u>Antiderivative</u>

$\ln(ab) = \ln a + \ln b \quad \rightsquigarrow$

$$f(x) = \ln p(x) = \ln(x - 1) + \ln(x + 3)$$

$\int \ln t \, \mathrm{d}t = t(\ln t - 1) + C \quad \rightsquigarrow$

$$F(x) = (x - 1)(\ln(x - 1) - 1) + (x + 3)(\ln(x + 3) - 1) + C$$

<u>Integral</u>

$$\begin{aligned}\int_2^6 f = [F]_2^6 &= [(x - 1)(\ln(x - 1) - 1) + (x + 3)(\ln(x + 3) - 1)]_{x=2}^{x=6} \\ &= 5(\ln 5 - 1) + 9(\ln 9 - 1) - 1(\ln 1 - 1) - 5(\ln 5 - 1) \\ &= 18 \ln 3 - 8 \approx 11.7750\end{aligned}$$

1.5 Integration of an Exponential

Determine an antiderivative of

$$f(x) = 2^{3x+4}$$

and compute $\int_0^1 f$.

Resources: Table of Elementary Integrals

Problem Variants

- $\int_0^1 3^{4-2x}$

??.?:

check

- $\int_{-2}^2 2^{2-x/2}$

??.?:

check

- $\int_1^2 10^{2x-1}$

???:

check

Solution

simplification of the integrand:

$$f(x) = 2^{4+3x} = 2^4 \cdot 2^{3x} = 16 \cdot 8^x = 16\,\mathrm{e}^{\ln 8\,x}$$

determination of an antiderivative F:
$\int \mathrm{e}^{ax}\,\mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax}$ (in general, replacing x by ax $\rightsquigarrow$ factor $1/a$ in the antiderivative) $\Longrightarrow$

$$F(x) = \int f(x)\,\mathrm{d}x = \frac{16}{\ln 8}\,\mathrm{e}^{\ln 8\,x} = \frac{16}{\ln 8}\,8^x$$

integral over $[0, 1]$:

$$\begin{aligned}\int_0^1 f &= \left[\frac{16}{\ln 8}8^x\right]_{x=0}^{x=1} = \frac{16}{\ln 8}\cdot 8 - \frac{16}{\ln 8}\cdot 1 \\ &= \frac{16}{\ln 8}\cdot 7 = \frac{112}{3\ln 2} \approx 53.86\end{aligned}$$

1.6 Antiderivative of a Trigonometric Polynomial

Determine an antiderivative of

$$f(x) = \cos(2x)\sin(5x)\,\mathrm{d}x\,.$$

Resources: Table of Elementary Integrals, Properties of the Integral

Problem Variants

- $f(x) = \sin(2x)\sin^2 x$

(?−? cos(?x)+???(?x))/16:

[ ] check

- $f(x) = \sin(2x)\cos(2x)$

−(?+???(?x))/8:

[ ] check

- $f(x) = \cos(2x)\cos(3x)$

(????(x)+???(?x))/10:

[ ] check

Solution

applying the trigonometric identity

$$2\sin\alpha\cos\beta = \sin(\alpha+\beta) + \sin(\alpha-\beta)$$

to $f(x) = \cos(2x)\sin(5x)$ with $\alpha = 5x$, $\beta = 2x$ $\quad\rightsquigarrow$

$$f(x) = \frac{1}{2}\sin(7x) + \frac{1}{2}\sin(3x)$$

integrating, noting that $\int \sin x \, \mathrm{d}x = -\cos x$ as well as $\int h(sx)\,\mathrm{d}x = \frac{1}{s}H(sx)$ with H an antiderivative of h $\quad\rightsquigarrow$

$$F(x) = -\frac{1}{14}\cos(7x) - \frac{1}{6}\cos(3x) + C$$

<u>Alternative Solution with Complex Analysis</u>

application of the **beautiful** formula of Euler-Moivre:

$$\mathrm{e}^{\mathrm{i}t} \underset{(1)}{=} \cos t + \mathrm{i}\sin t \quad\Longleftrightarrow\quad \cos t \underset{(2)}{=} \frac{\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}}{2},\ \sin t \underset{(3)}{=} \frac{\mathrm{e}^{\mathrm{i}t}-\mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}}$$

rewriting $f(x) = \cos(2x)\sin(5x)$, using (2) and (3) $\quad\rightsquigarrow$

$$f(x) = \frac{\mathrm{e}^{2\mathrm{i}x}+\mathrm{e}^{-2\mathrm{i}x}}{2}\,\frac{\mathrm{e}^{5\mathrm{i}x}-\mathrm{e}^{-5\mathrm{i}x}}{2\mathrm{i}} = \frac{\mathrm{e}^{7\mathrm{i}x}+\mathrm{e}^{3\mathrm{i}x}-\mathrm{e}^{-3\mathrm{i}x}-\mathrm{e}^{7\mathrm{i}x}}{4\mathrm{i}}$$

combining the first and the last, as well as the seond and the third exponential, using (3) $\quad\rightsquigarrow$

$$f(x) = \frac{1}{2}\left(\sin(7x) + \sin(3x)\right)$$

Remark

In this fashion, **every** product of sines and cosines can be converted to a linear combination of exponentials $\mathrm{e}^{\mathrm{i}kx}$, or, alternatively, to a linear combination of $\sin(kx)$ and $\cos(kx)$. For example,

$$\begin{aligned}\cos^2(3x)\sin(4x) &= \frac{\mathrm{e}^{10\mathrm{i}x}+2\mathrm{e}^{4\mathrm{i}x}-\mathrm{e}^{2\mathrm{i}x}+\mathrm{e}^{-2\mathrm{i}x}-2\mathrm{e}^{-4\mathrm{i}x}-\mathrm{e}^{-10\mathrm{i}x}}{8\mathrm{i}} \\ &= \frac{1}{4}\sin(10x) + \frac{1}{2}\sin(4x) - \frac{1}{4}\sin(2x)\end{aligned}$$

1.7 Integral of a Trigonometric Polynomial

Compute $\int_{-\pi}^{\pi} (\cos x - 3 \sin x)^4 \, dx$.

Resources: Table of Elementary Integrals, Properties of the Integral, Fundamental Theorem of Calculus

Problem Variants

■ $\int_{-\pi}^{\pi} (\sin(2x) - \cos x)^2 \, dx$

?:
[ ] check

■ $\int_{-\pi}^{\pi} (\cos(3x) + \sin(3x))^4 \, dx$

?:
[ ] check

■ $\int_{-\pi}^{\pi} (2 \cos(x) - \cos(3x))^2 \, dx$

?:
[ ] check

Solution

applying the binomial formula,

$$(a-b)^4 = \sum_{k=0}^{4} (-1)^k \binom{4}{k} a^{4-k} b^k = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \,,$$

to the integrand $f(x) = (\cos x - 3\sin x)^4 \quad \rightsquigarrow$

$$s = \int_{-\pi}^{\pi} f = \int_{-\pi}^{\pi} C^4 - 12C^3S + 54C^2S^2 - 108CS^3 + 81S^4 \,\mathrm{d}x$$

with $C = \cos x$, $S = \sin x$

The second and fourth term are odd functions ($f(-x) = -f(x)$). Hence, their integral over the symmetric interval $[-\pi, \pi]$ vanishes.

rewriting the remaining terms, using

$$C^4 = C^2(1-S^2), \quad S^4 = S^2(1-C^2), \quad CS = \frac{1}{2}\sin(2x)$$

$\rightsquigarrow$

$$s = \int_{-\pi}^{\pi} C^2 - \frac{1}{4}\sin^2(2x) + \frac{54}{4}\sin^2(2x) + 81S^2 - \frac{81}{4}\sin^2(2x)\,\mathrm{d}x$$

$\int_{-\pi}^{\pi} \cos^2(kx)\,\mathrm{d}x = \int_{-\pi}^{\pi} \sin^2(kx)\,\mathrm{d}x = \pi \qquad \Longrightarrow$

$$s = \pi\left(1 - \frac{1}{4} + \frac{54}{4} + 81 - \frac{81}{4}\right) = 75\pi \approx 235.6194$$

Alternative Solution with Complex Analysis

application of the **beautiful** formulas of Euler-Moivre:

$$\cos t = \frac{\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}}{2}, \; \sin t = \frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}}$$

The conversion to a linear combination $p(x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{\mathrm{i}kx}$ simplifies the computation of a definite integral over an interval of length 2π tremendously. Since, for $k \neq 0$,

$$\int_a^{a+2\pi} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = \left[\frac{\mathrm{e}^{\mathrm{i}kx}}{\mathrm{i}k}\right]_{x=a}^{x=a+2\pi} = \frac{\mathrm{e}^{\mathrm{i}ka}\,\overbrace{\mathrm{e}^{\mathrm{i}k(2\pi)}}^{=1}}{\mathrm{i}k} - \frac{\mathrm{e}^{\mathrm{i}ka}}{\mathrm{i}k} = 0\,,$$

only the summand $c_0 e^0 = c_0$ contributes: $\int_a^{a+2\pi} p = 2\pi c_0$.

For $f(x) = (\cos x - 3\sin x)^4$, Euler's formulas and the binomial theorem yield

$$\begin{aligned} f(x) &= \left(\frac{e^{ix}+e^{-ix}}{2} - 3\frac{e^{ix}-e^{-ix}}{2i}\right)^4 = \left(\frac{1}{2i}\right)^4 \left((-3+i)e^{ix} + (3+i)e^{-ix}\right)^4 \\ &= \frac{1}{16}\sum_{k=0}^{4}\binom{4}{k}(-3+i)^{4-k}e^{(4-k)ix}(3+i)^k e^{-kix}\,. \end{aligned}$$

The coeffient c_0 of the trivial exponential $e^{0ix} = 1$ corresponds to $k = 2$. Hence,

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi c_0 = 2\pi\,\frac{1}{16}\binom{4}{2}\underbrace{(-3+i)^2(3+i)^2)}_{(-10)^2} = 75\pi\,.$$

1.8 Antiderivatives of Expressions with Square Roots

Compute

a) $\int (x+1)\sqrt{1-x}\,\mathrm{d}x$ b) $\int \frac{x}{1+\sqrt{1-x}}\,\mathrm{d}x$ c) $\int \frac{1}{\sqrt{1+x}-\sqrt{x}}\,\mathrm{d}x$

Resources: Table of Elementary Integrals

Problem Variants

■ $\int \frac{5x}{\sqrt{2+x}-\sqrt{x}}\,\mathrm{d}x$

$x^{?/2} + (-?+?x)(2+x)^{?/2}/3 + C$:

[ ] check

■ $\int (2x-1)(2x+1)^{1/3}\,\mathrm{d}x$

$?(?x-?)(2x+1)^{4/3}/28 + C$:

[ ] check

■ $\int \frac{3x}{2-\sqrt{4+x}}\,\mathrm{d}x$

$-?(4+x)^{?/2}-?x + C$:

[ ] check

Solution

The expressions are rewritten so that the elementary antiderivatives

$$\begin{aligned}\int x^n \,\mathrm{d}x &= \frac{1}{n+1}\, x^{n+1} + C,\ n \neq -1 \\ \int (a \pm x)^{n/2} \,\mathrm{d}x &= \pm\frac{2}{n+2}(a \pm x)^{n/2+1} + C\end{aligned}$$

can be used.

a) $\int (1+x)\sqrt{1-x}\,\mathrm{d}x$

$(1+x) = -(1-x) + 2 \quad \rightsquigarrow$

$$\begin{aligned}&\int -(1-x)^{3/2} + 2(1-x)^{1/2}\,\mathrm{d}x = \frac{2}{5}(1-x)^{5/2} - \frac{4}{3}(1-x)^{3/2} + C \\ &\quad = -\frac{2}{15}\,(7+3x)(1-x)^{3/2} + C\end{aligned}$$

b) $\int \frac{x}{1+\sqrt{1-x}}\,\mathrm{d}x$

expanding the fraction with $1-\sqrt{1-x}$ and using the third binomial formula, $(a+b)(a-b) = a^2 - b^2 \quad \rightsquigarrow$

$$\int \frac{x(1-\sqrt{1-x})}{1-(1-x)}\,\mathrm{d}x = \int 1 - \sqrt{1-x}\,\mathrm{d}x = x + \frac{2}{3}(1-x)^{3/2} + C$$

c) $\int \frac{1}{\sqrt{1+x}-\sqrt{x}}\,\mathrm{d}x$

expanding with $\sqrt{1+x}+\sqrt{x} \quad \rightsquigarrow$

$$\int \frac{\sqrt{1+x}+\sqrt{x}}{(1+x)-x}\,\mathrm{d}x = \frac{2}{3}\left((1+x)^{3/2} + x^{3/2}\right) + C$$

1.9 Integrals of Roots of Linear Functions

Compute

$$\text{a)} \quad \int_0^3 \sqrt[3]{3x-1}\,\mathrm{d}x \qquad\qquad \text{b)} \quad \int_2^6 \frac{\mathrm{d}x}{\sqrt{4x+1}}$$

Resources: Fundamental Theorem of Calculus, Table of Elementary Integrals

Problem Variants

■ $\displaystyle\int_2^3 \sqrt{3x-5} + \frac{1}{\sqrt{3x-5}}\,\mathrm{d}x$

?.??:
[ ] check

■ $\displaystyle\int_1^5 \sqrt[3]{2x-2}\,\mathrm{d}x$

?:
[ ] check

■ $\displaystyle\int_1^6 \frac{\mathrm{d}x}{\sqrt[4]{3x-2}}$

?.??:
[ ] check

Solution

application of the formula for the antiderivative of a (fractional) power of a linear function:

$$F(x) = \int \underbrace{(ax+b)^r}_{f(x)} \, \mathrm{d}x = \frac{1}{a(r+1)}(ax+b)^{r+1} + C$$

for $r \neq -1$ and $a, b \in \mathbb{R}$, $a \neq 0$

a) $f(x) = \sqrt[3]{3x-1}$, $\int_0^3 f$

$r = 1/3$, $a = 3$, $b = -1$ $\quad\rightsquigarrow\quad$ antiderivative

$$F(x) = \frac{1}{3(1/3+1)}(3x-1)^{1/3+1} + C = \frac{1}{4}(3x-1)^{4/3} + C$$

fundamental theorem of calculus $\quad\rightsquigarrow$

$$\int_0^3 \underbrace{\sqrt[3]{3x-1}}_{f(x)} \, \mathrm{d}x = F(3) - F(0) = \frac{1}{4}8^{4/3} - \frac{1}{4}(-1)^{4/3} = 4 + \frac{1}{4} = \frac{17}{4}$$

b) $f(x) = 1/\sqrt{4x+1}$, $\int_2^6 f$

To apply the general formula, $f(x)$ is rewritten as

$$f(x) = (4x+1)^{-1/2},$$

i.e., $r = -1/2$, $a = 4$, $b = 1$, and

$$F(x) = \frac{1}{4 \cdot 1/2}(4x+1)^{1/2} + C = \frac{1}{2}\sqrt{4x+1} + C$$

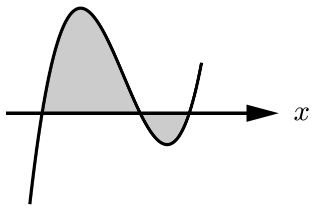
fundamental theorem of calculus $\quad\rightsquigarrow$

$$\int_2^6 f(x) \, \mathrm{d}x = F(6) - F(2) = \frac{1}{2}\sqrt{24+1} - \frac{1}{2}\sqrt{8+1} = \frac{5}{2} - \frac{3}{2} = 1$$

1.10 Area Bounded by the Graph of a Polynomial and the x-Axis

Compute the shaded area for the polynomial

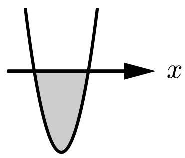
$$p(x) = x^3 - 2x^2 - x + 2\,.$$



Resources: Fundamental Theorem of Calculus, Table of Elementary Integrals

Problem Variants

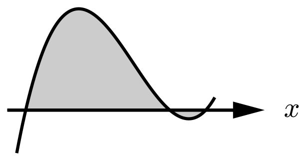
■ $$p(x) = 2x^2 - 5x + 2$$



?.???:

[ ] [check]

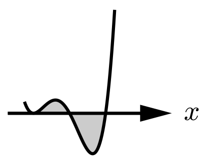
■ $$p(x) = x^3 - 7x + 6$$



??.??:

[ ] [check]

■ $$p(x) = x^4 - 3x^3 + 2x^2$$



?.?:

[ ] [check]

Solution

First, the zeros x_k of the polynomial p are determined. Then, the subareas $A_k = \left|\int_{x_k}^{x_\ell} p\right|$ between adjacent zeros are computed.

<u>Zeros</u>

An apparent zero of

$$p(x) = x^3 - 2x^2 - x + 2$$

is $x_0 = 1$.
dividing by the corresponding linear factor $x - 1 \quad \rightsquigarrow$

$$\begin{array}{rrrrl}
(\; x^3 & -2x^2 & -x & +2 \;) & : \; (x-1) \; = \; x^2 - x - 2 \\
x^3 & -x^2 & & & \\
\hline
 & -x^2 & -x & & \\
 & -x^2 & +x & & \\
\hline
 & & -2x & +2 & \\
 & & -2x & +2 & \\
\hline
 & & & 0 &
\end{array}$$

formula for the solutions of the quadratic equation $x^2 - x - 2 = 0 \quad \rightsquigarrow$ remaining zeros

$$x_{1,2} = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 1 \cdot (-2)}}{2} = \frac{1 \pm 3}{2},$$

i.e., $x_1 = -1$, $x_2 = 2$

<u>Area</u>

antiderivative of p: $P(x) = \frac{x^4}{4} - \frac{2x^3}{3} - \frac{x^2}{2} + 2x + C$

subareas:

- left part:

$$A_1 = \left|\int_{-1}^{1} p\right| = \left|[P]_{-1}^{1}\right| = \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right) - \left(\frac{1}{4} + \frac{2}{3} - \frac{1}{2} - 2\right) = \frac{8}{3}$$

- right part:

$$A_2 = \left|[P]_{1}^{2}\right| = \left|\left(4 - \frac{16}{3} - 2 + 4\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{1}{2} + 2\right)\right| = \frac{5}{12}$$

⚠ Taking absolute values is necessary, since for areas below the x-axis $\int_{x_k}^{x_\ell} f$ is negative.

total area: $A_1 + A_2 = \dfrac{8}{3} + \dfrac{5}{12} = \dfrac{37}{12}$

1.11 Area Between Two Functions

Compute the area of the region bounded by the graphs of the functions $f(x) = 4 - x$ and $g(x) = 3/x$.

Resources: Table of Elementary Integrals

Problem Variants

■ $f(x) = -x^2 + x - 4$, $g(x) = 1 - x$

??.?:

check

■ $f(x) = 3\sqrt{x}$, $g(x) = 2 + x$

?.?:

check

■ $f(x) = -x^2 + 5x + 3$, $g(x) = x^2 + 3x - 1$

?:

check

Solution

Description of the region

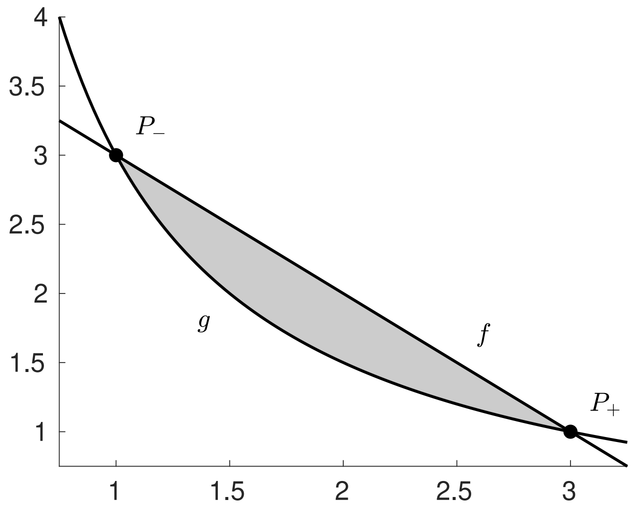
intersection of the graphs of $f(x) = 4 - x$ and $g(x) = 3/x$:
equating f and g,

$$4 - x = 3/x \iff x^2 - 4x + 3 = 0\,,$$

and applying the formula for a quadratic equation $\quad\leadsto$

$$x_\pm = \frac{4 \pm \sqrt{16 - 12}}{2}\,,$$

i.e., the intersections $P_- = (1, 3)$ and $P_+ = (3, 1)$



Area

Since $f(x) \geq g(x)$ for $1 \leq x \leq 3$, the area S equals $\int_1^3 f - g$ (area under f minus area under g), i.e.,

$$\begin{aligned} S &= \int_1^3 (4 - x) - \frac{3}{x}\,\mathrm{d}x = \left[4x - \frac{x^2}{2} - 3\ln x\right]_{x=1}^{x=3} \\ &= \left(12 - \frac{9}{2} - 3\ln 3\right) - \left(4 - \frac{1}{2} - 0\right) = 4 - 3\ln 3 \approx 0.7042 \end{aligned}$$

1.12 Limits via Integration

Compute

$$\lim_{n\to\infty}\sum_{k=1}^{3n}\frac{\sqrt{k+n}}{n\sqrt{n}}$$

by interpreting the sum as Riemann sum for approximating an integral.

Resources: Riemann Integral

Problem Variants

■ $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{2n}\frac{1}{\sqrt{n(k+2n)}}$

?.??:

[ ] check

■ $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{3n}\frac{k^2-n^2}{n^3}$

?:

[ ] check

■ $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{k}{2kn+n^2}$

?.??:

[ ] check

Solution

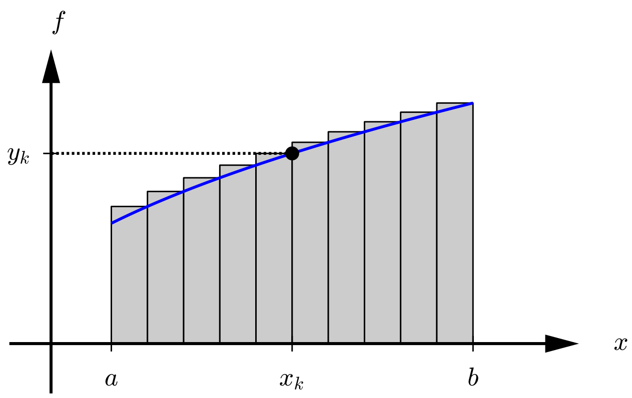
Riemann sum for $\int_a^b f$

special choice of a uniform partition

$$a, a+h, a+2h, \ldots, b-h, b$$

and sampling points at the subinterval endpoints[1], i.e., $x_k = kh \in [a+kh-h, a+kh]$ $\quad\rightsquigarrow$

$$\underbrace{h \sum_{a<a+kh\le b} f(a+kh)}_{\text{Riemann sum}} \underset{n\to\infty}{\longrightarrow} \int_a^b f(x)\,\mathrm{d}x$$



Application to the limit of $s_n = \sum_{k=1}^{3n} \frac{\sqrt{k+n}}{n\sqrt{n}}$ for $n \to \infty$

noting that $\sqrt{k+n}/\sqrt{n} = \sqrt{k/n+1}$ and writing the sum in the form of a Riemann sum with $h = 1/n$, $a = 0$, $b = 3$ $\quad\rightsquigarrow$

$$s_n = \frac{1}{n} \sum_{0<k/n\le 3} \sqrt{k/n+1}$$

Hence, s_n is a Riemann sum for $\int_0^3 \sqrt{x+1}\,\mathrm{d}x$, and, consequently,

$$\lim_{n\to\infty} s_n = \int_0^3 \sqrt{x+1}\,\mathrm{d}x = \left[\frac{2}{3}(x+1)^{3/2}\right]_{x=0}^{x=3} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}\,.$$

[1] suited for the given problem; $x_k = a + kh - h/2$ (midpoint) is the best choice for numerical purposes

Chapter 2

Rational Integrands

2.1 Integral of a Rational Function with a Single Pole

Compute $\int_0^4 \frac{x^2}{(x+1)^3}\,\mathrm{d}x$.

Resources: Elementary Rational Integrands, Table of Elementary Integrals

Problem Variants

- $\int_{-1}^{0} \frac{x}{(3x-1)^2}\,\mathrm{d}x$

−0.0???:

[ ] check

- $\int_{-1}^{1} \frac{(2-x)^2}{(3-x)^3}\,\mathrm{d}x$

?.??:

[ ] check

- $\int_{0}^{2} \frac{1+x^2}{x^2+2x+1}\,\mathrm{d}x$

?.??:

[ ] check

Solution

Simplification of the integrand

expanding the numerator $p(x) = x^2$ of the integrand $f(x) = \dfrac{x^2}{(x+1)^3}$ in terms of powers of $(x+1)$ (Taylor expansion at $x = -1$) $\rightsquigarrow$

$$x^2 = p(-1) + p'(-1)(x+1) + \frac{p''(-1)}{2}(x+1)^2 = 1 - 2(x+1) + (x+1)^2$$

and, dividing by $(x+1)^3$,

$$f(x) = \frac{1}{(x+1)^3} - \frac{2}{(x+1)^2} + \frac{1}{x+1}$$

Integration of the sum of elementary terms

$\int (x+a)^{-n}\,\mathrm{d}x = (x+a)^{1-n}/(1-n)+C$ for $n < -1$, $\int (x+a)^{-1}\,\mathrm{d}x = \ln|x+a|+C$ $\Longrightarrow$

$$\begin{aligned}
\int_0^4 f(x)\,\mathrm{d}x &= \int_0^4 \frac{1}{(x+1)^3} - \frac{2}{(x+1)^2} + \frac{1}{x+1}\,\mathrm{d}x \\
&= \left[-\frac{1}{2(x+1)^2} + \frac{2}{x+1} + \ln|x+1|\right]_{x=0}^{x=4} \\
&= \left(-\frac{1}{50} + \frac{2}{5} + \ln 5\right) - \left(-\frac{1}{2} + 2 + 0\right) \\
&= -28/25 + \ln 5 \approx 0.4894
\end{aligned}$$

Alternative solution

partial fraction decomposition with the ansatz

$$\frac{x^2}{(x+1)^3} = \frac{a}{(x+1)^3} + \frac{b}{(x+1)^2} + \frac{c}{x+1}$$

2.2 Antiderivative of a Rational Function with a Single Pole

Determine $\displaystyle\int \frac{x^2-1}{(2x-1)^3}\,\mathrm{d}x\,.$

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

■ $\displaystyle\int \frac{x^2}{(x-3)^2}\,\mathrm{d}x$

$?+?\ln|x-3|-?/(x-3)$:

[ ] check

■ $\displaystyle\int \frac{3x-1}{x^2-4x+4}\,\mathrm{d}x$

$?\ln|x-?|-?/(x-?)$:

[ ] check

■ $\displaystyle\int \frac{x}{(1+x)^3}\,\mathrm{d}x$

$1/(?(1+x)^?)??/(1+x)$:

[ ] check

Solution

Partial fraction decomposition

rational function $f(x) = (x^2-1)/(2x-1)^3$ with a triple pole at $x = 1/2$ $\rightsquigarrow$ ansatz

$$f(x) = \frac{x^2-1}{(2x-1)^3} = \frac{a}{x-1/2} + \frac{b}{(x-1/2)^2} + \frac{c}{(x-1/2)^3}$$

multiplying by the common denominator $(x-1/2)^3$ $\rightsquigarrow$

$$x^2/8 - 1/8 = a(x^2 - x + 1/4) + b(x - 1/2) + c$$

comparing the coefficients of x^2, x, and 1 $\rightsquigarrow$ linear equations

$$1/8 \underset{x^2}{=} a, \quad 0 \underset{x}{=} -a + b, \quad -1/8 \underset{1}{=} a/4 - b/2 + c$$

with the solution $a = 1/8$, $b = 1/8$, $c = -3/32$

Integration

$$\int \frac{\mathrm{d}x}{(x-x_0)^n} = -\frac{1}{n-1}\frac{1}{(x-x_0)^{n-1}} + C, \quad \int \frac{\mathrm{d}x}{x-x_0} = \ln|x-x_0| + C$$

$\rightsquigarrow$ antiderivative

$$\begin{aligned} F(x) &= \int \frac{1/8}{x-1/2} + \frac{1/8}{(x-1/2)^2} - \frac{3/32}{(x-1/2)^3}\,\mathrm{d}x \\ &= \frac{1}{8}\ln|x-1/2| - \frac{1/8}{x-1/2} + \frac{3/64}{(x-1/2)^2} + C \\ &= \frac{1}{8}\ln|2x-1| - \frac{1}{8x-4} + \frac{3}{(8x-4)^2} + \tilde{C} \end{aligned}$$

($\tilde{C} = C - (\ln 2)/8$)

Alternative solution

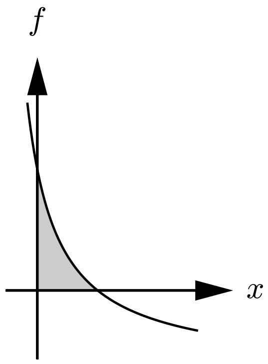
expanding the numerator of $f(x)$ in powers of $(x-1/2)$ (Taylor expansion at $x_0 = 1/2$):

$$x^2 - 1 = (x-1/2)^2 + (x-1/2) - 3/4$$

2.3 Area Bounded by the Graph of a Rational Function

Determine the gray area with curved boundary described by the graph of the function

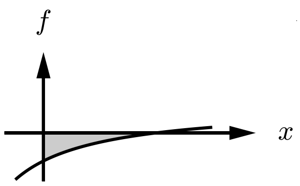
$$f(x) = \frac{3-2x}{x+1}\,.$$



Resources: Elementary Rational Integrands

Problem Variants

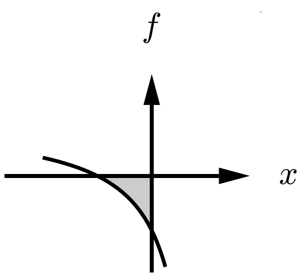
■ $f(x) = \dfrac{x/2-2}{2+x/2}$



?.??:

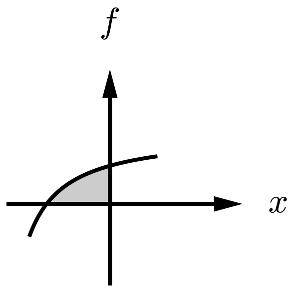
[ ] check

■ $f(x) = \dfrac{x+1}{x-1}$



?.??:

check



■ $f(x) = \dfrac{3x+4}{2x+5}$

?.??:

check

Solution

The area A below the graph of

$$f(x) = \frac{3-2x}{x+1} \,,$$

bounded by the vertical axis on the left, equals $\int_0^{x_\star} f(x)\,\mathrm{d}x$, where $x_\star$ is the intersection of the graph with the horizontal axis. Hence, $A = [F]_0^{x_\star}$ with F an antiderivative of f.

- intersection:

$$0 = f(x) \iff 0 = 3 - 2x \qquad \implies \qquad x_\star = 3/2$$

- antiderivative: writing

$$f(x) = \frac{5 - 2(x+1)}{x+1} = \frac{5}{x+1} - 2$$

$\rightsquigarrow$

$$F(x) = 5\ln|x+1| - 2x + C$$

- area:

$$A = [F]_0^{3/2} = (5\ln(5/2) - 3) - (5\underbrace{\ln 1}_{=0} - 0) = 5\ln 5 - 5\ln 2 - 3 \approx 1.5814$$

2.4 Integration of a Rational Function of Degree $(1, 2)$ with Real Poles

Determine an antiderivative of

$$f(x) = \frac{5x - 1}{x^2 - 4x + 3}$$

and compute $\int_{-1}^{0} f$.

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

■ $f(x) = \dfrac{3}{4 - x^2}$

$\int_0^1 f$ =?.??:

[ ] check

■ $f(x) = \dfrac{x}{2x^2 + 3x + 1}$

$\int_0^2 f$ =?.??:

[ ] check

■ $f(x) = \dfrac{x + 1}{x^2 + x}$

$\int_1^2 f$ =?.??:

[ ] check

Solution

Poles

setting the denominator of the integrand

$$f(x) = \frac{5x-1}{x^2-4x+3}$$

to zero, and applying the formula for solving a quadratic equation $\rightsquigarrow$ poles

$$x_{1,2} = \frac{4 \pm \sqrt{4^2 - 4 \cdot 1 \cdot 3}}{2} = \frac{4 \pm 2}{2}, \quad \text{i.e., } x_1 = 1,\ x_2 = 3$$

Partial fraction decomposition

ansatz for a rational function with two simple poles:

$$f(x) = \frac{5x-1}{(x-1)(x-3)} = \frac{a}{x-1} + \frac{b}{x-3}$$

- multiplying by $x-1$ and setting $x = 1$ $\implies$ $a = \dfrac{5 \cdot 1 - 1}{1-3} = -2$
- $\cdot(x-3)$, $x = 3$ $\implies$ $b = 7$

Antiderivative

$\displaystyle\int \frac{\mathrm{d}x}{x-p} = \ln|x-p| + C \quad \rightsquigarrow$

$$F(x) = \int -\frac{2}{x-1} + \frac{7}{x-3}\,\mathrm{d}x = -2\ln|x-1| + 7\ln|x-3| + C$$

Integral

$$\begin{aligned}\int_{-1}^{0} F &= [F]_{-1}^{0} = \Big(-2\underbrace{\ln 1}_{=0} + 7\ln 3\Big) - \Big(-2\ln 2 + 7\underbrace{\ln 4}_{=2\ln 2}\Big) \\ &= 7\ln 3 - 12\ln 2 \approx -0.6274\end{aligned}$$

2.5 Integral of a Rational Function of Degree $(1, 2)$ with Complex Conjugate Poles

Compute $\displaystyle\int_{-1}^{0} \frac{x+2}{x^2+2x+2}\,\mathrm{d}x$.

Resources: Elementary Rational Integrands

Problem Variants

■ $\displaystyle\int_{0}^{4} \frac{3-x}{x^2+1}\,\mathrm{d}x$

?.??:

check

■ $\displaystyle\int_{0}^{1} \frac{x+1}{2x^2-4x+3}\,\mathrm{d}x$

?.??:

check

■ $\displaystyle\int_{-1}^{1} \frac{x}{x^2-2x+5}\,\mathrm{d}x$

0.0??:

check

Solution

Transformation to standard form

rewriting the integrand f in the form

$$\frac{a(x-c)+b}{(x-c)^2+d^2}$$

in order to apply the formulas for integrating the elementary terms $\quad\rightsquigarrow$

$$f(x)=\frac{x+2}{x^2+2x+2}=\frac{(x+1)+1}{(x+1)^2+1}\,,$$

i.e., $a=b=d=1$, $c=-1$

Integration of the elementary terms

applying the formulas

$$\begin{aligned}\int\frac{x-c}{(x-c)^2+d^2}\,\mathrm{d}x &= \frac{\ln((x-c)^2+d^2)}{2}+C,\\ \int\frac{1}{(x-c)^2+d^2}\,\mathrm{d}x &= \frac{\arctan((x-c)/d)}{d}+C\end{aligned}$$

$\rightsquigarrow$ antiderivative

$$F(x)=\int f(x)\,\mathrm{d}x=\frac{\ln((x+1)^2+1)}{2}+\arctan(x+1)+C$$

and

$$\begin{aligned}\int_{-1}^{0} f(x)\,\mathrm{d}x &= F(0)-F(-1)\\ &= \left(\frac{\ln 2}{2}+\frac{\pi}{4}\right)-(0+0)=\ln\sqrt{2}+\frac{\pi}{4}\approx 1.1320\end{aligned}$$

2.6 Antiderivative of a Rational Function of Degree $(2, 2)$

Determine

$$\int \frac{(x-1)(x-2)}{x(x-3)}\,\mathrm{d}x\,.$$

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

■ $\displaystyle\int \frac{x^2}{2x^2 - x/2}\,\mathrm{d}x$

$x/? + \ln|?x{-}?|/?$:

[ ] check

■ $\displaystyle\int \frac{1-x^2}{x(x-2)}\,\mathrm{d}x$

$?? - \ln|x|/?{-}?\ln|x-2|/2$:

[ ] check

■ $\displaystyle\int \frac{x^2+x+1}{x^2+5x+6}\,\mathrm{d}x$

$?{+}?\ln|x+2|{-}?\ln|x{+}?|$:

[ ] check

Solution

Partial fraction decomposition

simple poles of

$$f(x) = \frac{(x-1)(x-2)}{x(x-3)} = \frac{p(x)}{q(x)}$$

at $x = 0$ and $x = 3$ and degree $p =$ degree q[1] $\rightsquigarrow$ ansatz

$$f(x) = \frac{(x-1)(x-2)}{x(x-3)} = a + \frac{b}{x} + \frac{c}{x-3}$$

determination of a, b, and c:

- $x \to \infty \implies a = 1$
- multiplying by x and setting $x = 0 \implies b = \frac{(0-1)(0-2)}{0-3} = -2/3$
- $\cdot(x-3)$ and $x = 3 \implies c = 2/3$

Antiderivative

integrating the elementary terms of the partial fraction decomposition $\rightsquigarrow$

$$\begin{aligned} \int f(x)\,\mathrm{d}x &= \int 1 - \frac{2/3}{x} + \frac{2/3}{x-3}\,\mathrm{d}x \\ &= x - \frac{2}{3}\ln|x| + \frac{2}{3}\ln|x-3| + C = x + \frac{2}{3}\ln\left|\frac{x-3}{x}\right| + C \end{aligned}$$

[1]In general, if degree $p =$ degree $q + \underbrace{n}_{>0}$, the constant a in the following ansatz is replaced by a polynomial of degree $\leq n$.

2.7 Integral of a Rational Function with Complex Conjugate and Real Poles

Compute $\displaystyle\int_1^2 \frac{4}{x^3+x}\,\mathrm{d}x\,.$

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

- $\displaystyle\int_{-1/2}^{1/2} \frac{3x^2}{x^4-1}\,\mathrm{d}x$

$-?.??$:

check

- $\displaystyle\int_1^2 \frac{2}{x^3-4x^2+5x}\,\mathrm{d}x$

$?.??$:

check

- $\displaystyle\int_0^1 \frac{4x}{(x+3)(x^2+1)}\,\mathrm{d}x$

$?.??$:

check

Solution

Partial Fraction Decomposition

The integrand

$$f(x) = \frac{4}{x^3 + x} = \frac{4}{x(x^2+1)} = \frac{4}{x(x-\mathrm{i})(x+\mathrm{i})}$$

has a pole at $x = 0$ and complex conjugate poles at $x = \pm\mathrm{i}$, and, therefore, the ansatz is

$$f(x) = \frac{4}{x(x^2+1)} = \frac{a}{x} + \frac{bx+c}{x^2+1}\,.$$

multiplication by x und setting $x = 0 \quad \Longrightarrow \quad a = 4$
subtracting the term $4/x$ from $f(x) \quad \rightsquigarrow$

$$\frac{bx+c}{x^2+1} = \frac{4}{x(x^2+1)} - \frac{4}{x} = \frac{4 - 4(x^2+1)}{x(x^2+1)} = -\frac{4x}{x^2+1}\,,$$

i.e., $b = -4$, $c = 0$

Integration of the elementary terms

applying the formulas

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + C, \quad \int \frac{x}{x^2+1}\,\mathrm{d}x = \frac{1}{2}\ln|x^2+1| + C\,,$$

for antiderivatives of the elementary terms $\quad \rightsquigarrow$

$$\int_1^2 f(x)\,\mathrm{d}x = \int_1^2 \frac{4}{x} - 4\frac{x}{x^2+1}\,\mathrm{d}x = \left[4\ln|x| - 2\ln|x^2+1|\right]_{x=1}^{x=2}$$
$$= (4\ln 2 - 2\ln 5) - (4\underbrace{\ln 1}_{=0} - 2\ln 2)$$

simplifying, using the rules $a\ln b = \ln b^a$, $\ln a + \ln b = \ln(ab)$, $-\ln a = \ln(1/a)$ $\rightsquigarrow$

$$\ln 16 - \ln 25 + \ln 4 = \ln(16\cdot 4/25) = \ln(64/25) \approx 0.9400$$

2.8 Integration of a Rational Function with a Polynomial Component

Determine the antiderivative of

$$r(x) = \frac{2x^3 + 5x^2 - 7x - 7}{2x^2 + 7x - 4}$$

and compute $\int_1^5 r(x)\,\mathrm{d}x$.

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

■ $\displaystyle\int_2^3 \frac{x^3}{x^2 - 1}\,\mathrm{d}x$

?.??:

[ ] check

■ $\displaystyle\int_{-1}^0 \frac{x^4}{2x^2 - 3x + 1}\,\mathrm{d}x$

0.???:

[ ] check

■ $\displaystyle\int_3^4 \frac{1 + x^4}{x^3 - x}\,\mathrm{d}x$

?.??:

[ ] check

Solution

Since the degree of the numerator of the rational function

$$r(x) = \frac{2x^3 + 5x^2 - 7x - 7}{2x^2 + 7x - 4}$$

is larger than the degree of the denominator, the ansatz for the partial fraction decomposition contains a polynomial component in addition to the elementary rational terms corresponding to the poles of r. All coefficients in the ansatz can be determined by multiplying with the common denominator and comparing the coefficients of the monomials. The expressions of the ansatz are integrated using the elementary tabulated antiderivatives.

Partial Fraction Decomosition

poles of r: $2x^2 + 7x - 4 = 0 \quad \rightsquigarrow$

$$x_{1,2} = \frac{-7 \pm \sqrt{7^2 - 4 \cdot 2 \cdot (-4)}}{2 \cdot 2} = -\frac{7}{4} \pm \frac{9}{4},$$

i.e., $x_1 = -4$, $x_2 = 1/2$

Since the degree of the numerator of r is by 1 larger than the degree of the denominator, the ansatz for the partial fraction decomposition contains a linear polynomial (degree 1):

$$r(x) = \frac{2x^3 + 5x^2 - 7x - 7}{2x^2 + 7x - 4} = ax + b + \frac{c}{x+4} + \frac{d}{x - 1/2}$$

multiplying with the denominator $2x^2 + 7x - 4 = 2(x+4)(x-1/2)$ of $r \quad \rightsquigarrow$

$$2x^3 + 5x^2 - 7x - 7 = (ax + b)(2x^2 + 7x - 4) + c(2x - 1) + d(2x + 8)$$

comparing coefficients of $x^k \quad \rightsquigarrow \quad$ linear system for a, b, c, d:

$$\begin{array}{rrcl} x^3: & 2 & = & 2a \\ x^2: & 5 & = & 7a + 2b \\ x: & -7 & = & -4a + 7b + 2c + 2d \\ 1: & -7 & = & -4b - c + 8d \end{array}$$

solution: $a = 1$, $b = -1$, $c = 3$, $d = -1$

Integration

antiderivative:

$$\begin{aligned} R(x) &= \int x - 1 + \frac{3}{x+4} - \frac{1}{x-1/2}\,\mathrm{d}x \\ &= \frac{1}{2}x^2 - x + 3\ln|x+4| - \ln|x-1/2| + C \end{aligned}$$

definite integral:

$$\begin{aligned} \int_1^5 r &= [R(x)]_1^5 = (25/2 - 1/2) - (5-1) + 3(\ln 9 - \ln 5) - (\ln(9/2) - \ln(1/2)) \\ &= 8 + 4\ln 3 - 3\ln 5 \approx 7.5661 \end{aligned}$$

2.9 Integration of a Rational Function of Degree $(3,4)$

The rational function

$$r(x) = \frac{5x^3 + 3x^2 + 7x - 6}{x^4 + 2x^3 + 5x^2 + 8x + 4}$$

has a simple pole at $z = 2\mathrm{i}$ and a double pole at $z = -1$. Determine the antiderivative and compute $\int_0^2 r$.

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

■ $r(x) = \dfrac{3x^3 - 7x^2 + 5x - 2}{x^4 - 5x^3 + 10x^2 - 10x + 4}$, poles at $z = 1, 2, 1 + \mathrm{i}$

$\int_{-1}^0 r = -?.??$:

[ ] check

■ $r(x) = \dfrac{1}{x^4 - 4x^3 + 5x^2 - 4x + 4}$, poles at $z = 2, 2, \mathrm{i}$

$\int_0^1 r = ?.??$:

[ ] check

■ $r(x) = \dfrac{x^3 - x}{x^4 + 5x^2 + 4}$, poles at $z = \mathrm{i}, 2\mathrm{i}$

$\int_0^2 r = 0.???$:

[ ] check

Solution

With a partial fraction decomposition, the rational function

$$r(x) = \frac{5x^3 + 3x^2 + 7x - 6}{x^4 + 2x^3 + 5x^2 + 8x + 4}$$

is expressed as a sum of elementary terms which are integrated using the tabulated antiderivatives.

<u>Partial fraction decomposition</u>

Since, for a real rational function, complex poles occur in complex conjugate pairs, the pole at $z_1 = 2\mathrm{i}$ corresponds to a pole at $z_2 = \bar{z}_1 = \overline{2\mathrm{i}} = -2\mathrm{i}$. With the double pole at $z_{3,4} = -1$, this leads to the factorization of the denominator and the following ansatz for the partial fraction decomposition:

$$r(x) = \frac{5x^3 + 3x^2 + 7x - 6}{(x^2+4)(x+1)^2} = \frac{ax+b}{x^2+4} + \frac{c}{x+1} + \frac{d}{(x+1)^2}$$

$((x - 2\mathrm{i})(x + 2\mathrm{i}) = x^2 + 4)$

multiplying by the common denominator $\quad\rightsquigarrow$

$$5x^3 + 3x^2 + 7x - 6 = (ax+b)(x+1)^2 + c(x^2+4)(x+1) + d(x^2+4)$$

comparing coefficients of the monomials x^k $\quad\rightsquigarrow\quad$ linear system for a, b, c, d:

$$\begin{array}{rcrcl} x^3 & : & 5 & = & a + c \\ x^2 & : & 3 & = & 2a + b + c + d \\ x & : & 7 & = & a + 2b + 4c \\ 1 & : & -6 & = & b + 4c + 4d \end{array}$$

solution: $a = 3$, $b = -2$, $c = 2$, $d = -3$, i.e.,

$$r(x) = \frac{3x}{x^2+4} - \frac{2}{x^2+4} + \frac{2}{x+1} - \frac{3}{(x+1)^2}$$

<u>Antiderivative</u>

$$\begin{aligned} R(x) &= \int \frac{3x}{x^2+4} - \frac{2}{x^2+2^2} + \frac{2}{x+1} - \frac{3}{(x+1)^2}\,\mathrm{d}x \\ &= \frac{3}{2}\ln(x^2+4) - \arctan(x/2) + 2\ln|x+1| + \frac{3}{x+1} + C \end{aligned}$$

Integral

$$
\begin{aligned}
\int_0^2 r(x)\,\mathrm{d}x &= [R(x)]_0^2 = \frac{3}{2}\ln(8/4) - (\pi/4 - 0) + 2\ln(3/1) + (3/3 - 3/1) \\
&= \ln(18\sqrt{2}) - \pi/4 - 2 \approx 0.4515
\end{aligned}
$$

used for simplifying the sum of logarithms: $p\ln a = \ln a^p$, $\ln a + \ln b = \ln(ab)$

Chapter 3

Integration by Parts

3.1 Integral of a Product of a Polynomial with Sine and Cosine

Compute $\int_0^{\pi} x \sin x \, \mathrm{d}x$.

Resources: Integration by Parts

Problem Variants

- $\int_0^{\pi} x \cos x \, \mathrm{d}x$

−?:

[ ] check

- $\int_{-\pi}^{\pi} (\pi - x) \sin(3x) \, \mathrm{d}x$

−?.??:

[ ] check

- $\int_0^{2\pi} x \sin x \cos x \, \mathrm{d}x$

−?.??:

[ ] check

Solution

applying the formula for integration by parts,

$$\int_a^b f(x)g'(x)\,\mathrm{d}x = [f(x)g(x)]_{x=a}^{x=b} - \int_a^b f'(x)g(x)\,\mathrm{d}x$$

with $f(x) = x$, $g'(x) = \sin x$, and noting that $g(x) = -\cos x$ is an antiderivative of $\sin x$ $\quad\rightsquigarrow$

$$\begin{aligned}\int_0^\pi x\sin x\,\mathrm{d}x &= [x(-\cos x)]_{x=0}^{x=\pi} - \int_0^\pi 1\,(-\cos x)\,\mathrm{d}x \\ &\underset{\cos\pi=-1}{=} (\pi - 0) + \int_0^\pi \cos x\,\mathrm{d}x = \pi\,,\end{aligned}$$

since

$$\int_0^\pi \cos x\,\mathrm{d}x = [\sin x]_{x=0}^{x=\pi} = 0 - 0 = 0$$

3.2 Antiderivative of a Product of a Polynomial with a Trigonometric Expression

Determine an antiderivative of $f(x) = x\,\cos^2 x\,\sin(2x)$.

Resources: Integration by Parts, Table of Elementary Integrals

Problem Variants

■ $f(x) = x\,\cos x\,\sin x$

$-?\cos(?x)/?+\sin(?x)/?$:

[ ] check

■ $f(x) = x^2\,\cos(4x)$

$-\sin(?x)/??+?\cos(?x)/?+x^?\sin(?x)/?$:

[ ] check

■ $f(x) = x\,\sin^2(2x)$

$-\cos(?x)/??-?\sin(?x)/?+x^?/?$:

[ ] check

Solution

To determine an antiderivative of a product of a polynomial p with a trigonometric expression f, f is first converted to the standard form of a trigonometric polynomial:

$$f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(kx) + b_k \sin(kx)\,.$$

If this is not easily possible via trigonometric identities, a method which always works uses Euler's formulas

$$\cos\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{e}^{-\mathrm{i}\varphi}}{2}, \quad \sin\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi}}{2\mathrm{i}}\,.$$

After expanding the resulting expression from the replacement of cosine and sine by exponentials, the coeffients a_k and b_k are easily identified. In a final step an antiderivative of pf is obtained by successively reducing the degree of p via integration by parts:

$$\int pf = pF - \int p'F$$

with F an antiderivative of f.

<u>Conversion of $f(x) = \cos^2 x\,\sin(2x)$ to standard form</u>

applying Euler's formulas with $\varphi = x$, $\varphi = 2x$ and using the abbreviation $z := \mathrm{e}^{\mathrm{i}x}$ ($\mathrm{e}^{\mathrm{i}kx} = z^k$, $\mathrm{e}^{-\mathrm{i}kx} = z^{-k}$) $\quad\rightsquigarrow$

$$\begin{aligned} f(x) &= \frac{(z+1/z)^2}{2^2}\frac{z^2-1/z^2}{2\mathrm{i}} \\ &= \frac{(z^2+2+1/z^2)(z^2-1/z^2)}{8\mathrm{i}} = \frac{z^4+2z^2-2/z^2-1/z^4}{8\mathrm{i}} \end{aligned}$$

converting back to real form, noting that $(z^k - 1/z^k)/(2\mathrm{i}) = \sin(kx)$ $\quad\rightsquigarrow$

$$f(x) = \frac{\sin(2x)}{2} + \frac{\sin(4x)}{4}$$

Antiderivative of $xf(x)$

integrating by parts $\quad\rightsquigarrow$

$$\int \underbrace{x}_{p} \underbrace{\left(\frac{\sin(2x)}{2} + \frac{\sin(4x)}{4}\right)}_{f} \, \mathrm{d}x$$
$$= \underbrace{x}_{p} \underbrace{\left(-\frac{\cos(2x)}{4} - \frac{\cos(4x)}{16}\right)}_{F} - \int \underbrace{1}_{p'} \underbrace{\left(-\frac{\cos(2x)}{4} - \frac{\cos(4x)}{16}\right)}_{F} \, \mathrm{d}x$$
$$= -\frac{x\cos(2x)}{4} - \frac{x\cos(4x)}{16} + \frac{\sin(2x)}{8} + \frac{\sin(4x)}{64}$$

3.3 Integrals of Products of Polynomials with Exponentials

Compute

$$\text{a)} \quad \int_0^1 x\,\mathrm{e}^x\,\mathrm{d}x \qquad \text{b)} \quad \int_1^2 (2x+1)^2\,\mathrm{e}^{2x}\,\mathrm{d}x$$

Resources: Integration by Parts, Table of Elementary Integrals

Problem Variants

■ a) $\int_0^1 (1+x)\,\mathrm{e}^{-x}\,\mathrm{d}x$, b) $\int_{-1}^0 x^2\,\mathrm{e}^x\,\mathrm{d}x$

---

a) ?.?? b) ?.??:

check

■ a) $\int_0^2 x\,\mathrm{e}^{-2x}\,\mathrm{d}x$, b) $\int_{-1}^1 (1+x)^2\,\mathrm{e}^{-x}\,\mathrm{d}x$

---

a) ?.?? b) ?.??:

check

■ a) $\int_0^1 (2+x)\,3^x\,\mathrm{d}x$, b) $\int_0^2 x^2\,2^x\,\mathrm{d}x$

---

a) ?.?? b) ?.??:

check

Solution

For a product of a function g with a polynomial f, integrating by parts,

$$\int_a^b fg' = [fg]_a^b - \int_a^b f'g\,, \tag{1}$$

reduces the degree ($f \to f'$) and thus simplifies the integral, **provided** that a simple antiderivative of g' ($g' \to g$) exists.

a) $\int_0^1 x\mathrm{e}^x\,\mathrm{d}x$

noting that the exponential function does not change when differentiated or integrated, applying (1) with $f(x) = x$, $g'(x) = \mathrm{e}^x \quad \rightsquigarrow$

$$\begin{aligned}\int_0^1 \underbrace{x}_{f}\underbrace{\mathrm{e}^x}_{g'}\,\mathrm{d}x &= \Big[\underbrace{x}_{f}\underbrace{\mathrm{e}^x}_{g}\Big]_{x=0}^{x=1} - \int_0^1 \underbrace{1}_{f'}\underbrace{\mathrm{e}^x}_{g}\,\mathrm{d}x \\ &= (\mathrm{e}-0) - [\mathrm{e}^x]_{x=0}^{x=1} = \mathrm{e} - (\mathrm{e}-1) = 1\end{aligned}$$

b) $\int_0^1 (2x+1)^2\mathrm{e}^{2x}\,\mathrm{d}x$

noting that

$$\frac{\mathrm{d}}{\mathrm{d}x}u(px+q) = pu'(px+q), \qquad \int u(px+q)\,\mathrm{d}x = \frac{1}{p}U(px+q)$$

with $U(y)$ an antiderivative of $u(y)$, applying (1) $\quad \rightsquigarrow$

$$\begin{aligned} s &:= \int_0^1 \underbrace{(2x+1)^2}_{f}\underbrace{\mathrm{e}^{2x}}_{g'}\,\mathrm{d}x \\ &= \underbrace{\big[(2x+1)^2\,\mathrm{e}^{2x}/2)\big]_{x=0}^{x=1}}_{A} - \underbrace{\int_0^1 2\cdot 2(2x+1)\,\mathrm{e}^{2x}/2\,\mathrm{d}x}_{B}\end{aligned}$$

- $A = 3^2\,\mathrm{e}^2/2 - 1/2 = 9\,\mathrm{e}^2/2 - 1/2$
- B is computed by integrating by parts a second time:

$$\begin{aligned} B &= \int_0^1 2(2x+1)\,\mathrm{e}^{2x}\,\mathrm{d}x = \big[2(2x+1)\,\mathrm{e}^{2x}/2\big]_{x=0}^{x=1} - \int_0^1 4\,\mathrm{e}^{2x}/2\,\mathrm{d}x \\ &= 3\,\mathrm{e}^2 - 1 - \big[\mathrm{e}^{2x}\big]_{x=0}^{1} = 3\,\mathrm{e}^2 - 1 - \,\mathrm{e}^2 + 1 = 2\mathrm{e}^2\end{aligned}$$

combining the results $\rightsquigarrow$

$$s = A - B = 9\,\mathrm{e}^2/2 - 1/2 - 2\,\mathrm{e}^2 = 5\,\mathrm{e}^2/2 - 1/2 \approx 17.9726$$

Remark
Noting that $a^x = \mathrm{e}^{(\ln a)x}$, integration by parts applies to integrands which are products of a^x with polynomials $p(x)$ as well.

3.4 Antiderivatives of Products of Polynomials with Exponentials

Determine the antiderivatives of

a) $f(x) = (3x+1)\,\mathrm{e}^{x/2}$ b) $g(x) = x^2\,\cosh(2x)$

Resources: Integration by Parts, Table of Elementary Integrals

Problem Variants

■ $f(x) = (3x-1)^2\,\mathrm{e}^{-3x}$

$-(?x^2+1)\,\mathrm{e}^{-3x}/?$:

[ ] check

■ $f(x) = (2x-3)\,\sinh(x/2)$

$(?x{-}?)\,\cosh(x/2){-}?\,\sinh(x/2)$:

[ ] check

■ $f(x) = x\,\cosh(2x-1)$

$?\,\sinh(2x-1)/? - \cosh(2x-1)/?$:

[ ] check

Solution

With integration by parts,

$$\int pu' = pu - \int p'u \tag{1}$$

the degree n of a polynomial factor p is reduced, and eliminated with at most n such integrations. This leads to a more elementary expression which is easily integrated.

<u>a) antiderivative of $f(x) = (3x+1)\,\mathrm{e}^{x/2}$</u>

applying (1) with $p(x) = 3x+1$, $u'(x) = \mathrm{e}^{x/2}$ $\quad\rightsquigarrow$

$$\begin{aligned} F(x) &= \int (3x+1)\,\mathrm{e}^{x/2}\,\mathrm{d}x = (3x+1)\,\underbrace{2\mathrm{e}^{x/2}}_{u(x)} - \int \underbrace{3}_{p'(x)}\,2\mathrm{e}^{x/2}\,\mathrm{d}x \\ &= (6x+2)\,\mathrm{e}^{x/2} - 12\,\mathrm{e}^{x/2} + C = (6x-10)\,\mathrm{e}^{x/2} + C \end{aligned}$$

<u>b) antiderivative of $g(x) = x^2\,\cosh(2x)$</u>

noting that

$$\cosh t = \frac{\mathrm{e}^t + \mathrm{e}^{-t}}{2}, \qquad \int \cosh t\,\mathrm{d}t = \sinh t + C = \frac{\mathrm{e}^t - \mathrm{e}^{-t}}{2} + C\,,$$

integration by parts $\quad\rightsquigarrow$

$$G(x) = \int \underbrace{x^2}_{p(x)}\,\underbrace{\cosh(2x)}_{u'(x)}\,\mathrm{d}x = \overbrace{x^2\,\underbrace{\frac{1}{2}\sinh(2x)}_{u(x)}}^{A} - \overbrace{\int \underbrace{2x}_{p'(x)}\,\frac{1}{2}\sinh(2x)\,\mathrm{d}x}^{B}$$

integrating the term B by parts, noting that $\int \sinh t\,\mathrm{d}t = \cosh t + C$ $\quad\rightsquigarrow$

$$\begin{aligned} B &= \int x\sinh(2x)\,\mathrm{d}x = x\frac{1}{2}\cosh(2x) - \int \frac{1}{2}\cosh(2x)\,\mathrm{d}x \\ &= \frac{1}{2}x\,\cosh(2x) - \frac{1}{4}\,\sinh(2x) + C \end{aligned}$$

adding the term A $\quad\rightsquigarrow$

$$G(x) = \frac{x^2}{2}\,\sinh(2x) + \frac{1}{4}\,\sinh(2x) - \frac{x}{2}\,\cosh(2x) + C$$

3.5 Integrals of Products of a Polynomial with Logarithms

Compute

$$\text{a)} \int_1^2 x^2 \ln x \, dx \qquad \text{b)} \int_1^2 x^3 \ln^2 x \, dx$$

Resources: Integration by Parts, Table of Elementary Integrals

Problem Variants

■ a) $\int_1^2 (3-x)\ln x \, dx$, b) $\int_1^2 \ln^2(2x-1)\, dx$

---

a) ?.?? b) ?.??:

check

■ a) $\int_0^1 \sqrt{x} \ln x \, dx$, b) $\int_0^1 x \ln^2 \sqrt{x}\, dx$

---

a) $-?/?$ b) $1/??$:

check

■ a) $\int_1^2 (\ln x)/x \, dx$, b) $\int_1^2 (\ln^2 x)/x^2 \, dx$

---

a) 0.?? b) 0.0??:

check

Solution

For a product of a monomial $f(x) = x^m$ with a power of a logarithm $g(x) = \ln^n x$, integrating by parts,

$$\int_a^b f'g = [fg]_a^b - \int_a^b fg' , \tag{1}$$

reduces the exponent of the logarithm ($g(x) = \ln^n x \to g'(x) = \frac{n}{x} \ln^{n-1} x$) and thus simplifies the integral. Repeating this procedure n times eliminates the logarithm.

a) $\int_1^2 x^2 \ln x \, \mathrm{d}x$

applying (1) with $f'(x) = x^2$, $g(x) = \ln x$ $\rightsquigarrow$

$$\begin{aligned} \int_1^2 \underbrace{x^2}_{f'} \underbrace{\ln x}_{g} \, \mathrm{d}x &= \Big[\underbrace{x^3/3}_{f} \underbrace{\ln x}_{g} \Big]_{x=1}^{x=2} - \int_1^2 \overbrace{\underbrace{(x^3/3)}_{f} \underbrace{(1/x)}_{g'}}^{x^2/3} \, \mathrm{d}x \\ &= 8(\ln 2)/3 - 0 - \big[x^3/9\big]_{x=1}^{x=2} = 8(\ln 2)/3 - 8/9 + 1/9 \\ &= 8 \ln 2/3 - 7/9 \approx 1.0706 \end{aligned}$$

b) $\int_1^2 x^3 \ln^2 x \, \mathrm{d}x$

applying (1) $\rightsquigarrow$

$$s := \int_1^2 \underbrace{x^3}_{f'} \underbrace{\ln^2 x}_{g} \, \mathrm{d}x = \underbrace{\big[(x^4/4) \ \ln^2 x\big]_{x=1}^{x=2}}_{A} - \underbrace{\int_1^2 (x^4/4) \, (2/x) \ln x \, \mathrm{d}x}_{B}$$

- $A = 4 \ln^2 2$, since $\ln 1 = 0$
- B is computed by integrating by parts a second time:

$$\begin{aligned} B &= \int_1^2 (x^3/2) \ \ln x \, \mathrm{d}x = \big[(x^4/8) \ \ln x\big]_{x=1}^{x=2} - \int_1^2 \underbrace{(x^4/8) \, (1/x)}_{x^3/8} \, \mathrm{d}x \\ &= 2 \ln 2 - \big[x^4/32\big]_{x=1}^{x=2} = 2 \ln 2 - 1/2 + 1/32 \end{aligned}$$

combining the results $\rightsquigarrow$

$$s = A - B = 4 \ln^2 2 - (2 \ln 2 - 15/32) = 15/32 - 2 \ln 2 + 4 \ln^2 2 \approx 1.0042$$

3.6 Antiderivatives of a Product with a Logarithm

Determine the antiderivative of $f(x) = \dfrac{\ln x}{(x+1)^2}$ for $x > 0$.

Resources: Integration by Parts, Table of Elementary Integrals

Problem Variants

■ $f(x) = \ln(x+1)/x^2$

$\ln x?(x??)\ln(x+1)/? + C$:

check

■ $f(x) = \ln(x)/\sqrt{x}$

$(?\ln x??)\sqrt{x} + C$:

check

■ $f(x) = \sqrt{x}\ln^2 x$

$?x^{?/2}(?\ln^2 x - ??\ln x + ?)/27 + C$:

check

Solution

Integrating by parts,

$$\int f'g = fg - \int fg' \,, \tag{1}$$

with g the expression with logarithms usually leads to a simpler integral, since differentiation replaces $\ln u(x)$ by $u'(x)/u(x)$ (by $1/x$ for $u(x) = x$).

<u>Antiderivative of $\ln x/(x+1)^2$</u>

applying (1) with $f'(x) = 1/(x+1)^2$ and $g(x) = \ln x$ $\quad\rightsquigarrow$

$$F(x) = \int \frac{1}{(x+1)^2}\,\ln x\,\mathrm{d}x = \underbrace{-\frac{1}{x+1}}_{f(x)}\ln x - \int -\frac{1}{x+1}\,\underbrace{\frac{1}{x}}_{g'(x)}\,\mathrm{d}x$$

partial fraction decomposition of the integrand of the last term,

$$\frac{1}{(x+1)x} = \frac{1}{x} - \frac{1}{x+1}$$

$\rightsquigarrow$

$$\begin{aligned} F(x) &= -\frac{\ln x}{x+1} + \int \frac{1}{x} - \frac{1}{x+1}\,\mathrm{d}x \\ &= -\frac{\ln x}{x+1} + \ln x - \ln(x+1) + C = \frac{x\ln x}{x+1} - \ln(x+1) + C \end{aligned}$$

3.7 Integral of a Product of Sine/Cosine with an Exponential

Compute $\displaystyle\int_0^{\pi/2} \sin^2 x\, e^{2x}\, dx$.

Resources: Integration by Parts

Problem Variants

■ $\displaystyle\int_0^{\pi} \cos^2 x\, e^{x}\, dx$

??.?:

[ ] check

■ $\displaystyle\int_0^{\pi} \cos x \sin x\, e^{x}\, dx$

−?.??:

[ ] check

■ $\displaystyle\int_0^{\pi} \sin^3 x\, e^{3x}\, dx$

???:

[ ] check

Solution

If $f'' = a + bf$ and $g = cg''$, integrating an integral I of the product $f(g')'$ by parts twice leads to an equation for I. This is the case, e.g., for $f(x) = \sin(kx)$, $\cos(kx)$ $\sin^2 x, \ldots$ and $g(x) = \mathrm{e}^{\ell x}$.

$I = \int_0^{\pi/2} \sin^2 x \, \mathrm{e}^{2x} \, \mathrm{d}x$

integrating by parts (differentiating $\sin^2 x$ and forming the antiderivative of e^{2x}) $\rightsquigarrow$

$$I = \int_0^{\pi/2} \underbrace{\sin^2 x}_{u(x)} \underbrace{\mathrm{e}^{2x}}_{v'(x)} \, \mathrm{d}x = \overbrace{\left[\sin^2 x \, \underbrace{\frac{\mathrm{e}^{2x}}{2}}_{v(x)}\right]_{x=0}^{x=\pi/2}}^{A} - \overbrace{\int_0^{\pi/2} \underbrace{2\sin x \cos x}_{u'(x)} \, \frac{\mathrm{e}^{2x}}{2} \, \mathrm{d}x}^{B}$$

- $\sin(\pi/2) = 1$, $\sin 0 = 0 \quad \Longrightarrow \quad A = \mathrm{e}^{2\pi/2}/2 - 0 = \mathrm{e}^{\pi}/2$
- a second integration by parts, noting that $\sin x \cos x = 0$ at the integration limits and $\cos^2 + \sin^2 = 1$ $\rightsquigarrow$

$$\begin{aligned} B &= \int_0^{\pi/2} \underbrace{\sin x \cos x}_{u(x)} \underbrace{\mathrm{e}^{2x}}_{v'(x)} \, \mathrm{d}x \\ &= \left[\sin x \cos x \, \underbrace{\frac{\mathrm{e}^{2x}}{2}}_{v(x)}\right]_{x=0}^{x=\pi/2} - \int_0^{\pi/2} \underbrace{(\cos^2 x - \sin^2 x)}_{u'(x) = 1 - 2\sin^2 x} \, \frac{\mathrm{e}^{2x}}{2} \, \mathrm{d}x \\ &= 0 - \int_0^{\pi/2} \frac{\mathrm{e}^{2x}}{2} \, \mathrm{d}x + \int_0^{\pi/2} \sin^2 x \, \mathrm{e}^{2x} \, \mathrm{d}x = -\left(\mathrm{e}^{\pi}/4 - 1/4\right) + I \end{aligned}$$

The original integral I appears on the right side!

combining the computations $\rightsquigarrow$

$$I = A - B = \mathrm{e}^{\pi}/2 + \mathrm{e}^{\pi}/4 - \frac{1}{4} - I$$

solving for I $\rightsquigarrow$

$$I = \left(3\,\mathrm{e}^{\pi} - 1\right)/8$$

3.8 Antiderivatives of Products of Sines/Cosines with Exponentials

Determine the antiderivatives of

$$\text{a) } f(x) = \cos(2x)\,\mathrm{e}^{4x} \qquad \text{b) } g(x) = \cos(4x)\,\sin(2x)$$

Resources: Integration by Parts

Problem Variants

■ $\int \sin(3x)\,\mathrm{e}^{-x}\,\mathrm{d}x$

$-(?\cos(3x) + \sin(3x))\,\mathrm{e}^{-x}/??$:

[ ] check

■ $\int \sin(2x)\,\sin(3x)\,\mathrm{d}x$

$(?\sin x ?\sin(?x)/??$:

[ ] check

■ $\int \cos x\,\sinh x\,\mathrm{d}x$

$\cos x ????x/?+???x \sinh x/?$:

[ ] check

Solution

For products $f(x)$ of sines/cosines and exponentials, an equation for the antiderivative $F(x)$ can be obtained by twofold integration by parts.

a) antiderivative of $f(x) = \cos(2x)\,\mathrm{e}^{4x}$

integration by parts $\rightsquigarrow$

$$F(x) = \int \underbrace{\cos(2x)}_{u'(x)} \underbrace{\mathrm{e}^{4x}}_{v(x)}\,\mathrm{d}x = \underbrace{\frac{\sin(2x)}{2}}_{u(x)}\,\mathrm{e}^{4x} - \overbrace{\int \frac{\sin(2x)}{2} \underbrace{4\mathrm{e}^{4x}}_{v'(x)}\,\mathrm{d}x}^{A}$$

a second integration by parts $\rightsquigarrow$

$$A = \int \underbrace{2\sin(2x)}_{u'(x)} \underbrace{\mathrm{e}^{4x}}_{v(x)}\,\mathrm{d}x = \underbrace{-\cos(2x)}_{u(x)}\,\mathrm{e}^{4x} - \overbrace{\int (-\cos(2x)) \underbrace{4\mathrm{e}^{4x}}_{v'(x)}\,\mathrm{d}x}^{-4F(x)}$$

combining the computations and solving for the antiderivative $F(x)$ $\rightsquigarrow$

$$F(x) = \frac{\sin(2x)}{2}\,\mathrm{e}^{4x} - A = \frac{\sin(2x)}{2}\,\mathrm{e}^{4x} + \cos(2x)\,\mathrm{e}^{4x} - 4F(x)\,,$$

i.e., $F(x) = (\sin(2x) + 2\cos(2x))\,\mathrm{e}^{4x}/10$

b) antiderivative of $g(x) = \cos(4x)\,\sin(2x)$

Instead of using integration by parts (which is possible), an alternative is to apply Euler's formulas:

$$\cos t = \frac{\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t}}{2}, \quad \sin t = \frac{\mathrm{e}^{\mathrm{i}t} - \mathrm{e}^{-\mathrm{i}t}}{2\mathrm{i}}\,.$$

substituting $t = 4x$ and $t = 2x$ $\rightsquigarrow$

$$\begin{aligned} G(x) &= \int \cos(4x)\,\sin(2x)\,\mathrm{d}x = \frac{1}{4\mathrm{i}} \int (\mathrm{e}^{4\mathrm{i}x} + \mathrm{e}^{-4\mathrm{i}x})(\mathrm{e}^{2\mathrm{i}x} - \mathrm{e}^{-2\mathrm{i}x})\,\mathrm{d}x \\ &= \frac{1}{4\mathrm{i}} \int \mathrm{e}^{6\mathrm{i}x} - \mathrm{e}^{2\mathrm{i}x} + \mathrm{e}^{-2\mathrm{i}x} - \mathrm{e}^{-6\mathrm{i}x}\,\mathrm{d}x \end{aligned}$$

applying Euler's formulas in the opposite direction by combining the first and last as well as the second and third summand of the integrand $\rightsquigarrow$

$$G(x) = \frac{1}{2} \int \sin(6x) - \sin(2x)\,\mathrm{d}x = -\frac{\cos(6x)}{12} + \frac{\cos(2x)}{4} + C$$

3.9 N-fold Integration by Parts

Compute $\displaystyle\int_{-1}^{1}(1-x^2)^n\,\mathrm{d}x\,.$

Resources: Integration by Parts

Problem Variants

■ $\int_0^2 x^{2n}(2-x)^n\,\mathrm{d}x$

---

integral for $n=10$: ?.??:

[ ] check

■ $\int_0^1 x^n \ln^n x\,\mathrm{d}x$

---

integral for $n=10$: 0.0000??:

[ ] check

■ $\int_0^\pi \sin^{2n} x\;(\cos^{2n} x\;\sin x)\;\mathrm{d}x$

---

integral for $n=5$: 0.000??:

[ ] check

Solution

writing the integrand as a product, $(1-x^2)^n = (1+x)^n(1-x)^n$, integration by parts $\rightsquigarrow$

$$\int_{-1}^{1} \underbrace{(1-x)^n}_{u'(x)} \underbrace{(1+x)^n}_{v(x)} \, \mathrm{d}x = \Big[\underbrace{-\frac{(1-x)^{n+1}}{n+1}}_{u(x)} \underbrace{(1+x)^n}_{v(x)} \Big]_{x=-1}^{x=1} - \int_{-1}^{1} \underbrace{-\frac{(1-x)^{n+1}}{n+1}}_{u(x)} \underbrace{n(1+x)^{n-1}}_{v'(x)} \, \mathrm{d}x \,,$$

where $[\ldots]_{x=-1}^{x=1} = 0$, since $(1-x)(1+x) = 0$ for $x = \pm 1$
integrating by parts one more time $\rightsquigarrow$

$$\frac{n}{n+1} \int_{-1}^{1} \underbrace{(1-x)^{n+1}}_{u'(x)} \underbrace{(1+x)^{n-1}}_{v(x)} \, \mathrm{d}x = 0 - \frac{n}{n+1} \int_{-1}^{1} \underbrace{-\frac{(1-x)^{n+2}}{n+2}}_{u(x)} \underbrace{(n-1)(1+x)^{n-1}}_{v'(x)} \, \mathrm{d}x$$

Again, the first term (omitted), $[\ldots]_{x=-1}^{x=1} = 0$, since the integrand contains the factors $(1-x)$ and $(1+x)$, vanishing at 1 and -1, respectively.
The general pattern is clear: integrating by parts $n-2$ more times $\rightsquigarrow$

$$\frac{n}{n+1} \frac{n-1}{n+2} \cdots \frac{1}{2n} \int_{-1}^{1} (1-x)^{2n} \, \mathrm{d}x = \frac{n!}{(2n)!/n!} \left[-\frac{(1-x)^{2n+1}}{2n+1} \right]_{x=-1}^{x=1} = \frac{(n!)^2}{(2n+1)!} 2^{2n+1}$$

Chapter 4

Substitution

4.1 Integration, Using the Chain Rule

Compute $\displaystyle\int_0^9 \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}\,\mathrm{d}x\,.$

Resources: Substitution

Problem Variants

■ $\displaystyle\int_0^1 x^2(x^3-1)^4\,\mathrm{d}x$

1/??:

[ ] check

■ $\displaystyle\int_0^\pi \frac{\sin x}{2+\cos x}\,\mathrm{d}x$

?.??:

[ ] check

■ $\displaystyle\int_1^{\mathrm{e}} \frac{\sqrt{\ln x}}{x}\,\mathrm{d}x$

?.??:

[ ] check

Solution

By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}x} F(g(x)) = f(g(x))g'(x), \quad f = F',$$

i.e., for an antiderivative F of f, $F(g(x))$ is an antiderivative of $f(g(x))g'(x)$. Hence, by the fundamental theorem of calculus,

$$\int_a^b f(g(x))g'(x)\,\mathrm{d}x = \Big[F(\underbrace{g(x)}_{y})\Big]_{x=a}^{x=b} = [F(y)]_{y=g(a)}^{y=g(b)}\,. \tag{1}$$

Application to $\int_0^9 \sqrt{1+\sqrt{x}}\Big/\sqrt{x}\,\mathrm{d}x$

- rewrite the integrand in the form $f(g(x))g'(x)$:

$$\sqrt{1+\sqrt{x}}\Big/\sqrt{x} = 2\Big(1+\underbrace{\sqrt{x}}_{g(x)}\Big)^{1/2}\underbrace{\frac{1}{2}x^{-1/2}}_{g'(x)}, \quad f(y) = 2(1+y)^{1/2}$$

- apply the formula (1) with $F(y) = \dfrac{4}{3}(1+y)^{3/2}$ an antiderivative of f:

$$\left[\frac{4}{3}(1+y)^{3/2}\right]_{y=\sqrt{0}}^{y=\sqrt{9}} = \frac{32}{3} - \frac{4}{3} = \frac{28}{3}$$

4.2 Antiderivatives, Using the Chain Rule

Determine the antiderivatives of

$$\text{a) } \frac{x}{x^2+3} \qquad \text{b) } \mathrm{e}^{4x}\sqrt{1+\mathrm{e}^{4x}} \qquad \text{c) } \sin(2x)\cos(x)$$

with respect to the variable x.

Resources: Substitution

Problem Variants

■ $\exp(x + \exp x)$

$\exp(??????) + C$:

check

■ $\cos(\cos(2x)\sin(2x))$

$????(\cos(2x))/? + C$:

check

■ $\dfrac{1}{x+\sqrt{x}}$

$?\ln(? + \sqrt{x}) + C$:

check

Solution

If an integrand can be written in the form

$$g(x) = c\, f(y(x))\, y'(x)\,, \tag{1a}$$

then, by the chain rule, the antiderivative of g is

$$G(x) = c\, F(y(x)) \tag{1b}$$

with F an antiderivative of f with respect to the variable y.

<u>a) $g(x) = \dfrac{x}{x^2+3}\,\mathrm{d}x$</u>

identify y, y', f, and c:

$$g(x) = \left(\frac{1}{2}\right)\frac{1}{x^2+3}(2x)$$

$\implies$ $y(x) = x^2+3,\ y'(x) = 2x,\ f(y) = 1/y,\ c = 1/2$

⚠ Note that the factors (1/2) and 2 have been added to obtain $2x$ as derivative of x^2+3.

applying (1) with $F(y) = \displaystyle\int \frac{1}{y}\,\mathrm{d}y = \ln|y| + C$ the antiderivative of f $\rightsquigarrow$

$$G(x) = \frac{1}{2}F(y)\bigg|_{y=x^2+3} = \frac{1}{2}\ln|y|\bigg|_{y=x^2+3} + C = \frac{1}{2}\ln(x^2+3) + C$$

<u>b) $g(x) = \mathrm{e}^{4x}\sqrt{1+\mathrm{e}^{4x}}$</u>

adding an additional factor, $g(x) = (1/4)\sqrt{1+\mathrm{e}^{4x}}\,(4\,\mathrm{e}^{4x})$ and

$$y(x) = \mathrm{e}^{4x},\quad y'(x) = 4\mathrm{e}^{4x},\quad f(y) = (1+y)^{1/2},\quad c = 1/4$$

$F(y) = \displaystyle\int (1+y)^{1/2}\,\mathrm{d}y = \frac{2}{3}(1+y)^{3/2} + C$ $\rightsquigarrow$ antiderivative

$$\frac{1}{4}F(y)\bigg|_{y=y(x)} = \frac{1}{4}\frac{2}{3}(1+y)^{3/2}\bigg|_{y=\mathrm{e}^{4x}} + C = \frac{1}{6}(1+\mathrm{e}^{4x})^{3/2} + C$$

c) $g(x) = \sin(2x)\cos(x)$

$\sin(2x) = 2\sin x \cos x \quad \rightsquigarrow$

$$g(x) = (2\sin x \cos x)\cos x = (-2)(\cos^2 x)(-\sin x)\,,$$

i.e.,

$$y(x) = \cos x, \quad y'(x) = -\sin x, \quad f(y) = y^2, \quad c = -2$$

substitution, according to (1) $\quad \rightsquigarrow \quad$ antiderivative

$$-\,2F(y)|_{y=y(x)} = -2 \cdot \frac{1}{3}y^3\Big|_{y=\cos x} + C = -\frac{2}{3}\cos^3 x + C$$

4.3 Antiderivative of a Rational Function Involving Square Roots

Determine $\int \frac{x}{\sqrt{2x+3}}\,\mathrm{d}x$.

Resources: Special Substitutions

Problem Variants

- $\int \frac{x}{1+\sqrt{x+1}}\,\mathrm{d}x$

$?(x+1)^{?/2}-??$:

[ ] check

- $\int \frac{\sqrt{2x-1}}{x}\,\mathrm{d}x$

$?\sqrt{2x-1}-?\arctan\sqrt{2x-1}$:

[ ] check

- $\int \frac{x+\sqrt{x}}{x-\sqrt{x}}\,\mathrm{d}x$

$x+?\sqrt{x}+?\ln(\sqrt{x}-?)$:

[ ] check

Solution

For an indefinite integral $F(x) = \int r(x, \sqrt{px+q})\,\mathrm{d}x$ with a rational function r, the square roots are eliminated by the substitution

$$y = \sqrt{px+q}, \quad \mathrm{d}y = \frac{p}{2y}\,\mathrm{d}x\,.$$

This yields the indefinite integral

$$G(y) = \int \underbrace{r((y^2-q)/p, y)}_{\tilde{r}(y)} \frac{2y}{p}\,\mathrm{d}y$$

with a rational function $\tilde{r}$, and $F(x) = G(\sqrt{px+q})$.

<u>Application to $\displaystyle\int x/\sqrt{2x+3}\,\mathrm{d}x$</u>

substituting $y = \sqrt{2x+3} \iff x = (y^2-3)/2$ with $\mathrm{d}x = y\,\mathrm{d}y$ $\quad\rightsquigarrow$

$$G(y) = \int \underbrace{\frac{(y^2-3)/2}{y}}_{\tilde{r}(y)} y\,\mathrm{d}y = \int \frac{y^2-3}{2}\,\mathrm{d}y = \frac{1}{6}y^3 - \frac{3}{2}y + C$$

backsubstitution $\quad\rightsquigarrow\quad$ antiderivative

$$\begin{aligned} F(x) &= G(\sqrt{2x+3}) = \frac{1}{6}(2x+3)^{3/2} - \frac{3}{2}(2x+3)^{1/2} + C \\ &= (x/3-1)\sqrt{2x+3} + C \end{aligned}$$

<u>Alternative solution</u>

expressing the numerator x in terms of $2x+3$, i.e., writing

$$x = \frac{1}{2}(2x+3) - \frac{3}{2}$$

$\rightsquigarrow\quad$ integrand consisting of elementary terms:

$$\int \frac{x}{\sqrt{2x+3}}\,\mathrm{d}x = \int \frac{1}{2}(2x+3)^{1/2} - \frac{3}{2}(2x+3)^{-1/2}\,\mathrm{d}x$$

4.4 Substitution for Integrating a Rational Function of Fractional Powers

Compute

$$\int_1^{64} \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}}.$$

Resources: Special Substitutions, Elementary Rational Integrands

Problem Variants

■ $\displaystyle\int_1^{16} \frac{\sqrt{x}}{1+\sqrt[4]{x}}\,\mathrm{d}x$

??.?:

[ ] check

■ $\displaystyle\int_0^{1} \frac{x^{-1/2}}{4-x^{1/3}}\,\mathrm{d}x$

?.??:

[ ] check

■ $\displaystyle\int_0^{1} \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x^2}}$

?.??:

[ ] check

Solution

Rational functions of x, $x^{1/p}$, and $x^{1/q}$ can be transformed to a rational function of y with the substitution

$$x = y^r, \quad \mathrm{d}x = r y^{r-1}\,\mathrm{d}y\,,$$

where r is the least common multiple of p and q.

Application to $\displaystyle\int_1^{64} \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}}$

- substitution:

  least common multiple of $p=2$ and $q=3$: $r=6$

  substituting $x=y^6$, $\mathrm{d}x = 6y^5\,\mathrm{d}y$, and transforming the limits of integration, $x=1 \to y=1$, $x=64 \to y=\sqrt[6]{64}=2$ $\quad\rightsquigarrow$

$$\int_1^{64} \frac{\mathrm{d}x}{\sqrt{x}+\sqrt[3]{x}} = \int_1^2 \frac{6y^5}{y^3+y^2}\,\mathrm{d}y = 6\underbrace{\int_1^2 \frac{y^3}{y+1}\,\mathrm{d}y}_{A}$$

- computation of the rational integral:

  expanding $p(y)=y^3$ in terms of powers of $(y+1)$ (cubic Taylor polynomial at $y_0=-1$) $\quad\rightsquigarrow$

$$y^3 = \sum_{k=0}^{3} \frac{p^{(k)}(-1)}{k!}\,(y+1)^k = -1 + 3(y+1) - 3(y+1)^2 + (y+1)^3$$

  dividing by $(y+1)$ $\quad\rightsquigarrow$

$$\begin{aligned} 6A &= 6\int_1^2 -\frac{1}{y+1} + \underbrace{3 - 3(y+1) + (y+1)^2}_{1-y+y^2}\,\mathrm{d}y \\ &= \left[-6\ln|y+1| + 6y - 3y^2 + 2y^3\right]_{y=1}^{y=2} \\ &= -6(\ln 3 - \ln 2) + (12-6) - (12-3) + (16-2) \\ &= 11 - 6\ln(3/2) \approx 8.5672 \end{aligned}$$

4.5 Substitution $\mathbf{y} = \exp(\mathbf{x})$

Determine an antiderivative of $f(x) = \dfrac{1}{e^x - 3 + 2\,e^{-x}}$

Resources: Substitution

Problem Variants

■ $f(x) = \dfrac{1 + e^x}{2 + e^x}$

$(\ln(? + e^x){+}?)/?$:

[ ] check

■ $f(x) = \exp(\exp(x))\exp(2x)$

$(\exp(?x){-}?)\exp(\exp(?x))$:

[ ] check

■ $f(x) = \dfrac{e^{2x}}{\sqrt{3 + e^x}}$

$(2e^x/?{-}?)\sqrt{? + e^x}$:

[ ] check

Solution

The substitution

$$y = \mathrm{e}^x, \quad \mathrm{d}y = \mathrm{e}^x\,\mathrm{d}x \iff \mathrm{d}x = \mathrm{d}y/y$$

eliminates exponential functions in an integrand, which usually leads to simplifications.

applying this substitution to $\displaystyle\int \frac{\mathrm{d}x}{\mathrm{e}^x - 3 + 2\,\mathrm{e}^{-x}} \quad \rightsquigarrow$

$$\int \frac{\mathrm{d}y/y}{y - 3 + 2/y} = \int \underbrace{\frac{1}{y^2 - 3y + 2}}_{g(y)}\,\mathrm{d}y$$

construction of an antiderivative of g with partial fraction decomposition: poles at $y_1 = 1$ and $y_2 = 2 \quad \rightsquigarrow \quad$ ansatz

$$g(y) = \frac{1}{(y-1)(y-2)} = \frac{a}{y-1} + \frac{b}{y-2}$$

- multiplying by $y - 1$ and setting $y = 1 \quad \implies \quad a = \dfrac{1}{1-2} = -1$
- $\cdot(y-2)$, $y = 2 \quad \implies \quad b = 1$

integrating the elementary summands $\quad \rightsquigarrow \quad$ antiderivative

$$G(y) = \int -\frac{1}{y-1} + \frac{1}{y-2}\,\mathrm{d}y = -\ln|y-1| + \ln|y-2| + C = \ln\left|\frac{y-2}{y-1}\right| + C$$

reversing the substitution $y = \mathrm{e}^x \quad \rightsquigarrow \quad$ antiderivative of $f(x) = \dfrac{1}{\mathrm{e}^x - 3 + 2\,\mathrm{e}^{-x}}$:

$$F(x) = \ln\left|\frac{\mathrm{e}^x - 2}{\mathrm{e}^x - 1}\right| + C$$

4.6 Trigonometric Substitution for an Integrand Containing the Expression $\sqrt{x^2 + a^2}$

Determine the antiderivative of

$$f(x) = \frac{1}{x^2\sqrt{x^2+9}}$$

and compute $\int_3^{3\sqrt{3}} f$.

Resources: Substitution, Special Substitutions

Problem Variants

- $\displaystyle\int_0^1 x^3\sqrt{x^2+1}\,\mathrm{d}x$

?.??:

[ ] check

- $\displaystyle\int_0^2 \frac{x^3}{\sqrt{x^2+4}}\,\mathrm{d}x$

?.??:

[ ] check

- $\displaystyle\int_{1/2}^{\sqrt{3}/2} \frac{\sqrt{4x^2+1}}{x^4}\,\mathrm{d}x$

?.??:

[ ] check

Solution

Integrands, containing the expression $\sqrt{x^2+a^2}$, can be simplified with the trigonometric substitution

$$x = a\tan y, \quad \mathrm{d}x = \frac{a}{\cos^2 y}\,\mathrm{d}y, \quad \sqrt{x^2+a^2} = \frac{a}{\cos y}\,.$$

Application to $\displaystyle\int_3^{3\sqrt{3}} \frac{\mathrm{d}x}{x^2\sqrt{x^2+9}}$

- antiderivative of the integrand $f(x) = \dfrac{1}{x^2\sqrt{x^2+9}}$:

  substituting

  $$x = 3\tan t, \quad \mathrm{d}x = \frac{3}{\cos^2 t}\,\mathrm{d}t, \quad \sqrt{x^2+9} = \frac{3}{\cos t}$$

  $\rightsquigarrow$

  $$F(x) = \int \frac{\mathrm{d}x}{x^2\sqrt{x^2+9}} = \int \frac{1}{(9\tan^2 t)(3/\cos t)}\frac{3}{\cos^2 t}\,\mathrm{d}t \underset{\tan t = \frac{\sin t}{\cos t}}{=} \int \underbrace{\frac{\cos t}{9\sin^2 t}}_{g(t)}\,\mathrm{d}t$$

  Using that, by the chain rule, $-1/u(t)$ is an antiderivative of $u'(t)/u(t)^2$, the antiderivative of $g(t)$ is

  $$G(t) = -\frac{1}{9\sin t} + C$$

  reversing the substitution $x = 3\tan t \quad \rightsquigarrow \quad$ antiderivative of f:

  $$F(x) = -\frac{1}{9\sin(\underbrace{\arctan(x/3)}_{t})} + C$$

  Not an obvious expression!

- $\int_3^{3\sqrt{3}} f$:

  Noting that $\tan(\pi/4) = 1$, $\tan(\pi/3) = \sqrt{3}$, the integral equals

  $$\left[-\frac{1}{9\sin(\arctan(x/3))}\right]_{x=3}^{x=3\sqrt{3}} = -\frac{1}{9\sin(\arctan\sqrt{3})} + \frac{1}{9\sin(\arctan 1)}$$
  $$= -\frac{1}{9\sin(\pi/3)} + \frac{1}{9\sin(\pi/4)} = -\frac{2}{9\sqrt{3}} + \frac{2}{9\sqrt{2}} \approx 0.0288\,.$$

Remark
If the antiderivative of f is not of interest, a faster way to compute $\int_3^{3\sqrt{3}} f$ is to transform the integration limits,

$$x = 3 \to t = \arctan x/3 = \arctan 1 = \pi/4, \quad x = 3\sqrt{3} \to t = \pi/3\,,$$

and compute the transformed integral $\int_{\pi/4}^{\pi/3} g(t)\,\mathrm{d}t$ instead,

$$\int_3^{3\sqrt{3}} f = [G(t)]_{t=\pi/4}^{t=\pi/3} = \left[-\frac{1}{9\sin t}\right]_{t=\pi/4}^{t=\pi/3} = \cdots\,,$$

leading (obviously) to the same result.

4.7 Trigonometric Substitution for an Integrand Containing the Expression $\sqrt{\mathbf{a^2 - x^2}}$

Determine an antiderivative of

$$f(x) = \frac{x^3}{\sqrt{4 - x^2}}$$

and compute $\int_0^2 f$.

Resources: Substitution, Special Substitutions

Problem Variants

■ $\displaystyle\int_0^3 x^3\sqrt{9 - x^2}\,\mathrm{d}x$

??.?:

check

■ $\displaystyle\int_0^{1/\sqrt{2}} \frac{x^2}{\sqrt{1 - x^2}}\,\mathrm{d}x$

?.??:

check

■ $\displaystyle\int_{-2}^2 (4 - x^2)^{3/2}\,\mathrm{d}x$

??.?:

check

Solution

An integrand, which contains the expression $\sqrt{a^2 - x^2}$, can in many cases be simplified with the substitution

$$x = a \sin t \text{ with } t \in [-\pi/2, \pi/2], \quad \sqrt{a^2 - x^2} = a \cos t, \quad \mathrm{d}x = a \cos t \, \mathrm{d}t \,.$$

<u>Application to $\displaystyle\int_0^2 \frac{x^3}{\sqrt{4 - x^2}} \, \mathrm{d}x$</u>

- antiderivative $F(x)$ of the integrand $f(x) = \dfrac{x^3}{\sqrt{4 - x^2}}$:

  substituting

  $$x = 2 \sin t, \quad \sqrt{4 - x^2} = 2 \cos t, \quad \mathrm{d}x = 2 \cos t \, \mathrm{d}t$$

  $\rightsquigarrow$

  $$F(x) = \int \frac{(2 \sin t)^3}{2 \cos t} \, 2 \cos t \, \mathrm{d}t = 8 \int \sin^3 t \, \mathrm{d}t$$

  Writing $\sin^3 t = (1 - \cos^2 t) \sin t$ and substituting $u = \cos t$, $\mathrm{d}u = -\sin t \, \mathrm{d}t \quad \rightsquigarrow$

  $$F(x) = 8 \int u^2 - 1 \, \mathrm{d}u = 8(u^3/3 - u) + C$$

  reversing the substitutions,

  $$u = \cos t = \sqrt{1 - \sin^2 t} = \frac{\sqrt{4 - (2 \sin t)^2}}{2} = \frac{\sqrt{4 - x^2}}{2}$$

  $\rightsquigarrow$

  $$F(x) = \frac{(4 - x^2)^{3/2}}{3} - 4\sqrt{4 - x^2} = -\frac{x^2 + 8}{3} \sqrt{4 - x^2} + C$$

- $\int_0^2 f$:

  fundamental theorem of calculus $\quad \rightsquigarrow$

  $$\int_0^2 f = [F]_0^2 = \left[-\frac{x^2 + 8}{3} \sqrt{4 - x^2} \right]_{x=0}^{x=2} = -0 + \frac{8}{3}\sqrt{4} = \frac{16}{3}$$

4.8 Hyperbolic Substitution for an Integrand Containing the Expression $\sqrt{x^2 - a^2}$

Compute $\displaystyle\int_3^5 \frac{\sqrt{x^2-9}}{x^2}\,\mathrm{d}x$.

Resources: Substitution, Special Substitutions

Problem Variants

- $\displaystyle\int_1^2 x^2\sqrt{x^2-1}\,\mathrm{d}x$

?.??:

[ ] check

- $\displaystyle\int_2^3 (x^2-4)^{3/2}\,\mathrm{d}x$

?.??:

[ ] check

- $\displaystyle\int_1^2 \frac{x^3}{\sqrt{x^2-1}}\,\mathrm{d}x$

?.??:

[ ] check

Solution

An integrand, which contains the expression $\sqrt{x^2-a^2}$, can in many cases be simplified with the substitution

$$x = a\cosh t, \quad \sqrt{x^2-a^2} = a\sinh t, \quad \mathrm{d}x = a\sinh t\,\mathrm{d}t\,. \tag{1}$$

Application to $\displaystyle\int_3^5 \frac{\sqrt{x^2-9}}{x^2}\,\mathrm{d}x$

substitution (1) with $a = 3$ $\rightsquigarrow$

$$\int_0^b \frac{3\sinh t}{(3\cosh t)^2}\,3\sinh t\,\mathrm{d}t = \int_0^b \frac{\sinh^2 t}{\cosh^2 t}\,\mathrm{d}t$$

since $3\cosh 0 = 3$, and with $b = \operatorname{arccosh}(5/3) > 0 \leftrightarrow 5 = 3\cosh b$

integrating by parts $\rightsquigarrow$

$$\int_0^b \underbrace{\sinh t}_{u(t)}\,\underbrace{\frac{\sinh t}{\cosh^2 t}}_{v'(t)}\,\mathrm{d}t = \Big[\underbrace{\sinh t}_{u(t)}\,\underbrace{\frac{-1}{\cosh t}}_{v(t)}\Big]_{t=0}^{t=b} - \int_0^b \underbrace{\cosh t}_{u'(t)}\,\underbrace{\frac{-1}{\cosh t}}_{v(t)}\,\mathrm{d}t =: A + B$$

- first term: $\sinh 0 = 0$, $\cosh^2 - \sinh^2 = 1 \implies$

$$A = -\frac{\sinh b}{\cosh b} = -\frac{\sqrt{\cosh^2 b - 1}}{\cosh b} \underset{b=\operatorname{arccos}(5/3)}{=} -\frac{\sqrt{(5/3)^2-1}}{5/3} = -\frac{4}{5}$$

- second term: $B = -\int_0^b -1\,\mathrm{d}t = b$ where

$$\cosh b = \frac{\mathrm{e}^b + \mathrm{e}^{-b}}{2} = 5/3 \underset{z:=\mathrm{e}^b}{\iff} z + 1/z = 10/3 \iff 3z^2 - 10z + 3 = 0$$

  solving this quadratic equation $\rightsquigarrow$

$$z = \frac{10 \pm \sqrt{100 - 4\cdot 3\cdot 3}}{2\cdot 3} = \frac{10\pm 8}{6}$$

  with $z = 3$ leading to a positive value $b = \ln 3$

adding the results $\rightsquigarrow$ $A + B = -4/5 + \ln 3 \approx 0.2986$ as value of the integral

4.9 Integration of a Rational Trigonometric Function

Compute

$$\int_0^{\pi/2} \frac{2\,\mathrm{d}t}{4\cos t\,+\,3\sin t}\,.$$

Resources: Integration of Rational Trigonometric Functions, Partial Fractions

Problem Variants

- $\displaystyle\int_{\pi/3}^{\pi/2} \frac{1}{\sin^2 t}\,\mathrm{d}t$

?.??:

[ ] check

- $\displaystyle\int_0^{\pi} \frac{4}{3+\cos t}\,\mathrm{d}t$

?.??:

[ ] check

- $\displaystyle\int_{-\pi}^{\pi} \frac{3-\cos t}{2+\sin t}\,\mathrm{d}t$

??.?:

[ ] check

Solution

Substitution

standard substitution $x = \tan(t/2)$ for an integrand, which is a rational function of $\sin t$ and $\cos t$ $\quad\rightsquigarrow$

$$\cos t = \frac{1-x^2}{1+x^2}, \quad \sin t = \frac{2x}{1+x^2}, \quad \mathrm{d}x = \frac{1+x^2}{2}\,\mathrm{d}t$$

transformation of the boundaries

$$t = 0 \mapsto x = \tan 0 = 0, \quad t = \pi/2 \mapsto x = \tan \pi/4 = 1$$

application to the integrand $2/(3\sin t + 4\cos t)$ $\quad\rightsquigarrow$

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \frac{2\,\mathrm{d}t}{3\sin t \,+\, 4\cos t} &= \int_0^1 \frac{2(1+x^2)}{3(2x)+4(1-x^2)}\,\frac{2}{1+x^2}\,\mathrm{d}x \\ &= \int_0^1 \underbrace{\frac{-1}{x^2-\frac{3}{2}x-1}}_{r(x)}\,\mathrm{d}x\end{aligned}$$

Partial fraction decomposition

$x^2 - \frac{3}{2}x - 1 = (x-2)(x+1/2) \quad \Longrightarrow$
The integrand r has simple poles at $x = -1/2$ and $x = 2$, and, decomposing into partial fractions, can be represented in the form

$$r(x) = -\frac{1}{(x+1/2)(x-2)} = \frac{a}{x+1/2} + \frac{b}{x-2}\,.$$

multiplying by $(x+1/2)$ and setting $x = -1/2$ $\quad\rightsquigarrow$

$$a = -\frac{1}{-1/2-2} = \frac{2}{5}$$

multiplying by $(x-2)$ and setting $x = 2$ $\quad\rightsquigarrow$

$$b = -\frac{1}{2+1/2} = -\frac{2}{5}$$

Integration

$\int \mathrm{d}x/(x-p) = \ln|x-p| + C$, $\ln p - \ln q = \ln p/q$ $\quad\rightsquigarrow\quad$ antiderivative

$$R(x) = \int r(x)\,\mathrm{d}x = \frac{2}{5}\ln|x+1/2| - \frac{2}{5}\ln|x-2| + C = \frac{2}{5}\,\ln\left|\frac{x+1/2}{x-2}\right| + C$$

inserting the boundaries $\quad\rightsquigarrow$

$$\int_0^1 r(x)\,\mathrm{d}x = [R(x)]_0^1 = \frac{2}{5}\,\left(\ln(3/2) - \ln(1/4)\right) = \frac{2}{5}\,\ln 6 \approx 0.7167$$

Chapter 5

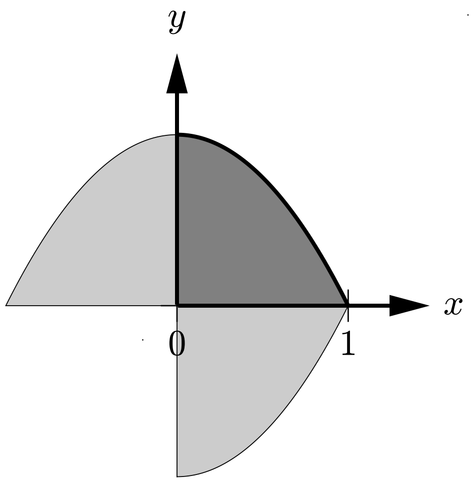
Solids of Revolution

5.1 Volume of Solids of Revolution with Respect to Different Axes

Determine the volumes of the solids resulting from rotation of the graph of the function

$$y = 1 - x^2, \quad 0 \le x \le 1\,,$$

about the x- and y-axis, respectively.
The cross sections of the solids are depicted in the figure, and have the dark gray area in common.



Resources: Volume and Surface of a Solid of Revolution

Problem Variants

■ $y(x) = 1 - \sqrt{x}$, $0 \le x \le 1$

$V_x = ?.??$, $V_y = ?.??$:

[ ] check

■ $y(x) = \cos x$, $0 \le x \le \pi/2$

$V_x = ?.??$, $V_y = ?.??$:

[ ] check

■ $y(x) = \mathrm{e}^{-x}$, $0 \le x \le 1$

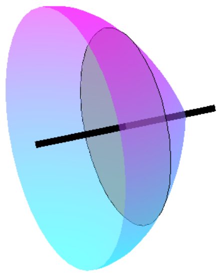
$V_x = ?.??$, $V_y = ?.??$:

[ ] check

Solution

Rotation about the x-axis

As depicted in the figure, the solid S can be described as the union of discs D_x, $0 \le x \le 1$, with radius $y(x)$.

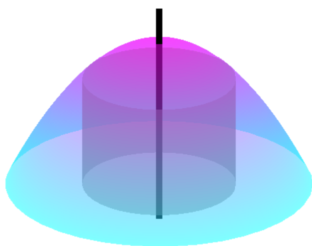


Hence,

$$\begin{aligned}\operatorname{vol} S &= \int_0^1 \underbrace{\pi y(x)^2}_{\text{area } D_x} \,\mathrm{d}x = \pi \int_0^1 (1-x^2)^2 \,\mathrm{d}x = \pi \int_0^1 1 - 2x^2 + x^4 \,\mathrm{d}x \\ &= \pi \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 \right]_{x=0}^{x=1} = \pi \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{8}{15}\pi \approx 1.6755\,.\end{aligned}$$

Rotation about the y-axis

As depicted in the figure, the solid S can be described as the union of cylinders C_x, $0 \le x \le 1$, with height $y(x)$ and radius x.

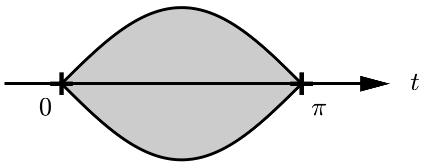


Hence,

$$\begin{aligned}
\operatorname{vol} S &= \int_0^1 \underbrace{2\pi x y(x)}_{\operatorname{area} C_x} \mathrm{d}x = 2\pi \int_0^1 x(1-x^2)\,\mathrm{d}x = 2\pi \int_0^1 x - x^3\,\mathrm{d}x \\
&= 2\pi \left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_{x=0}^{x=1} = 2\pi\left(\frac{1}{2} - \frac{1}{4}\right) = \pi/2 \approx 1.5707
\end{aligned}$$

5.2 Surface of a Solid of Revolution

Determine the surface of the solid of revolution with cross section and rotation axis depicted in the figure, which is generated by the radius function



$$r(t) = \sin t, \quad 0 \le t \le \pi .$$

Resources: Volume and Surface of a Solid of Revolution

Problem Variants

■ $r(t) = t^3$, $0 \le t \le 1$

?.??:

check

■ $r(t) = \sqrt{t}$, $0 \le t \le 1$

?.??:

check

■ $r(t) = \mathrm{e}^t$, $0 \le t \le 1$

??.?:

check

Solution

applying the formula

$$\operatorname{area} S = 2\pi \int_a^b r(t)\,\sqrt{1+r'(t)^2}\,\mathrm{d}t$$

for the surface of a solid of revolution S with radius function $r(t)$, $a \le t \le b$, to $r(t) = \sin t$, $0 \le t \le \pi$ $\quad\rightsquigarrow$

$$\operatorname{area} S = 2\pi \int_0^\pi \sin t\,\sqrt{1+\cos^2 t}\,\mathrm{d}t \underset{\text{symmetry}}{=} 4\pi \int_0^{\pi/2} \sin t\,\sqrt{1+\cos^2 t}\,\mathrm{d}t$$

substituting $x = \cos t$, $\mathrm{d}x = -\sin t\,\mathrm{d}t$, and noting that $t = 0 \leftrightarrow x = 1$, $t = \pi/2 \leftrightarrow x = 0$ $\quad\rightsquigarrow$

$$4\pi \int_0^1 \sqrt{1+x^2}\,\mathrm{d}x\,,$$

since reversing the order of integration changes the sign of the integrand

a second substitution, $x = \sinh u$, $\mathrm{d}x = \cosh u\,\mathrm{d}u$, noting that $1 + \sinh^2 u = \cosh^2 u$ $\quad\rightsquigarrow$

$$4\pi \int_0^{\operatorname{arcsinh}1} \cosh u\,(\cosh u\,\mathrm{d}u) = 4\pi \int_0^{\operatorname{arcsinh}1} \left(\frac{\mathrm{e}^u + \mathrm{e}^{-u}}{2}\right)^2 \mathrm{d}u$$

$$= \pi \int_0^{\operatorname{arcsinh}1} \mathrm{e}^{2u} + 2 + \mathrm{e}^{-2u}\,\mathrm{d}u = \underbrace{\frac{\pi}{2}\left[\mathrm{e}^{2u} + 4u - \mathrm{e}^{-2u}\right]_{u=0}^{u=\operatorname{arcsinh}1}}_{=:A}$$

$u = \operatorname{arcsinh} 1 \iff \sinh u = (\mathrm{e}^u - \mathrm{e}^{-u})/2 = 1$, i.e., with $E = \mathrm{e}^u > 0$,

$$E - 1/E = 2 \implies E^2 - 2E - 1 = 0 \underset{E>0}{\implies} E = 1 + \sqrt{2}$$

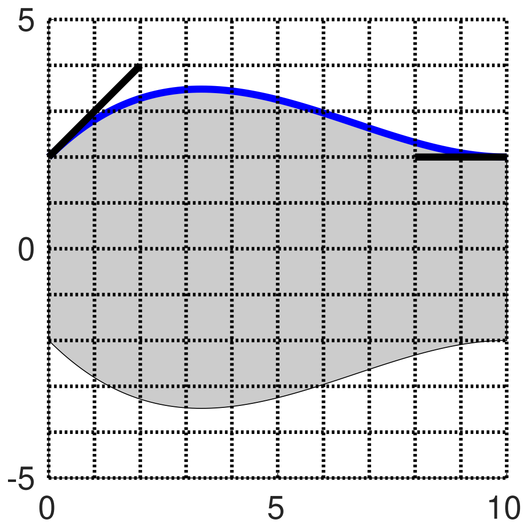
substituting $u = \ln E = \ln(1+\sqrt{2})$ for the upper limit in A $\quad\rightsquigarrow$

$$\begin{aligned} \operatorname{area} S &= \frac{\pi}{2}\left((1+\sqrt{2})^2 + 4\ln(1+\sqrt{2}) - 1/(1+\sqrt{2})^2\right) \\ &\underset{(\star)}{=} 2\pi\left(\sqrt{2} + \ln(1+\sqrt{2})\right) \approx 14.4235 \end{aligned}$$

$$(\star)\ \frac{1}{(1+\sqrt{2})^2} = \frac{(1-\sqrt{2})^2}{((1+\sqrt{2})(1-\sqrt{2}))^2} = \frac{1-2\sqrt{2}+2}{(1-2)^2} = (1+\sqrt{2})^2 - 4\sqrt{2}$$

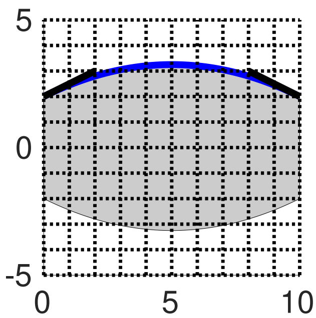
5.3 Profile and Volume of a Vase

Model the profile of the depicted vase with the graph of a cubic polynomial and compute the volume.



Resources: Volume and Surface of a Solid of Revolution

Problem Variants

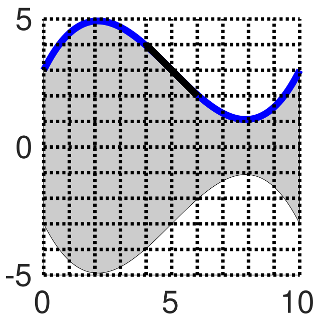


■

---

volume: ???.?:

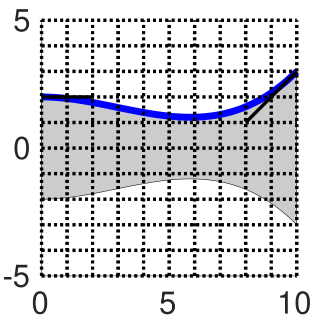
check



■

volume: ???.?:

check



volume: ??.??:

check

Solution

Profile

radius 2 at $x = 0$ and $x = 10$ $\quad\rightsquigarrow\quad$ cubic polynomial

$$p(x) - 2 = (ax + b)x(x - 10)$$

with derivative $p'(x) = (2ax + b)(x - 10) + (ax^2 + bx)$

slopes at $x = 0$ and $x = 10$ $\quad\rightsquigarrow$

$$\begin{array}{rcccl} 1 & = & p'(0) & = & -10b \\ 0 & = & p'(10) & = & 100a + 10b \end{array}$$

first equation $\quad\Longrightarrow\quad$ $b = -1/10$
substituting this value into the second equation $\quad\Longrightarrow\quad$ $a = 1/100$

Volume

substituting the polynomial, modeling the profile,

$$p(x) = 2 + \left(\frac{x}{100} - \frac{1}{10}\right) x(x - 10) = 2 + \frac{1}{100}x(x - 10)^2$$

into the formula for the volume of a solid of revolution $\quad\rightsquigarrow$

$$V = \pi \int_0^{10} p(x)^2 \, \mathrm{d}x = \pi \int_0^{10} 4 + \frac{4}{100}x(x - 10)^2 + \frac{1}{10000}x^2(x - 10)^4 \, \mathrm{d}x$$

integrating by parts, noting that the integrands vanish at the limits of integration, causing the term $[\ldots]_0^{10}$ to vanish $\quad\rightsquigarrow$

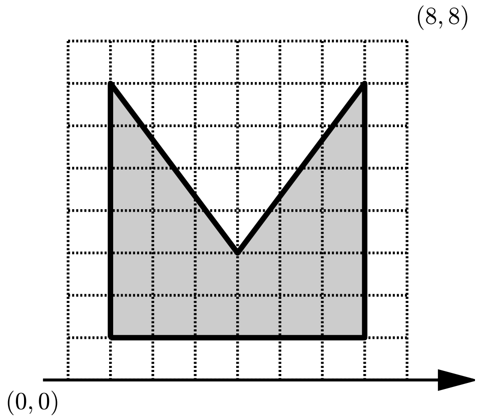
$$\begin{aligned} \int_0^{10} \underbrace{x}_{u(x)} \underbrace{(x - 10)^2}_{v'(x)} \, \mathrm{d}x &= -\int_0^{10} \underbrace{1}_{u'(x)} \underbrace{\frac{1}{3}(x - 10)^3}_{v(x)} \, \mathrm{d}x = \frac{10000}{12} \\ \int_0^{10} \underbrace{x^2}_{u(x)} \underbrace{(x - 10)^4}_{v''(x)} \, \mathrm{d}x &= \int_0^{10} \underbrace{2}_{u''(x)} \underbrace{\frac{1}{30}(x - 10)^6}_{v(x)} \, \mathrm{d}x = \frac{2000000}{21} \end{aligned}$$

substituting into the expression for the volume $\quad\rightsquigarrow$

$$V = \pi \left(40 + \frac{100}{3} + \frac{200}{21}\right) = \pi \frac{580}{7} \approx 260.3$$

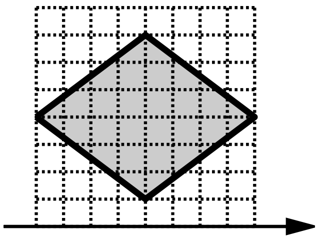
5.4 Surface Generated by Rotating a Polygon

Determine the area of the surface, generated by rotating the depicted polygon about the horizontal axis.



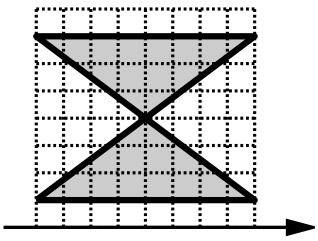
Resources: Guldin's Rules

Problem Variants

■ 

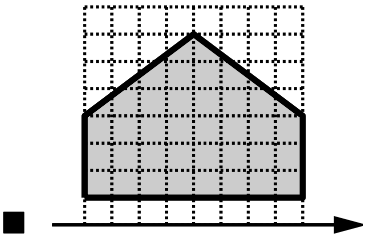
???π:

check

■ 

???π:

check



???π:

check

Solution

According to Guldin's rule for surfaces, the area of the surface, generated by rotating a polygon P about an nonintersecting axis, is equal to the product of the length of P and the length of the circle, traversed by the center of gravity of P.

The center of gravity c of a polygon with vertices $c_0, \ldots, c_n$ is a weighted sum of the midpoints of the edges:

$$c = \sum_{k=1}^{n} \underbrace{|c_k - c_{k-1}|}_{\text{edge length}} \underbrace{(c_k + c_{k-1})/2}_{\text{midpoint}} \Big/ \underbrace{\sum_{k=1}^{n} |c_k - c_{k-1}|}_{\text{total length}}, \tag{1}$$

where $c_n = c_0$.

Center of gravity

The vertices of the given polygon are

$$c_0 = (1,1),\ (7,1),\ (7,7),\ (4,3),\ (1,7),\ (1,1) = c_5 = c_0\,.$$

edge lengths: 6, 6, $\sqrt{3^2 + 4^2} = 5$, 5, 6
total length: $6 + 6 + 5 + 5 + 6 = 28$
midpoints: $(4,1)$, $(7,4)$, $(5.5,5)$, $(2.5,5)$, $(1,4)$

By symmetry, the first component of the center of gravity equals $(1+7)/2 = 4$.

computing the second component with the formula (1) $\leadsto$

$$(6 \cdot 1 + 6 \cdot 4 + 5 \cdot 5 + 5 \cdot 5 + 6 \cdot 4) \Big/ 28 = 26/7$$

$\implies \quad c = (4, 26/7)$

Area

The radius of the circle, traversed by the center of gravity, equals 26/7 (distance of c to the rotation axis). Hence, Guldin's rule yields the area

$$28 \cdot 2\pi(26/7) = 208\pi \approx 653.4513\,.$$

5.5 Solid Generated by Rotating a Triangle

Determine the volume of the solid, generated by rotating the triangle with vertices $(-4,3)$, $(-2,8)$, $(3,4)$ about the straight line $g:\ 3x-4y=2$.

Resources: Guldin's Rules

Problem Variants

■ $(0,0)$, $(1,0)$, $(0,1)$, $\quad g:\ x-y=2$

?.??:

[ ] check

■ $(1,0)$, $(0,1)$, $(-2,-2)$, $\quad g:\ x+y=2$

??.?:

[ ] check

■ $(1,0)$, $(3,0)$, $(2,4)$, $\quad g:\ x=0$

??.?:

[ ] check

Solution

The volume of a solid S, generated by rotating a set D with center of gravity c about an axis g with $D \cap g = \emptyset$, can be computed with Guldin's rule for solids of revolution:

$$\operatorname{vol} S = 2\pi \operatorname{dist}(c, g) \operatorname{area} D \,.$$

Center of gravity c

The center of gravity of a triangle with vertices a_k is $c = \dfrac{1}{3}\displaystyle\sum_{k=1}^{3} a_k$.

substituting $a_1 = (-4, 3)$, $a_2 = (-2, 8)$, $a_3 = (3, 4)$ $\quad\rightsquigarrow$

$$c = ((-4, 3) + (-2, 8) + (3, 4))/3 = (-1, 5)$$

Distance d to the rotation axis

The distance of a point c to a straight line $g :\ p_1x + p_2y = q$ can be computed by substituting $(x, y) = (c_1, c_2)$ into the equation of the line:

$$d = |(p_1c_1 + p_2c_2) - q|/|p|, \quad |p| = \sqrt{p_1^2 + p_2^2}\,.$$

$p_1 = 3$, $p_2 = -4$, $q = 2$, $c = (-1, 5)$ $\quad\rightsquigarrow$

$$d = |(3 \cdot (-1) - 4 \cdot 5) - 2|/\sqrt{3^2 + 4^2} = |-25|/5 = 5$$

Area A of the triangle

The area of a triangle is equal to half of the absolute value of the determinant of two spanning vectors, i.e., for the given triangle,

$$\begin{aligned} A &= \frac{1}{2}\left|\det\left(\begin{pmatrix} -2 \\ 8 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} - \begin{pmatrix} -4 \\ 3 \end{pmatrix}\right)\right| \\ &= \frac{1}{2}\left|\det\left(\begin{pmatrix} 2 \\ 5 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \end{pmatrix}\right)\right| = \frac{1}{2}\,|2 \cdot 1 - 5 \cdot 7| = \frac{33}{2} \end{aligned}$$

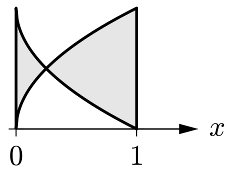
Volume V of the solid

Guldin's rule $\quad \Longrightarrow$

$$V = 2\pi d \, \cdot \, A = 2\pi \, 5 \, \cdot \, \frac{33}{2} = 165\pi \approx 518.3628$$

5.6 Solid of Revolution Generated by the Graphs of Two Functions

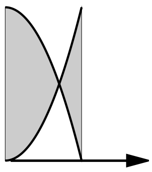
Determine the volume of the solid, generated by rotation about the x-axis of the gray area, bounded by parts of the graphs of the functions $f(x) = \sqrt{x}$ and $g(x) = 1 - \sqrt{x}$.



Resources: Volume and Surface of a Solid of Revolution

Problem Variants

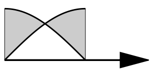
- $f(x) = x^2$, $g(x) = 4 - x^2$, $0 \leq x \leq 2$



??.?:

check

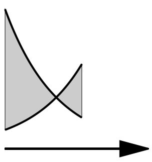
- $f(x) = \sin x$, $g(x) = \cos x$, $0 \leq x \leq \pi/2$



?.??:

check

- $f(x) = \mathrm{e}^x$, $g(x) = \mathrm{e}^{-x}$, $-1 \leq x \leq 1/2$



??.?:

check

Solution

application of the formula for the volume of a solid S, generated by rotating the set $A:\ a \le x \le b,\ f(x) \le y \le g(x)$ about the x-axis:

$$\operatorname{vol} S = \pi \int_a^b f(x)^2 - g(x)^2 \, \mathrm{d}x$$

Since, in the given problem, the inequality between the two functions is reversed at the intersection of their graphs, the left and right part of the volume has to be computed separately.

<u>Intersection of the graphs</u>

$$\sqrt{x} = f(x) = g(x) = 1 - \sqrt{x}$$

$\implies \quad x = 1/4$

<u>Left part of the solid</u>

$0 \le x \le 1/4,\ f(x) \le g(x)$:

$$\begin{aligned} \operatorname{vol} S_{\text{left}} &= \pi \int_0^{1/4} g(x)^2 - f(x)^2 \, \mathrm{d}x = \pi \int_0^{1/4} \underbrace{(1-\sqrt{x})^2 - (\sqrt{x})^2}_{1-2\sqrt{x}} \, \mathrm{d}x \\ &= \pi \left[x - \frac{4}{3} x^{3/2} \right]_{x=0}^{x=1/4} = \pi \left(\frac{1}{4} - \frac{4}{3}\frac{1}{8} \right) = \frac{1}{12}\,\pi \end{aligned}$$

<u>Right part of the solid</u>

$1/4 \le x \le 1,\ g(x) \le f(x)$:

$$\operatorname{vol} S_{\text{right}} = \int_{1/4}^{1} (\sqrt{x})^2 - (1-\sqrt{x})^2 \, \mathrm{d}x = \frac{5}{12}\,\pi$$

adding the volumes $\quad \rightsquigarrow$

$$\operatorname{vol} S = \operatorname{vol} S_{\text{left}} + \operatorname{vol} S_{\text{right}} = \frac{1}{12}\pi + \frac{5}{12}\pi = \frac{\pi}{2}$$

5.7 Volume and Center of Gravity of a Hyperboloid

Determine the center of gravity of the hyperboloid

$$S: \sqrt{1+x^2+y^2} \le z \le 3 .$$

Resources: Volume and Surface of a Solid of Revolution, Center of Gravity of a Solid of Revolution

Problem Variants

- paraboloid $S: x^2+y^2 \le z \le 1$

---

z-component of the center of gravity: ?.??:

[ ] check

- cone $S: x^2+y^2 \le z^2 \le 1,\ 0 \le z$

---

z-component of the center of gravity: ?.??:

[ ] check

- ellipsoid $S: x^2+y^2+2z^2 \le 2,\ 0 \le z$

---

z-component of the center of gravity: ?.??:

[ ] check

Solution

application of the formulas for the volume and the z-component of the center of gravity of a solid of revolution S with radius function $r(z)$, $a \le z \le b$:

$$\begin{aligned} \operatorname{vol} S &= \pi \int_a^b r(z)^2 \, \mathrm{d}z \\ c_z &= \pi \int_a^b z\, r(z)^2 \, \mathrm{d}z \Big/ \operatorname{vol} S \end{aligned}$$

<u>Radius function</u>

$S:\ \sqrt{1+x^2+y^2} \le z \le 3 \quad \Longleftrightarrow$

$$z \ge 1 \quad \wedge \quad x^2+y^2 \le z^2-1 \le 8\,,$$

i.e., $r(z) = \sqrt{z^2-1}$, $1 \le z \le 3$

<u>Volume</u>

$$\begin{aligned} \operatorname{vol} S &= \pi \int_1^3 r(z)^2 \, \mathrm{d}z = \pi \int_1^3 z^2 - 1 \, \mathrm{d}z \\ &= \pi \left[\frac{z^3}{3} - z \right]_{z=1}^{z=3} = \pi \left((9-3) - \left(\frac{1}{3} - 1 \right) \right) = \frac{20}{3}\pi \end{aligned}$$

<u>Center of gravity</u>

z-component

$$\begin{aligned} \operatorname{vol} S\, c_z &= \pi \int_1^3 z r(z)^2 \, \mathrm{d}z = \pi \int_1^3 (z^3 - z) \, \mathrm{d}z = \\ &= \pi \left[\frac{z^4}{4} - \frac{z^2}{2} \right]_{z=1}^{z=3} = \pi \left(\left(\frac{81}{4} - \frac{9}{2} \right) - \left(\frac{1}{4} - \frac{1}{2} \right) \right) = 16\pi\,, \end{aligned}$$

i.e., $c_z = 16 \Big/ \dfrac{20}{3} = \dfrac{12}{5}$

Chapter 6

Improper Integrals

6.1 Convergence of an Improper Integral over $[0, \infty)$

Decide whether the integral

$$\int_0^\infty \frac{\sin x}{\sqrt{x}} \, \mathrm{d}x$$

converges absolutely, converges (but not absolutely), or diverges.

Resources: Improper Integral, Majorant, Minorant

Problem Variants

■ $\displaystyle\int_0^\infty \frac{\cos x}{1+x^2} \, \mathrm{d}x$

---

absolutely convergent (a), convergent (but not absolutely convergent) (c), divergent (d):

[ ] check

■ $\displaystyle\int_0^\infty \frac{\sin^2 x}{x} \, \mathrm{d}x$

---

absolutely convergent (a), convergent (but not absolutely convergent) (c), divergent (d):

[ ] check

■ $\displaystyle\int_0^\infty \frac{\sin x}{x} \, \mathrm{d}x$

---

absolutely convergent (a), convergent (but not absolutely convergent) (c), divergent (d):

[ ] check

Solution

It is shown that the integral

$$\int_0^\infty \frac{\sin x}{\sqrt{x}}\,\mathrm{d}x$$

converges, but does not converge absolutely.

Absolute convergence

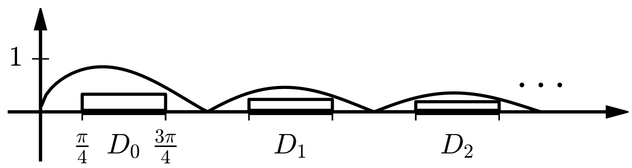
The divergence of $\int_0^\infty |f|$ is proved by constructing a suitable minorant, i.e., a nonnegative function, which is smaller than $|f|$ and not integrable.

$|\sin(x)| = |\sin(x + k\pi)|$ and monotonicity of the square root $\quad\Longrightarrow$

$$\frac{|\sin x|}{\sqrt{x}} \geq \frac{|\sin(\pi/4)|}{\sqrt{k\pi + 3\pi/4}} \geq \frac{1/\sqrt{2}}{\sqrt{\pi(k+1)}} \geq a_k = \frac{1}{3\sqrt{k+1}}$$

for $x \in D_k = \pi k + [\pi/4, 3\pi/4]$, $k = 0, 1, \ldots$

$\rightsquigarrow$ possible choice of a minorant: the step function s, depicted in the figure, which is equal to a_k on D_k and zero otherwise



integral of the step function:

$$\int_0^{n\pi} s = \sum_{k=0}^{n-1} \underbrace{(\mathrm{length} D_k)}_{\pi/2} \frac{1}{3\sqrt{k+1}} = \frac{\pi}{6} \sum_{k=1}^{n} k^{-1/2} \underset{n\to\infty}{\longrightarrow} \infty$$

$\Longrightarrow \quad$ divergence

Convergence

$\sin x \leq x$, $x \geq 0 \quad \Longrightarrow$

$$\frac{|\sin x|}{\sqrt{x}} \leq \sqrt{x}\,,$$

i.e., $\sqrt{x}$ is an integrable majorant on any **bounded** interval

To show the integrability on $[0, \infty)$, the convergence of $F(b) = \displaystyle\int_1^b \frac{\sin x}{\sqrt{x}}\,\mathrm{d}x$ for $b \to \infty$ is established.

integration by parts $\quad\rightsquigarrow$

$$\int_1^b \underbrace{\sin x}_{u'(x)}\, \underbrace{x^{-1/2}}_{v(x)}\, \mathrm{d}x = \Big[\underbrace{-\cos x}_{u(x)}\; x^{-1/2}\Big]_1^b - \int_1^b -\cos x\; \underbrace{(-x^{-3/2}/2)}_{v'(x)}\, \mathrm{d}x$$

The first term converges to $\cos 1$ for $b \to \infty$, and the absolute value of the integrand of the second term is majorized by $x^{-3/2}/2$ with

$$\int_0^\infty x^{-3/2}/2\, \mathrm{d}x = \left[-x^{-1/2}\right]_{x=1}^\infty = 0 - (-1) = 1\,.$$

$\Longrightarrow \quad$ convergence

Remark
invoking Maple^{TM} :

```
int(sin(x)/sqrt(x),x=0..infinity)
```

$\rightsquigarrow \quad \sqrt{\pi/2}$

6.2 Convergence of Improper Integrals over $[0, 1]$

Decide whether the integrals

$$\text{a)} \int_0^1 \frac{e^{\sqrt{x}} - 1}{x}\, dx \quad \text{b)} \int_0^1 \sqrt{x}\, e^{1/x}\, dx$$

exist or not.

Resources: Improper Integral, Majorant, Minorant

Problem Variants

■ $\displaystyle\int_0^1 \frac{\sqrt{x}}{e^x - 1}\, dx$

---

exists (e), does not exist (n): ?:

[ ] check

■ $\displaystyle\int_0^1 \frac{x}{1 - \cos x}\, dx$

---

exists (e), does not exist (n): ?:

[ ] check

■ $\displaystyle\int_0^1 \frac{\sqrt{1 - x^2}}{1 - x^3}\, dx$

---

exists (e), does not exist (n): ?:

[ ] check

Solution

An integral of a continuous function f on a bounded interval $(a,b]$ with a singularity of f at a is absolutely convergent (divergent) if

$$|f(x)| \le \underbrace{c(x-a)^r}_{\text{majorant}},\ r > -1 \qquad \left(|f(x)| \ge \underbrace{c(x-a)^r}_{\text{minorant}},\ r \le -1\right)$$

for $x \in (a, a+\delta]$. To establish either of the two inequalities, a Taylor expansion at $x = a$ or the computation of an appropriate limit for $x \to a$ with the rule of L'Hôpital is useful.

a) $\displaystyle\int_0^1 \frac{\mathrm{e}^{\sqrt{x}} - 1}{x}\,\mathrm{d}x$

substituting $t = \sqrt{x}$ into the Taylor expansion

$$\mathrm{e}^t = 1 + t + t^2/2 + t^3/6 + O(t^4)$$

$\rightsquigarrow$

$$\frac{\mathrm{e}^{\sqrt{x}} - 1}{x} = \frac{x^{1/2} + x/2 + x^{3/2}/6 + O(x^2)}{x} = x^{-1/2} + \left[1/2 + x^{1/2}/6 + O(x)\right]$$

Hence, since $|[\ldots]|$ can be bounded by a constant c, the integrable function $x^{-1/2} + c$ is a majorant for f on the interval $[0,1]$.

$\Longrightarrow$ absolute integrability

b) $\displaystyle\int_0^1 \sqrt{x}\,\mathrm{e}^{1/x}\,\mathrm{d}x$

The function $\mathrm{e}^{1/x}$ grows **extremely fast** as $x \to 0$. In fact,

$$\lim_{x\to 0} x^k\,\mathrm{e}^{1/x} = \infty \tag{1}$$

for **any** k. To prove divergence of the given integral, the case $k = 2$ suffices, since, as a consequence, for x in a sufficiently small interval $(0,\delta]$,

$$x^2\,\mathrm{e}^{1/x} \ge 1 \quad \Longleftrightarrow \quad \sqrt{x}\,\mathrm{e}^{1/x} \ge x^{-3/2}\,,$$

i.e., $x^{-3/2}$ is a nonintegrable minorant.

Turning to the proof of (1) for $k = 2$, the limit can be computed with the rule of L'Hôpital:

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}\,,$$

if **both** functions tend to ∞ (or to 0) for $x \to a$.

applying this formula with $a = 0$, $f(x) = \mathrm{e}^{1/x}$, $g(x) = 1/x^2$ $\implies$

$$\lim_{x\to 0} \frac{\mathrm{e}^{1/x}}{1/x^2} = \lim_{x\to 0} \frac{\mathrm{e}^{1/x}(-1/x^2)}{-2/x^3} = \frac{1}{2} \lim_{x\to 0} \frac{\mathrm{e}^{1/x}}{1/x}$$

applying the rule of L'Hôpital again $\implies$

$$\lim_{x\to 0} \frac{\mathrm{e}^{1/x}}{1/x} = \lim_{x\to 0} \frac{\mathrm{e}^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to 0} \frac{\mathrm{e}^{1/x}}{1} = \infty\,,$$

confirming (1) for $k = 2$

6.3 Integral over a Bounded Interval with an Endpoint Singularity

Prove the existence of the integral

$$\int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}}\,\mathrm{d}x\,.$$

Resources: Improper Integral, Majorant

Problem Variants

- $\displaystyle\int_0^2 \ln(\sin x)\,\mathrm{d}x$
- $\displaystyle\int_0^1 \frac{\sin(\ln x)}{\sqrt{x}}\,\mathrm{d}x$
- $\displaystyle\int_0^1 \frac{\sqrt{1-x^2}}{1-x^3}\,\mathrm{d}x$

Just to appreciate 19^{th} century mathematics: **All** integrals can be computed explicity! Use Maple™ to confirm.

Solution

To prove existence of an integral $\int_a^b f$, a majorant is constructed, i.e., a function g, with $|f(x)| \le g(x)$ and finite integral $\int_a^b g$.

Construction of a majorant

expanding the integrand $f(x) = \sqrt{1-x}/(1-\sqrt{x})$ with $1+\sqrt{x}$ and noting that $(1-\sqrt{x})(1+\sqrt{x}) = 1-x \quad \rightsquigarrow$

$$\int_0^1 f = \int_0^1 \frac{\sqrt{1-x}}{1-\sqrt{x}}\,\mathrm{d}x = \int_0^1 \frac{\sqrt{1-x}(1+\sqrt{x})}{1-x}\,\mathrm{d}x = \int_0^1 \frac{1+\sqrt{x}}{\sqrt{1-x}}\,\mathrm{d}x$$

estimating $1+\sqrt{x}$ by 2 $\quad \rightsquigarrow \quad$ majorant

$$g(x) = 2/\sqrt{1-x}$$

Integral of the majorant

$$\int_0^1 |g| = \int_0^1 \frac{2}{\sqrt{1-x}}\,\mathrm{d}x \underset{y=1-x}{=} \int_0^1 \frac{2}{\sqrt{y}}\,\mathrm{d}y = [4\sqrt{y}]_{y=0}^{y=1} = 4$$

integrable majorant $\quad \Longrightarrow \quad$ convergence

6.4 Improper Integral with Parameter

For which values of the parameter $r > 0$ does the improper integral

$$\int_0^{\pi} \frac{1 - \cos(x^{2r})}{x^3}\,\mathrm{d}x$$

exist?

Resources: Majorant, Minorant

Problem Variants

- $\displaystyle\int_0^{\pi/2} \frac{x^2}{\sin(x^r)}\,\mathrm{d}x$

$r <?$:

[ ] check

- $\displaystyle\int_1^{\infty} \frac{(x+1)^2}{x^{3r}+4}\,\mathrm{d}x$

$r >?$:

[ ] check

- $\displaystyle\int_1^{2} \frac{\ln^2 x}{(x-1)^r}$

$r <?$:

[ ] check

Solution

To prove the existence (non-existence) of $\int_a^b f$, $|f|$ is bounded from above (below) by an integrable (non-integrable) majorant (minorant).

<u>Construction of a majorant and minorant</u>

Taylor expansion of the integrand $f(x) = (1-\cos(x^{2r}))/x^3 \quad \rightsquigarrow$

$$f(x) = \frac{1-(1-(x^{2r})^2/2+O(x^{2r\cdot 4}))}{x^3} = x^{4r-3}/2 + O(x^{8r-3})$$

Hence, for sufficiently small $\delta > 0$,

$$\underbrace{x^{4r-3}/4}_{\text{minorant}} \le \underbrace{f(x)}_{\ge 0} \le \underbrace{x^{4r-3}}_{\text{majorant}}, \quad 0 \le x \le \delta\,.$$

existence of $\int_0^\delta x^s\,\mathrm{d}x$ for $s > -1$ (non-existence for $s \le -1$) $\quad\rightsquigarrow\quad$ condition for r:

$$\text{existence} \iff 4r-3 > -1 \iff r > 1/2$$

(non-existence for $1/2 \ge r > 0$)

6.5 Improper Integral of a Rational Function of Degree $[0, 2]$

Compute $\displaystyle\int_2^\infty \frac{2}{3x^2 - 4x + 1}$.

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

- $\displaystyle\int_3^\infty \frac{1}{x^2 + x}\,\mathrm{d}x$

?.??:

check

- $\displaystyle\int_0^\infty \frac{1}{x^2 + 3x + 2}\,\mathrm{d}x$

?.??:

check

- $\displaystyle\int_{-\infty}^{-2} \frac{1}{x^2 - 1}\,\mathrm{d}x$

?.??:

check

Solution

First, the poles of the rational integrand

$$f(x) = \frac{2}{3x^2 - 4x + 1}$$

are determined, in order to select the appropriate ansatz for a partial fraction decomposition. Then, the resulting elementary terms are integrated over the interval $[2, \infty)$.

Poles

solving $3x^2 - 4x + 1 = 0 \quad \rightsquigarrow$

$$x_{1,2} = \frac{4 \pm \sqrt{4^2 - 4 \cdot 3 \cdot 1}}{2 \cdot 3} = \frac{2}{3} \pm \frac{1}{3}\,,$$

i.e., $x_1 = 1$, $x_2 = 1/3$
corresponding factorization of the denominator of f: $3(x-1)(x-1/3)$

Partial fraction decomposition

ansatz for an integrand with two simple poles:

$$f(x) = \frac{2}{3(x-1)(x-1/3)} = \frac{a}{x-1} + \frac{b}{x-1/3}$$

- multiplying by $x - 1$ and setting $x = 1 \quad \Longrightarrow \quad a = \dfrac{2}{3(1-1/3)} = 1$
- $\cdot(x - 1/3)$, $x = 1/3 \quad \Longrightarrow \quad b = -1$

⚠ It is no coincidence that $b = -a$. There must be some cancelation to ensure that the sum of the two terms decays like $1/x^2$.

Integration of the elementary terms

$$\int \frac{1}{t - t_0}\,\mathrm{d}t = \ln|t - t_0| + C \text{ and } \ln u - \ln v = \ln(u/v) \quad \rightsquigarrow$$

$$\begin{aligned} \int_2^\infty f &= \lim_{c\to\infty} \int_2^c \frac{1}{x-1} - \frac{1}{x-1/3}\,\mathrm{d}x = \lim_{c\to\infty} \left[\ln|x-1| - \ln|x-1/3|\right]_{x=2}^c \\ &= \lim_{c\to\infty} \ln \underbrace{\frac{|c-1|}{|c-1/3|}}_{A} - \ln\frac{1}{5/3} = \ln 5 - \ln 3 \approx 0.5108\,, \end{aligned}$$

since $A \to 1$ for $c \to \infty$ and $\lim_{A\to 1} \ln A = 0$

6.6 Integration of a Rational Function with Denominator of Degree 3 over $\mathbb{R}_+$

Determine the antiderivative of

$$f(x) = \frac{2-x}{1+x^3}$$

and compute $\int_0^\infty f$.

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

- $f(x) = \dfrac{1}{x^3 - x^2}$,

$\int_2^\infty f$ =?.??:

[ ] check

- $f(x) = \dfrac{3+x}{x^3 + x}$,

$\int_1^\infty f$ =?.??:

[ ] check

- $f(x) = \dfrac{1}{x^3 + 3x^2 + 2x}$,

$\int_1^\infty f$ =?.??:

[ ] check

Solution

First, the poles of

$$f(x) = \frac{2-x}{1+x^3}$$

are determined to select the ansatz for the partial fraction decomposition. After expressing f as a sum of simple terms, the antiderivative is obtained from tabulated elementary integrals. Then, the fundamental theorem of calculus yields $\int_0^\infty f$ as limit of the integral over $[0, b]$ for $b \to \infty$.

<u>Poles and Factorization</u>

An obvious pole is $x_1 = -1$.

division by the corresponding linear factor $(x+1)$:

$$\begin{array}{llllll}
(& x^3 & & & +1 &) : (x+1) = \underbrace{x^2 - x + 1}_{q(x)} \\
 & x^3 & +x^2 & & & \\ \hline
 & & -x^2 & & & \\
 & & -x^2 & -x & & \\ \hline
 & & & x & +1 & \\
 & & & x & +1 & \\ \hline
 & & & & 0 &
\end{array}$$

The quotient

$$q(x) = (x - 1/2)^2 + 3/4$$

has no real zeros. Hence, there are no further real poles (already apparent from the monotonicity of $1+x^3$), and $q(x)$ is a factor of the real factorization of f:

$$f(x) = \frac{2-x}{(x+1)((x-1/2)^2 + 3/4)}.$$

<u>Partial fraction decomposition</u>

ansatz

$$f(x) = \frac{2-x}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{b(x-1/2)+c}{(x-1/2)^2+3/4}$$

multiplying by $x+1$ and setting $x = -1 \quad \Longrightarrow \quad a = 1$

subtraction of the term $1/(x+1)$ from $f(x)$ $\rightsquigarrow$ $b=-1$, $c=1/2$ and, hence,

$$f(x)=\sum_{k=1}^{3} r_k(x)=\frac{1}{x+1}-\frac{x-1/2}{(x-1/2)^2+3/4}+\frac{1/2}{(x-1/2)^2+(\sqrt{3}/2)^2}$$

Antiderivative

integration of the elementary rational expressions (cf. the table of elementary rational integrals):

- $R_1(x)=\ln|x+1|+C$
- $R_2(x)=-\ln\left|(x-1/2)^2+3/4\right|/2+C$
- $R_3(x)=\dfrac{1}{\sqrt{3}}\arctan\left((2x-1)/\sqrt{3}\right)+C$

adding the three expressions, noting that $\ln u-(\ln v)/2 \underset{(\star)}{=} \ln(u/\sqrt{v})$ $\rightsquigarrow$

$$F(x)=\int f(x)\,\mathrm{d}x=\underbrace{\ln\left|\frac{x+1}{\sqrt{x^2-x+1}}\right|}_{\underset{(\star)}{=}R_1(x)+R_2(x)}+\frac{1}{\sqrt{3}}\arctan\left((2x-1)/\sqrt{3}\right)+C$$

Integral over $[0,\infty)$

noting that $\lim_{A\to 1}\ln A=0$ and $\lim_{x\to\infty}\arctan x=\pi/2$ $\rightsquigarrow$

$$\int_0^\infty f=[F]_0^\infty=\left(\ln 1+\frac{1}{\sqrt{3}}\frac{\pi}{2}\right)-\left(\ln 1-\frac{1}{\sqrt{3}}\frac{\pi}{6}\right)=\frac{2\sqrt{3}}{9}\pi\approx 1.2092$$

6.7 Integral of a Rational Function with Two Pairs of Complex Conjugate Poles over $(-\infty, \infty)$

Compute $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{x^4+4}$.

Resources: Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

- $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(x^2+1)(x^2+4)}$

$\pi/?$:

[ ] check

- $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{(2x^2+1)(x^2+2)}$

$\pi\sqrt{?}/?$:

[ ] check

- $\displaystyle\int_{-\infty}^{\infty} x\,\frac{x}{(x^2+1)^2}\,\mathrm{d}x$

Hint: Use integration by parts

$\pi/?$:

[ ] check

Solution

First, the poles of the rational integrand

$$f(x) = \frac{1}{x^4+4}$$

are determined, the denominator is factored accordingly, and the appropriate ansatz for a partial fraction decomposition is selected. Then, the resulting elementary terms are integrated over the interval $(-\infty, \infty)$.

Poles

$x^4 + 4 = 0 \quad \Longrightarrow$

$$x^2 = \sqrt{-4} = \pm 2\mathrm{i}$$

To compute the square roots of $\pm\mathrm{i}$, the polar form of complex numbers is used.
writing $2\mathrm{i} = 2\,\mathrm{e}^{\mathrm{i}\pi/2} \quad \rightsquigarrow$

$$x_{1,2} = \pm\sqrt{2\mathrm{i}} = \pm\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4} = \pm\sqrt{2}\,(\cos(\pi/4) + \mathrm{i}\sin(\pi/4)) = \pm(1+\mathrm{i})$$

similarly,

$$x_{3,4} = \pm\sqrt{-2\mathrm{i}} = \pm(1-\mathrm{i})$$

factorization:

$$\begin{aligned} x^4 + 4 &= \prod_{k=1}^{4}(x - x_k) \\ &= (x-1-\mathrm{i})(x+1+\mathrm{i})(x-1+\mathrm{i})(x+1-\mathrm{i}) \\ &\underset{(\star)}{=} ((x-1)^2+1)((x+1)^2+1) \end{aligned}$$

$(\star)$: combining the first and third as well as the second and fourth factor, using the third binomial formula

Partial fraction decomposition

ansatz, corresponding to the factorization:

$$f(x) = \frac{1}{((x-1)^2+1)((x+1)^2+1)} = \frac{a(x-1)+b}{(x-1)^2+1} + \frac{c(x+1)+d}{(x+1)^2+1}$$

multiplying by the common denominator $\quad \rightsquigarrow$

$$\begin{aligned} 1 &= (a(x-1)+b)((x+1)^2+1) + (c(x+1)+d)((x-1)^2+1) \\ &= (a+c)x^3 + (a+b-c+d)x^2 + (2b-2d)x + (-2a+2b+2c+2d) \end{aligned}$$

comparing the coefficients of x^3, x, x^2, and 1 $\implies$

$$a = -1/8,\ b = c = d = 1/8\,,$$

and, hence,

$$f(x) = \frac{1}{8}\left(\frac{-(x-1)}{(x-1)^2+1} + \frac{1}{(x-1)^2+1} + \frac{(x+1)}{(x+1)^2+1} + \frac{1}{(x+1)^2+1}\right) \tag{1}$$

Integration

Noting that a shift is irrelevant when integrating over $(-\infty, \infty)$, i.e.,

$$\int_{-\infty}^{\infty} u(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} u(x - x_0)\,\mathrm{d}x\,,$$

the integrals of the first and third term cancel (opposite sign), and the integrals of the second and fourth term both equal $\int_{-\infty}^{\infty} \frac{1/8}{x^2+1}\,\mathrm{d}x$. Hence,

$$\int_{-\infty}^{\infty} f = 2\int_{-\infty}^{\infty} \frac{1/8}{x^2+1}\,\mathrm{d}x = \frac{1}{4}\left[\arctan x\right]_{x=-\infty}^{x=\infty} = \frac{1}{4}\left(\pi/2 - (-\pi/2)\right) = \frac{\pi}{4}\,,$$

where the shorthand notation $[u]_{-\infty}^{\infty}$ was used for obvious limits $\lim_{x\to\pm\infty} u(x)$.

Surprisingly simple![1]

[1] Even simpler, and more elegant, is to use Complex Analysis → cf. *MathTraining: Problems and Solutions for Complex Analysis.*

6.8 Improper Integral of a Product of a Polynomial with an Exponential Function

Compute $\displaystyle\int_1^\infty (x+1)\mathrm{e}^{1-x}\,\mathrm{d}x$.

Resources: Integration by Parts, Improper Integral

Problem Variants

■ $\displaystyle\int_0^\infty x^2\,\mathrm{e}^{-2x}\,\mathrm{d}x$

1/?:

[ ] check

■ $\displaystyle\int_0^\infty x^3\,\mathrm{e}^{-x^2}\,\mathrm{d}x$

1/?:

[ ] check

■ $\displaystyle\int_0^\infty x\,4^{-x}\,\mathrm{d}x$

?.??:

[ ] check

Solution

First, the integral is simplified with integration by parts. Then the limit $\lim_{b\to\infty}\int_1^b f$ is computed.

Integration by parts

applying the formula

$$\int_a^b uv' = [uv]_a^b - \int u'v$$

with $u(x) = x+1$ and $v'(x) = \mathrm{e}^{1-x}$ $\quad\rightsquigarrow$

$$\int_1^b (x+1)\mathrm{e}^{1-x}\,\mathrm{d}x = \Big[(x+1)\underbrace{\left(-\mathrm{e}^{1-x}\right)}_{v(x)}\Big]_{x=1}^{x=b} - \int_1^b \underbrace{1}_{u'(x)}\cdot\left(-\mathrm{e}^{1-x}\right)\mathrm{d}x$$

$$= \big(-(b+1)\,\mathrm{e}^{1-b}+2\big) + \big[-\mathrm{e}^{1-x}\big]_{x=1}^{x=b} = \big(-(b+1)\,\mathrm{e}^{1-b}+2\big) + \big(-\mathrm{e}^{1-b}+1\big)$$

Computation of the limit

Since $\lim_{b\to\infty} b^n\mathrm{e}^{-b} = 0$ for any $n > 0$,

$$\int_1^b (x+1)\,\mathrm{e}^{1-x}\,\mathrm{d}x \underset{b\to\infty}{\longrightarrow} (-0+2)+(-0+1) = 3\,.$$

Remark

In view of the simplicity of the limits, it is legitimate to shorten the presentation by writing

$$\int_1^\infty f = \big[(x+1)(-\mathrm{e}^{1-x})\big]_{x=1}^{x=\infty} - \int_1^\infty 1\cdot\left(-\mathrm{e}^{1-x}\right)\mathrm{d}x = 2 + \big[-\mathrm{e}^{1-x}\big]_{x=1}^{x=\infty} = 3\,,$$

thus incorporating the limits directly in the integration process.

6.9 Improper Integral of a Product of Sine/-Cosine with an Exponential Function

Compute $\int_0^\infty \sin^2 x \, e^{-2x} \, dx$.

Resources: Integration by Parts, Improper Integral

Problem Variants

- $\int_0^\infty \sin(2x) \, e^{-x} \, dx$

2/?:

[ ] check

- $\int_0^\infty \cos^2(2x) \, e^{-x} \, dx$

9/??:

[ ] check

- $\int_0^\infty \cos(3x) \, e^{-x} \, dx$

1/??:

[ ] check

Solution

integrating by parts,

$$\int_a^b uv' = [uv]_a^b - \int_a^b u'v\,,$$

with $u(x) = \sin^2 x$ and $v'(x) = \mathrm{e}^{-2x}$ $\quad\rightsquigarrow$

$$\begin{aligned}
I &= \int_0^\infty \sin^2 x\,\mathrm{e}^{-2x}\,\mathrm{d}x \\
&= \underbrace{\left[\sin^2 x\,\overbrace{\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)}^{v(x)}\right]_{x=0}^{x=\infty}}_{=0} - \int_0^\infty \overbrace{2\sin x\cos x}^{u'(x)}\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\,\mathrm{d}x \\
&= \underbrace{\left[\sin x\cos x\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\right]_{x=0}^{x=\infty}}_{=0} - \int_0^\infty \underbrace{(\cos^2 x - \sin^2 x)}_{=1-2\sin^2 x}\left(-\frac{1}{2}\mathrm{e}^{-2x}\right)\,\mathrm{d}x \\
&= \frac{1}{2}\int_0^\infty \mathrm{e}^{-2x}\,\mathrm{d}x - I
\end{aligned}$$

solving for I $\quad\rightsquigarrow$

$$2I = \frac{1}{2}\int_0^\infty \mathrm{e}^{-2x}\,\mathrm{d}x = \left[-\frac{1}{4}\mathrm{e}^{-2x}\right]_{x=0}^{x=\infty} = \frac{1}{4}, \quad \text{i.e., } I = \frac{1}{8}$$

<u>Alternative solution</u>

applying Euler's formula, $\sin x = \dfrac{\mathrm{e}^{\mathrm{i}x} - \mathrm{e}^{-\mathrm{i}x}}{2\mathrm{i}}$, no integration by parts is necessary:

$$\begin{aligned}
\int_0^\infty \sin^2 x\,\mathrm{e}^{-2x}\,\mathrm{d}x &= \frac{1}{(2\mathrm{i})^2}\int_0^\infty \left(\mathrm{e}^{2\mathrm{i}x} - 2 + \mathrm{e}^{-2\mathrm{i}x}\right)\,\mathrm{e}^{-2x}\,\mathrm{d}x \\
&= -\frac{1}{4}\left[\frac{\mathrm{e}^{2\mathrm{i}x-2x}}{2\mathrm{i}-2} + \mathrm{e}^{-2x} + \frac{\mathrm{e}^{-2\mathrm{i}x-2x}}{-2\mathrm{i}-2}\right]_{x=0}^{x=\infty} \\
&= -\frac{1}{4}\left(-\frac{1}{2\mathrm{i}-2} - 1 - \frac{1}{-2\mathrm{i}-2}\right) = -\frac{1}{4}\left(\frac{1+\mathrm{i}}{4} - 1 + \frac{1-\mathrm{i}}{4}\right) = \frac{1}{8}
\end{aligned}$$

An advertisement for Complex Analysis!

6.10 Improper Integral with Square Roots over $[0,1]$

Compute $\displaystyle\int_0^1 \frac{\sqrt{x+1}}{\sqrt{x}}\,\mathrm{d}x$.

Resources: Improper Integral, Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

- $\displaystyle\int_0^1 \sqrt{\frac{x}{x+1}}\,\mathrm{d}x$

?.??:

check

- $\displaystyle\int_0^1 x\sqrt{2x+3}\,\mathrm{d}x$

?.??:

check

- $\displaystyle\int_0^1 \frac{\sqrt{1+x}}{\sqrt{1-x}}\,\mathrm{d}x$

?.??:

check

Solution

The function

$$f(x) = p(x)\sqrt{r(x)}, \quad r(x) = \frac{ax+b}{cx+d},$$

with a polynomial p is absolutely integrable over any finite interval since it grows at most like $C/\sqrt{x-x_\star}$ at a pole $x_\star$ of the linear rational function r. With the substitution $y^2 = r(x)$, $\int f$ can be transformed to an integral of a rational function (no longer involving square roots), which can be computed using partial fraction decomposition.

<u>Substitution</u>

For the function $f(x) = \sqrt{\dfrac{x+1}{x}}$, the substitution $y^2 = (x+1)/x \iff x = 1/(y^2-1)$, $2y\,\mathrm{d}y = -1/x^2\,\mathrm{d}x \quad \rightsquigarrow$

$$\int_0^1 \sqrt{\frac{x+1}{x}}\,\mathrm{d}x = \int_\infty^{\sqrt{2}} y\,\underbrace{(-2y)\frac{1}{(y^2-1)^2}\,\mathrm{d}y}_{\mathrm{d}x} = \int_{\sqrt{2}}^\infty \underbrace{\frac{2y^2}{(y^2-1)^2}}_{g(y)}\,\mathrm{d}y$$

(a sign change because of the interchange of the integration limits)

<u>Partial fraction decomposition</u>

ansatz for a function with two real double poles:

$$g(y) = \frac{2y^2}{(y-1)^2(y+1)^2} = \frac{a}{(y-1)^2} + \frac{b}{y-1} + \frac{c}{(y+1)^2} + \frac{d}{y+1}$$

The constants a, b, c, d can be determined by multiplying with the common denominator and comparing the coefficients of y^k. Simpler is the following alternative.

- multiplying by $(y-1)^2$ and setting $y=1 \quad \Longrightarrow \quad a = 1/2$
- $\cdot(y+1)^2$, $y=-1 \quad \Longrightarrow \quad c = 1/2$

The constants b and d can now be determined by evaluating at two points:

- $y = 0 \quad \Longrightarrow \quad 0 = 1/2 - b + 1/2 + d$
- $y = 2 \quad \Longrightarrow \quad 8/9 = 1/2 + b + 1/18 + d/3$

solving the two equations $\quad \rightsquigarrow \quad b = 1/2,\ d = -1/2$

Integration

computing the integral

$$\int_{\sqrt{2}}^{\infty} g(y)\,\mathrm{d}y = \int_{\sqrt{2}}^{\infty} \frac{1/2}{(y-1)^2} + \frac{1/2}{y-1} + \frac{1/2}{(y+1)^2} - \frac{1/2}{y+1}\,\mathrm{d}y$$

using the antiderivatives

$$\int \frac{\mathrm{d}y}{y-y_0} = \ln|y-y_0|, \qquad \int \frac{\mathrm{d}y}{(y-y_0)^2} = -\frac{1}{y-y_0}$$

as well as the formula $\ln u - \ln v = \ln(u/v)$ $\quad\rightsquigarrow$

$$\begin{aligned}\int_{\sqrt{2}}^{\infty} g &= \left[-\frac{1}{2(y-1)} + \frac{1}{2}\ln|y-1| - \frac{1}{2(y+1)} - \frac{1}{2}\ln|y+1|\right]_{y=\sqrt{2}}^{y=\infty} \\ &= \left[-\frac{y}{y^2-1} + \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right|\right]_{y=\sqrt{2}}^{y=\infty} = 0 - \left(-\sqrt{2} + \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1}\right) \\ &= \;= \sqrt{2} - \frac{1}{2}\ln(3-2\sqrt{2}) \underset{(\star)}{=} \sqrt{2} + \ln(1+\sqrt{2}) \approx 2.2956\end{aligned}$$

⚠ The identity

$$-\frac{1}{2}\ln(3-2\sqrt{2}) \underset{(\star)}{=} \ln(1+\sqrt{2})$$

is puzzling at first, but easily confirmed. Multiplying by 2, and noting that $-\ln u = \ln(1/u)$, $2\ln u = \ln u^2$, it is equivalent to

$$\frac{1}{3-2\sqrt{2}} = (1+\sqrt{2})^2 = 3+2\sqrt{2}\,,$$

valid in view of $(u+v)(u-v) = u^2 - v^2$.

6.11 Improper Integral with Square Roots over an Infinite Interval

Compute $\displaystyle\int_4^\infty \frac{\sqrt{x}}{x^2-1}\,\mathrm{d}x$.

Resources: Improper Integral, Partial Fraction Decomposition, Elementary Rational Integrands

Problem Variants

■ $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x(1+\sqrt{x})}$

?.??:

check

■ $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{\sqrt{x}(1+x)}$

?.??:

check

■ $\displaystyle\int_0^\infty \frac{\sqrt{x+1}}{x^2}\,\mathrm{d}x$

?.??:

check

Solution

After showing that $f(x) = \dfrac{\sqrt{x}}{x^2-1}$ is absolutely integrable on $[4,\infty)$, $\int_4^\infty f$ is transformed into an integral of a rational function with the substitution

$$x = y^2 \leftrightarrow y = \sqrt{x}, \quad \mathrm{d}x = 2y\,\mathrm{d}y\,. \tag{1}$$

Then, the partial fraction decomposition of the integrand is determined, and the elementary terms are integrated, using tabulated antiderivatives.

Absolute integrability

$1 \le x^2/2$ for $x \ge 4 \quad \implies$

$$|f(x)| = \left|\frac{\sqrt{x}}{x^2-1}\right| \le \left|\frac{\sqrt{x}}{x^2-x^2/2}\right| \le 2x^{-3/2}$$

$\implies$ existence of $\int_4^\infty |f|$ since $x^{-3/2}$ is integrable on $[4,\infty)$

In general, a quotient of sums of fractional powers is absolutely integrable on an interval $D=[a,\infty)$, if the largest exponent in the denominator is by more than 1 larger than the largest exponent in the numerator (in the problem: $2 > 1/2+1$), and the denominator has no zeros in D.

Substitution

applying the substitution (1), noting that $x=4 \leftrightarrow y=2 \quad \rightsquigarrow$

$$\int_4^\infty \frac{\sqrt{x}}{x^2-1}\,\mathrm{d}x = \int_2 \underbrace{\frac{y}{y^4-1}\,2y}_{g(y)}\,\mathrm{d}y$$

Partial fraction decomposition

factorization of the denominator of g:

$$y^4-1 = (y^2-1)(y^2+1) = (y-1)(y+1)(y^2+1)$$

corresponding ansatz:

$$g(y) = \frac{2y^2}{(y-1)(y+1)(y^2+1)} = \frac{a}{y-1} + \frac{b}{y+1} + \frac{cy+d}{y^2+1}$$

The constants a, b, c, d could be determined by multiplying by the denominator of g and comparing coefficients of y^k. Simpler (and more elegant) are the following considerations.

- multiplying by $(y-1)$ and setting $y=1 \quad \Longrightarrow \quad a = \dfrac{2\cdot 1}{(1+1)(1^2+1)} = \dfrac{1}{2}$
- $\cdot(y+1),\ y=-1 \quad \Longrightarrow \quad b = -\dfrac{1}{2}$
- setting $y=0 \quad \Longrightarrow \quad 0 = -\dfrac{1}{2} - \dfrac{1}{2} + d$, i.e., $d=1$
- comparing the coefficient of y^3 after multiplying by the denominator of $g \quad \Longrightarrow \quad 0 = \dfrac{1}{2} - \dfrac{1}{2} + c$, i.e., $c=0$

summarizing:

$$g(y) = \frac{1}{2(y-1)} - \frac{1}{2(y+1)} + \frac{1}{y^2+1}$$

Integration

forming the antiderivative of g, using the tabulated integrals of the elementary terms, and combining the logarithms by applying the formula $\ln u - \ln v = \ln(u/v) \quad \rightsquigarrow$

$$G(y) = \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| + \arctan y + C$$

integration, $\lim_{t\to 1}\ln t = 0$, $\ln(1/u) = -\ln u \quad \rightsquigarrow$

$$\begin{aligned}\int_2^\infty g &= [G]_2^\infty = \left(0+\frac{\pi}{2}\right) - \left(\frac{1}{2}\ln(1/3) + \arctan 2\right)\\ &= \frac{\pi}{2} + \frac{\ln 3}{2} - \arctan 2 \approx 1.0129\end{aligned}$$

Chapter 7

Calculus Highlights

7.1 Error Estimate for the Midpoint Rule

Determine the constant c in the error estimate

$$\int_0^1 \mathrm{e}^x \, \mathrm{d}x - \frac{1}{n} \sum_{k=0}^{n-1} \mathrm{e}^{(k+1/2)/n} = cn^{-2} + O(n^{-3})$$

for the midpoint rule.

Resources: Riemann Integral

Solution

exact integral:

$$S = \int_0^1 \mathrm{e}^x \,\mathrm{d}x = [\mathrm{e}^x]_{x=0}^{x=1} = \mathrm{e} - 1$$

approximation using the midpoint rule:

formula for a geometric sum, $\sum_{k=0}^{n-1} q^k = \dfrac{q^n - 1}{q - 1}$, with $q = \mathrm{e}^{1/n}$ $\quad\rightsquigarrow$

$$\begin{aligned} S_n &= \frac{1}{n}\sum_{k=0}^{n-1} \mathrm{e}^{(k+1/2)/n} = \frac{\mathrm{e}^{1/(2n)}}{n}\sum_{k=0}^{n-1} \mathrm{e}^{k/n} \\ &= \frac{\mathrm{e}^{1/(2n)}}{n}\,\frac{\mathrm{e}^{n/n} - 1}{\mathrm{e}^{1/n} - 1} = \frac{\mathrm{e} - 1}{n\big(\mathrm{e}^{1/(2n)} - \mathrm{e}^{-1/(2n)}\big)} \end{aligned}$$

Taylor expansion of the denominator with $\mathrm{e}^t = 1 + t + t^2/2 + t^3/6 + O(t^4)$, $t = 1/(2n)$ and $t = -1/(2n)$ $\quad\rightsquigarrow$

$$\begin{aligned} &n\Big(1 + \frac{1}{2n} + \frac{1}{2(2n)^2} + \frac{1}{6(2n)^3} + O(n^{-4}) \\ &\qquad -1 + \frac{1}{2n} - \frac{1}{2(2n)^2} + \frac{1}{6(2n)^3} + O(n^{-4})\Big) = 1 + \frac{1}{24}n^{-2} + O(n^{-3}) \end{aligned}$$

$1/(1+\varepsilon) = 1 - \varepsilon + O(\varepsilon^2) \quad \Longrightarrow$

$$S - S_n = (\mathrm{e} - 1)\left(1 - \frac{1}{1 + \frac{1}{24}n^{-2} + O(n^{-3})}\right) = (\mathrm{e} - 1)\left(\frac{1}{24}n^{-2} + O(n^{-3})\right)$$

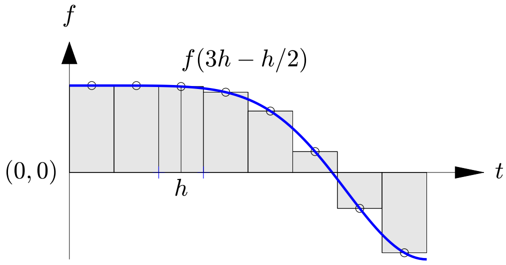
$\rightsquigarrow$ $\quad$ constant $c = (\mathrm{e} - 1)/24 \approx 0.0716$

7.2 Illustration of the Rapid Convergence of the Midpoint Rule for Periodic Integrands with Maple™

Approximate the Bessel integral

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(nt - x\sin t)\,\mathrm{d}t$$

for $n = 1$, $x = 1$ with the midpoint rule for $2, 4, 8, \ldots, 64$ sampling points in 100-digit floating point arithmetic. Compare with the exact value $J_1(1)$.



Resources: Riemann Integral

Solution

```
# parameter n,x and number of digits
n := 1: x := 1: Digits := 100:

# Bessel integrand
f := t -> cos(n*t-x*sin(t))/Pi
```

$$f := t \mapsto \cos(nt - x\sin(t))/\pi$$

```
# midpoint rule with 2,4,8,...,64 sampling points
# numerical computation with floating point arithmetic
for m from 1 to 6 do
   h := Pi/2^m;
   S := h*sum(evalf(f(k*h-h/2)), k = 1 .. 2^m);
   print(S);
end do:
```

0.4593626849327842188921157625623308759050862399972858042991770262687068328251374530693296706994088502

0.4400520828215007499018578129032659892475057685789014483983050014421659463163010140393748144874186333

0.4400505857449335389138437701418267377018996647446125018211077065914649396877937643871822977684362097

0.4400505857449335159596822037189149131273723581689399721214421297459197341554642992152021394474568453

0.4400505857449335159596822037189149131273723019927652511367581717801382224780155479307965923811982541

0.4400505857449335159596822037189149131273723019927652511367581717801382224780155479307965923811982541

```
# comparison with the exact value of the Bessel function
evalf(BesselJ(1,1))
```

0.4400505857449335159596822037189149131273723019927652511367581717801382224780155479307965923811982541

7.3 Weights of a Quadrature Formula

Determine the weights w_k of the approximation (Simpson's rule)

$$\int_{-h/2}^{h/2} f(x)\,\mathrm{d}x = h(w_{-1}f(-h/2) + w_0 f(0) + w_1 f(h/2)) \,+\, O(h^5)\,,$$

which integrates polynomials of degree ≤ 3 exactly. Test the accuracy for the integrals $\int_0^1 \mathrm{e}^x\,\mathrm{d}x$ and $\int_0^1 \sqrt{x}\,\mathrm{d}x$ by applying the approximation on an increasing number of subintervals.

Resources: Riemann Integral

Solution

Weights

exact integration of polynomials of degree $\leq 3 \quad \Longleftrightarrow$

$$\int_{-h/2}^{h/2} x^k \, \mathrm{d}x = h(w_{-1}\,(-h/2)^k + w_0\, 0^k + w_1\,(h/2)^k)$$

for $k = 0, 1, 2, 3$

$$\begin{array}{lll} k=0 & \Longrightarrow & h = h(w_{-1} + w_0 + w_1) \\ k=1 & \Longrightarrow & 0 = h(w_{-1}\,(-h/2) + w_1\,(h/2)),\text{ i.e. } w_{-1} = w_1 \\ k=2 & \Longrightarrow & \dfrac{2}{3}(h/2)^3 = h(w_{-1}\,(-h/2)^2 + w_1\,(h/2)^2) \end{array}$$

substituting $w_{-1} = w_1$ into the last equation $\quad \Longrightarrow \quad w_{\pm 1} = 1/6$
first equation $\quad \Longrightarrow \quad w_0 = 1 - w_{-1} - w_1 = 2/3$

symmetry of the weights $\quad \Longrightarrow \quad$ exactness for $f(x) = x^3$ and all other monomials with odd exponents

Alternative solution

The weights can also be determined as integrals of the quadratic Lagrange polynomials p_k, corresponding to the points $x_{-1} = -h/2$, $x_0 = 0$, $x_1 = h/2$. They are defined by the interpolation conditions $p_k(x_\ell) = \delta_{k,\ell}$. For example,

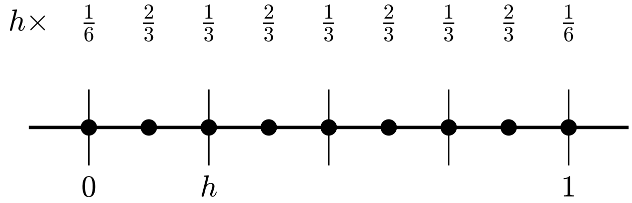
$$p_0(x) = \frac{(x - x_{-1})(x - x_1)}{(x_0 - x_{-1})(x_0 - x_1)}$$

$(p_0(x_0) = 1,\ p_0(x_{\pm 1}) = 0)$. The corresponding weight is

$$w_0 = \frac{1}{h}\int_{-h/2}^{h/2} \frac{(x - x_{-1})(x - x_1)}{(x_0 - x_{-1})(x_0 - x_1)}\,\mathrm{d}x = \frac{1}{h}\int_{-h/2}^{h/2} \frac{(x + h/2)(x - h/2)}{(0 + h/2)(0 - h/2)}\,\mathrm{d}x = \frac{2}{3}$$

Approximation

The figure shows the weights if Simpson's rule is applied on a partition of the integration interval into subintervals of length h. Note, that adjacent intervals share the weight $h/6$, resulting in a combined weight $h/3$.



MATLAB® script for approximation of $\int_0^1 f$ with subinterval length h

```
% contribution from the subinterval endpoints
s = h * sum(f(0:h:1))/3;
s = s-h*(f(0)+f(1))/6;  % correction at 0 and 1

% contribution from the subinterval midpoints
s = s + 2*h*sum(f(h/2:h:1))/3
```

approximations for $f(x) = \exp(x)$ and $f(x) = \sqrt{x}$ and $h = 1, 1/2, 1/4, 1/8, 1/16$:

```
f = @(x) exp(x)      f = @(x) sqrt(x)
----------           ----------
1.71886115           0.63807118
1.71831884           0.65652626
1.71828415           0.66307928
1.71828197           0.66539818
1.71828183           0.66621818
```

For $\sqrt{x}$ the approximation is much less accurate because of the singularity of the derivatives at $x = 0$.

7.4 Gauß Parameters with MATLAB®

Write a MATLAB® function which computes the points x_k and the weights w_k of the Gauß approximation

$$\int_0^1 f(x)\,\mathrm{d}x \approx \sum_{k=1}^{n} w_k f(x_k)$$

of order n.

The points are the zeros of the nth orthogonal polynomial $p(x) = x^n + p_n x^{n-1} + \cdots + p_1$, and the weights are determined from the exactness of the formula for polynomials of degree $< n$[1], i.e.,

$$\int_0^1 p(x)x^{\ell-1}\,\mathrm{d}x = 0, \quad \sum_{k=1}^{n} w_k x_k^{\ell-1} = \int_0^1 x^{\ell-1}\,\mathrm{d}x, \quad \ell = 1, \ldots, n\,.$$

Resources: Integration with MATLAB®

[1]Because of the special choice of the points, the approximation is exact for polynomials of degree $< 2n$; the reason for the high accuracy of the Gauß formulas.

Solution

Linear system for the orthogonal polynomial p

$\int_0^1 (x^n + p_n x^{n-1} + \cdots + p_1) x^{\ell-1}\,\mathrm{d}x = 0,\ \ell = 1,\dots,n \quad \Longleftrightarrow$

$$\sum_{k=1}^{n} \underbrace{\left(\int_0^1 x^{k-1} x^{\ell-1}\,\mathrm{d}x\right)}_{\frac{1}{k+\ell-1} =: h_{\ell,k}} p_k = -\underbrace{\int_0^1 x^n x^{\ell-1}\,\mathrm{d}x}_{\frac{1}{n+\ell} =: h_{\ell,n+1}}, \quad \ell = 1,\dots,n\,,$$

with the Hilbert matrix

$$H = \begin{pmatrix} 1/1 & 1/2 & 1/3 & \cdots \\ 1/2 & 1/3 & 1/4 & \cdots \\ 1/3 & 1/4 & 1/5 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Linear system for the weights w

$\sum_{k=1}^n w_k x_k^{\ell-1} = \int_0^1 x^{\ell-1}\,\mathrm{d}x,\ \ell = 1,\dots,n \quad \Longleftrightarrow$

$$\sum_{k=1}^{n} \underbrace{x_k^{\ell-1}}_{=: v_{\ell,k}} w_k = \underbrace{\frac{1}{\ell}}_{h_{\ell,1}}, \quad \ell = 1,\dots,n\,,$$

with the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & 1 & \cdots \\ x_1 & x_2 & x_3 & \cdots \\ x_1^2 & x_2^2 & x_3^2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

MATLAB® function

```
function [x,w] = GaussPar(n)

% Hilbert matrix and solution of the linear system for p
H = hilb(n+1); p = -H(1:n,1:n)\H(1:n,n+1);

% reordering p_k and appending the highest coefficient 1
```

```
p = [1; p(end:-1:1)];

% zeros of the orthogonal polynomial
x = roots(p);

% Vandermonde matrix and solution of the linear system for w
V = (ones(n,1)*x').^([0:n-1]'*ones(1,n)); w = V\H(1:n,1);
```

Striking simplicity of MATLAB® code!

points and weights for $n = 5$ (exactness up to degree 9)

$x_1, x_2, x_3:$	0.95308992296	0.76923465505	0.50000000000
$w_1, w_2, w_3:$	0.11846344252	0.23931433524	0.28444444444

symmetry $\implies$ $x_4 = 1 - x_2$, $x_5 = 1 - x_1$ and $w_4 = w_2$, $w_5 = w_1$

```
% test case: int_0^1 exp(x) dx
% Gauss approximation (A) for n=5 and exact value (E)

[x,w] = GaussPar(5);
A = w'*exp(x); E = exp(1)-1;

format long; [A; E]
   1.718281828458391
   1.718281828459046
```

12 correct digits!

Remark
Of course, the Gauß parameters are not computed in order to approximate an integral. Instead, the points and weights are tabulated. Just a linear transformation to the integration interval $[a, b]$ is necessary. Usually, the order is chosen not too large, and the accuracy is increased by subdividing $[a, b]$ (piecewise Gauß approximation).

7.5 Romberg Extrapolation with MATLAB®

Denote by $S(1,1), S(2,1), \ldots$ the approximations for an integral with the trapezoidal rule, generated by successively halving the stepsize:

$$\int_a^b f \approx S(j,1) = h_j \left(\frac{1}{2} f(a) + f(a+h_j) + \cdots + f(b-h_j) + \frac{1}{2} f(b) \right)$$

with $h_j = h2^{1-j}$ and $h = h_1$ the stepsize of the first approximation. For smooth integrands f, the accuracy can be significantly improved with the recursion

$$S(j,k) = \frac{4^{k-1} S(j,k-1) - S(j-1,k-1)}{4^{k-1}-1}, \quad k = 2, \ldots, j\,.$$

The last entry $S(j,j)$ in the jth row of this triangular scheme is an approximation of order $O(h^{2j})$.

The triangular array

$$\begin{array}{ccccc} S(1,1) & & & & \\ & \searrow & & & \\ S(2,1) & \rightarrow & S(2,2) & & \\ & \searrow & & \searrow & \\ S(3,1) & \rightarrow & S(3,2) & \rightarrow & S(3,3) \\ \ldots & & & & \end{array}$$

is generated row by row. After halving the step size ($h_{j-1} \rightarrow h_j$) and an additional approximation $S(j,1)$ of the trapezoidal rule, the extrapolated values $S(j,2), \ldots S(j,j)$ are computed.

Write a MATLAB® function `S = romberg(f,a,b,h,tol,steps)` which implements this so-called Romberg algorithm.

Test your program for

$$f(x) = \sin(\exp(x)), \quad a = 0,\ b = 1,\ h = 1/8\,,$$

and an estimated error $\leq \mathrm{tol} = 1.0\mathrm{e}-8\,.$

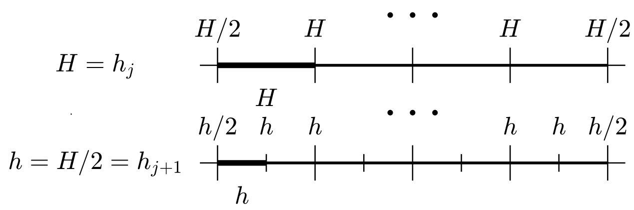
Resources: Integration with MATLAB®

Solution

Trapezoidal approximation

The figure shows the points and weights of two consecutive trapezoidal approximations $S(j,1)$ and $S(j+1,1)$. Since at the points $a, a+H, a+2H, \ldots$ all weights change by the same factor $\mathbf{\frac{1}{2}}$, when refining the stepsize, the integrand has to be evaluated only at the points $a+h, a+h+H, a+h+2H$ when computing $S(j+1,1)$:

$$S(j+1) = h_{j+1}\,(f(y+h) + f(a+h+H) + f(a+h+2H) + \cdots) + \frac{\mathbf{1}}{\mathbf{2}}\,S(j,1)\,.$$



Matlab® function

```
function S = romberg(f,a,b,h,tol,steps)

% trapezoidal rule with stepsize h
x = [a+h:h:b-h];
S(1,1) = h*(f(a)/2+sum(f(x))+f(b)/2);

% extrapolation steps
for j=2:steps
   h = h/2;
   % additional function values
   % f(a:2*h:b) was already computed for S(j-1,1)
   fx = f(a+h:2*h:b-h);
   % trapezoidal rule, using values of the last approximation
   S(j,1) = h*sum(fx)+S(j-1,1)/2;
   for k=2:j
      % extrapolation step
      S(j,k) = (4^(k-1)*S(j,k-1)-S(j-1,k-1))/(4^(k-1)-1);
   end
   if abs(S(j,j)-S(j,j-1))<=tol
      S = S(j,j); return   % accuracy reached
```

```
   end
end
% no convergence after steps extrapolations
S = 'no convergence'
```

Test case

data

f = @(x) sin(exp(x)), a=0, b=1, h=1/8, tol=1.0e-8, steps=16

$\rightsquigarrow$ matrix S, generated by the algorithm with return value (bold) as approximation for $\int_a^b f$

0.871023590187			
0.873974369576	0.874957962707		
0.874711528008	0.874957247485	0.874957199804	
0.874895783381	0.874957201839	0.874957198796	**0.874957198780**

Remark

Without extrapolation, 12 steps of the trapezoidal rule with $8 \cdot 2^{11} + 1 = 16385$ function evaluations are required, compared to $8 \cdot 2^3 + 1 = 65$ function evaluations for the Romberg scheme. The operations to generate the triangular scheme are negligible.

These considerable savings are possible only for smooth functions, i.e., functions with continuous derivatives of very high order.

By contrast, if derivatives of low order cease to be continuous, the accuracy deteriorates considerably. The function $f(x) = \sin(\sqrt{x})$ is a typical example.

7.6 Antiderivative of the Logarithm of a Polynomial

Determine

$$\int \ln(x^3 - x^2 - x + 1)\,\mathrm{d}x\,,$$

assuming $x > 1$.

Resources: Table of Elementary Integrals, Properties of the Integral

Solution

Simplification

factoring the argument $p(x)$ of the integrand $f(x) = \ln(\underbrace{x^3 - x^2 - x + 1}_{p(x))}$:

dividing by the linear factor $x-1$, corresponding to the apparent zero $x_1 = 1$ of p,

$$\begin{array}{rrrrl} (x^3 & -x^2 & -x & +1 &) \; : \; (x-1) = x^2 - 1 \\ x^3 & -x^2 & & & \\ \hline & 0 & -x & +1 & \\ & & -x & +1 & \\ \hline & & & 0 & \end{array}$$

$\rightsquigarrow$ additional zeros (zeros of $x^2 - 1$) $x_2 = 1$ (double zero), $x_3 = -1$
resulting factorization:

$$p(x) = (x - x_1)(x - x_2)(x - x_3) = (x-1)^2(x+1)$$

applying the rules $\ln(ab) = \ln a + \ln b$, $\ln a^2 = 2\ln a$ $\rightsquigarrow$

$$f(x) = \ln(\underbrace{(x-1)^2(x+1)}_{p(x)}) = 2\ln(x-1) + \ln(x+1)$$

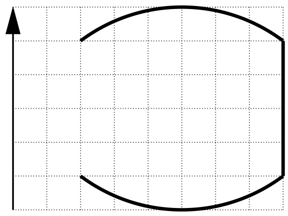
Indefinite integral

applying the formula $\int \ln t \, \mathrm{d}t = t\ln t - t$ with $t = x - 1$ and $t = x + 1$ (shifts are irrelevant) $\rightsquigarrow$ antiderivative

$$\begin{aligned} F(x) &= 2((x-1)\ln(x-1) - (x-1)) + (x+1)\ln(x+1) - (x+1) + C \\ &= -3x + 2(x-1)\ln(x-1) + (x+1)\ln(x+1) + \tilde{C} \end{aligned}$$

$(\tilde{C} = C + 1)$

7.7 Surface of a Car Tire



Determine the area of the surface, generated by rotating the depicted curve, consisting of two circular segments and a line segment, about the vertical axis.

The grid width in the figure corresponds to one length unit.

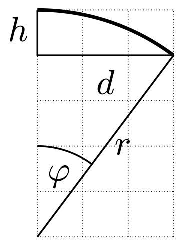
Resources: Guldin's Rules

Solution

Radius r and angle φ of the circular segments

theorem of Pythagoras $\implies$

$$r^2 = d^2 + (r-h)^2 \iff 2rh = d^2 + h^2$$



substituting the concrete values $\rightsquigarrow$

$$r = (d^2 + h^2)/(2h) = (3^2 + 1^2)/(2 \cdot 1) = 5$$

and $\varphi = \arctan(d/(r-h)) = \arctan(3/4)$

Area A_1 of the surface part, generated by the straight line segment

shell of a cylinder with Radius $R = 8$ and height $H = 4$:

$$A_1 = 2\pi R\, H = 64\pi$$

Area A_2 of the surface part generated by one of the circular segments

application of Guldin's rule for surfaces:
"The area of a surface, generated by rotating a curve C about a nonintersecting axis, is equal to the product of the length of C and the length of the circle, traversed by the center of gravity of C."

length of each of the two circular segments (angle of the sector $= 2\varphi$):

$$L = 2\varphi\, r = 2\arctan(3/4) \cdot 5 = 10\arctan(3/4)$$

symmetry $\implies$
distance of the center of gravity of a circular segment: $d = 5$
length of the traversed circle: $U = 2\pi d = 10\pi$
$\rightsquigarrow$

$$A_2 = UL = 10\pi \cdot 10\arctan(3/4) = 100\pi\arctan(3/4)$$

area of the combined surface:

$$A_1 + 2A_2 = 64\pi + 200\pi\arctan(3/4) \approx 605.39$$

7.8 Improper Integral of a Quotient of a Logarithm and a Polynomial

Determine an antiderivative of $f(x) = \ln(x^2 + 1)/(x + 1)^2$ and compute $\int_0^\infty f(x)\,\mathrm{d}x$.

Resources: Improper Integral, Integration by Parts, Partial Fraction Decomposition

Solution

Integrating by parts, the logarithm in the numerator of the integrand

$$f(x) = \frac{\ln(x^2+1)}{(x+1)^2}$$

can be eliminated. A partial fraction decomposition expresses the resulting rational function as a sum of elementary terms, which are easily integrated.

<u>Antiderivative</u>

integration by parts $\quad\rightsquigarrow$

$$\begin{aligned} F(x) &= \int \underbrace{(x+1)^{-2}}_{u'}\underbrace{\ln(x^2+1)}_{v}\,\mathrm{d}x \\ &= \overbrace{\underbrace{-(x+1)^{-1}}_{u}\underbrace{\ln(x^2+1)}_{v}}^{=:F_1(x)} - \int \underbrace{-(x+1)^{-1}}_{u}\underbrace{\frac{2x}{x^2+1}}_{v'}\,\mathrm{d}x \end{aligned}$$

partial fraction decomposition of $-uv'$ with the ansatz

$$g(x) := -u(x)v'(x) = \frac{2x}{(x+1)(x^2+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2+1}$$

multiplying with $(x+1)$ and setting $x=-1 \quad\Longrightarrow\quad \dfrac{2(-1)}{(-1)^2+1} = -1 = a$

subtracting the term $-1/(x+1) \quad\rightsquigarrow$

$$bx+c = (x^2+1)\left(\frac{2x}{(x+1)(x^2+1)} - \frac{-1}{x+1}\right) = \frac{2x+x^2+1}{x+1} = x+1\,,$$

i.e., $b=c=1$

resulting antiderivatives of the elementary terms:

$$\begin{aligned} G(x) = -\int u(x)v'(x)\,\mathrm{d}x &= \int -\frac{1}{x+1} + \frac{x}{x^2+1} + \frac{1}{x^2+1}\,\mathrm{d}x \\ &= -\ln|x+1| + \frac{1}{2}\ln(x^2+1) + \arctan x + C \end{aligned}$$

simplification with $(1/2)\ln p = \ln\sqrt{p}$, $\ln p - \ln q = \ln p/q$ and addition of $F_1(x) = u(x)v(x) \quad\rightsquigarrow$

$$F(x) = -\frac{\ln(x^2+1)}{x+1} + \underbrace{\ln\sqrt{\frac{x^2+1}{(x+1)^2}}}_{=:F_2(x)} + \underbrace{\arctan x}_{=:F_3(x)}$$

Integral over $[0,\infty)$

- $\ln(x^2+1) \le \ln(x+1)^2 = 2\ln(x+1)$, $\ln x/x \to 0$ für $x \to \infty \implies \lim_{x\to\infty} F_1(x) = 0$
- $\dfrac{x^2+1}{(x+1)^2} = \dfrac{1+1/x^2}{1+2/x+1/x^2} \to 1$ for $x \to \infty \implies \lim_{x\to\infty} F_2(x) = \ln\sqrt{1} = 0$
- $\lim_{x\to\infty} \underbrace{\arctan x}_{F_3(x)} = \pi/2$

definition of an improper integral, substitution of the limits, and $F_k(0) = 0$, $k = 1,2,3 \quad \rightsquigarrow$

$$\int_0^\infty f(x)\,\mathrm{d}x = \lim_{r\to\infty} \int_0^r f(x)\,\mathrm{d}x = \lim_{r\to\infty} F(r) - F(0) = \pi/2 - 0 = \pi/2$$

7.9 Antiderivative and Integral of a Trigonometric Expression

Determine an antiderivative of

$$\cos^2(2t)\sin^2 t$$

and compute the integral over $[0, 2\pi]$.

Resources: Table of Elementary Integrals

Solution

Conversion to a complex trigonometric polynomial

applying the formulas of Euler-Moivre,

$$\cos\varphi \underset{(C)}{=} \frac{e^{i\varphi}+e^{-i\varphi}}{2}, \quad \sin\varphi \underset{(S)}{=} \frac{e^{i\varphi}-e^{-i\varphi}}{2i},$$

to $f(t) = \cos^2(2t)\sin^2 t \quad \rightsquigarrow$

$$f(t) = \frac{(e^{2it}+e^{-2it})^2(e^{it}-e^{-it})^2}{2^2(2i)^2}$$

expanding, with $z = e^{it} \quad \rightsquigarrow$

$$\begin{aligned} f(t) &= -\frac{(z^2+z^{-2})^2(z-z^{-1})^2}{16} = \frac{(z^4+2+z^{-4})(-z^2+2-z^{-2})}{16} \\ &= \frac{-z^6+2z^4-3z^2+4-3z^{-2}+2z^{-4}-z^{-6}}{16} \end{aligned}$$

Antiderivative

converting back to a real trigonometric polynomial, using (C) in the reverse direction, i.e., replacing $(z^k+z^{-k})/2$ by $\cos(kt) \quad \rightsquigarrow$

$$f(t) = \frac{1}{4} - \frac{3}{8}\cos(2t) + \frac{1}{4}\cos(4t) - \frac{1}{8}\cos(6t)$$

$\int \cos(kt)\,dt = \frac{1}{k}\sin(kt) + C \quad \rightsquigarrow \quad$ antiderivative

$$F(t) = \frac{1}{4}t - \frac{3}{16}\sin(2t) + \frac{1}{16}\sin(4t) - \frac{1}{48}\sin(6t)$$

Integral

$\int_0^{2\pi} \cos(kt)\,dt = 0$ for $k > 0 \quad \Longrightarrow$

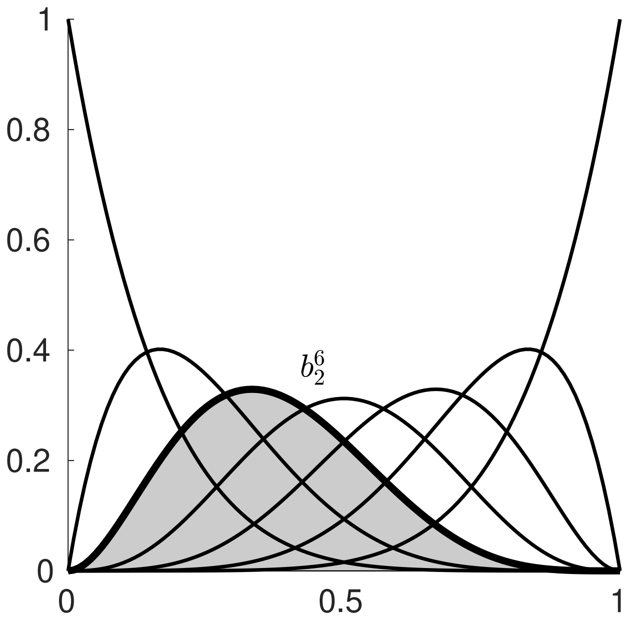
$$\int_0^{2\pi} f = \int_0^{2\pi} \frac{1}{4}\,dt = \frac{\pi}{2}$$

7.10 Integrals of Bernstein Polynomials

The figure shows the Bernstein polynomials[2]

$$b_k^n(x) = \binom{n}{k} (1-x)^{n-k} x^k, \quad 0 \le k \le n,$$

for $n = 6$.



Compute $\int_0^1 b_k^n(x)\,\mathrm{d}x$.

Resources: Integration by Parts, Fundamental Theorem of Calculus

[2]These polynomials play a fundamental role in Computer Aided Design, as was first realized by P. Bézier (Renault) and P. de Casteljau (Citroën); cf. K. Höllig, J. Hörner: *Approximation and Modeling with B-Splines*, SIAM, OT132, 2013, for an introduction to the basic modeling techniques.

Solution

The integral of

$$b_k^n(x) = \binom{n}{k}(1-x)^{n-k}x^k$$

over $[0,1]$ can be computed using integration by parts:

$$\int_0^1 b_k^n(x)\,\mathrm{d}x = \binom{n}{k}\int_0^1 \underbrace{(1-x)^{n-k}}_{u(x)}\,\underbrace{x^k}_{v'(x)} =$$

$$\binom{n}{k}\Big[\underbrace{(1-x)^{n-k}}_{u(x)}\,\underbrace{\frac{x^{k+1}}{k+1}}_{v(x)}\Big]_{x=0}^{x=1} - \binom{n}{k}\int_0^1 \underbrace{-(n-k)(1-x)^{n-k-1}}_{u'(x)}\,\underbrace{\frac{x^{k+1}}{k+1}}_{v(x)}\,\mathrm{d}x$$

For $0 \le k < n$, the first term on the right side vanishes, and, since

$$\binom{n}{k}\frac{n-k}{k+1} = \frac{n!(n-k)}{(n-k)!\,k!\,(k+1)} = \frac{n!}{(n-k-1)!\,(k+1)!} = \binom{n}{k+1},$$

the second term equals

$$\binom{n}{k+1}\int_0^1 (1-x)^{n-(k+1)}x^{k+1}\,\mathrm{d}x = \int_0^1 b_{k+1}^n(x)\,\mathrm{d}x\,.$$

Hence, starting with $k=0$, the integrals of all Bernstein polynomials are equal:

$$\int_0^1 b_0^n = \int_0^1 b_1^n = \cdots = \int_0^1 b_n^n\,,$$

where the last integral is easily computed:

$$\int_0^1 b_n^n(x)\,\mathrm{d}x = \int_0^1 x^n\,\mathrm{d}x = \frac{1}{n+1}\,.$$

7.11 Taylor Expansion and Integration by Parts

Prove the identity[3]

$$\int_0^1 x^x \,\mathrm{d}x = S = \sum_{n=1}^{\infty} (-1)^n n^{-n} \,,$$

and compute the integral with an error less than 0.001.

Resources: Integration by Parts, Table of Elementary Integrals

[3]discovered 1697 by Bernoulli and, together with the variant $\int_0^1 x^{-x}\,\mathrm{d}x = \sum_{n=1}^{\infty} n^{-n}$ referred to as „Sophomore's Dream“ (cf. J. Borwein, D. Bailey, R. Girgensohn: *Experimentation in Mathematics: Computational Path to Discovery*, CRC Press (2004))

Solution

Equivalence of the integral and the sum

Taylor expansion of the exponential function $\quad\rightsquigarrow$

$$x^x = \mathrm{e}^{x\ln x} = \sum_{n=0}^{\infty} \frac{1}{n!}\,(x\ln x)^n$$

In order to integrate the infinite series term by term, first, $I_{n,k} = \int_0^1 x^n (\ln x)^k \,\mathrm{d}x$ is computed:
integration by parts with $u'(x) = x^n$ and $v(x) = \ln^k x \quad\rightsquigarrow$

$$\begin{aligned} I_{n,k} &= \Big[\underbrace{\frac{x^{n+1}}{n+1}}_{u(x)}\underbrace{(\ln x)^k}_{v(x)}\Big]_{x=0}^{x=1} - \int_0^1 \underbrace{\frac{x^{n+1}}{n+1}}_{u(x)}\underbrace{k(\ln x)^{k-1}\frac{1}{x}}_{v'(x)}\,\mathrm{d}x \\ &= 0 - \frac{k}{n+1}\int_0^1 x^n(\ln x)^{k-1}\,\mathrm{d}x = -\frac{k}{n+1}\,I_{n,k-1}\,, \end{aligned}$$

where the first term, $[\ldots]_{x=0}^{x=1}$, is equal to 0 since $x\ln x$ vanishes at $x=0$ and $x=1$

recursive application with $I_{n,0} = \int_0^1 x^n\,\mathrm{d}x = \frac{1}{n+1} \qquad\Longrightarrow$

$$\begin{aligned} I_{n,n} &= \left(-\frac{n}{n+1}\right) I_{n,n-1} = \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+1}\right) I_{n,n-2} \\ &= \left(-\frac{n}{n+1}\right)\left(-\frac{n-1}{n+1}\right)\cdots\left(-\frac{1}{n+1}\right)\underbrace{\left(\frac{1}{n+1}\right)}_{I_{n,0}} = (-1)^n\frac{n!}{(n+1)^{n+1}} \end{aligned}$$

substituting into the Taylor expansion $\quad\rightsquigarrow$

$$\int_0^1 x^x\,\mathrm{d}x = \sum_{n=0}^{\infty}\frac{1}{n!}I_{n,n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^n}$$

Error estimate

Since the summands alternate in sign and converge to 0, the truncation error of a finite sum can be bounded by the absolute value of the first summand omitted (Leibniz criterion):

$$\left| S - \sum_{n=1}^{N}\frac{(-1)^{n-1}}{n^n}\right| \le (N+1)^{-N-1}$$

$N = 4 \quad \rightsquigarrow \quad$ approximation

$$S \approx S_4 = 1 - \frac{1}{4} + \frac{1}{27} - \frac{1}{256} = 0.7831\dots$$

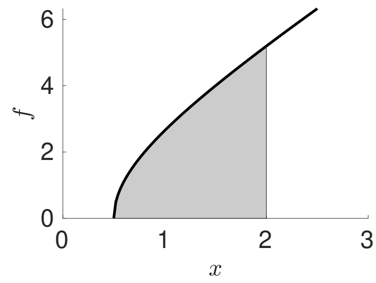
and $|\int_0^1 x^x \, \mathrm{d}x - S_4| \leq 5^{-5} = 1/3125 < 0.001$

7.12 Area Bounded by the Square Root of a Parabola

Compute the depicted area, which is bounded by the graph of the function

$$f(x) = \sqrt{4x^2 + 8x - 5}$$

for $1/2 \leq x \leq 2$.



Resources: Special Substitutions

Solution

The area A is equal to the integral

$$\int_{1/2}^{2} f, \quad f(x) = \sqrt{4x^2 + 8x - 5}\,.$$

With a linear substitution, the integrand is first transformed to the standard form $\sqrt{y^2 - 1}$ and then computed with the substitution

$$y = \cosh z,\ \mathrm{d}y = \sinh z\,\mathrm{d}z, \quad z = \operatorname{arcosh} y = \ln(y + \sqrt{y^2 - 1})\,. \tag{1}$$

<u>Transformation to standard form</u>

completing the square $\rightsquigarrow$

$$4x^2 + 8x - 5 = (2x + 2)^2 - 9 = 9\left(\left(\frac{2x + 2}{3}\right)^2 - 1\right)$$

linear substitution $\rightsquigarrow$

$$y = \frac{2x + 2}{3},\ \mathrm{d}y = \frac{2}{3}\,\mathrm{d}x, \quad x = 1/2 \leftrightarrow y = 1,\ x = 2 \leftrightarrow y = 2$$

and

$$A = \int_{1/2}^{2} \sqrt{4x^2 + 8x - 5}\,\mathrm{d}x = \int_{1}^{2} 3\sqrt{y^2 - 1}\,\frac{3}{2}\mathrm{d}y$$

<u>Computation of the Integral</u>

Applying the substitution (1), the integration limits $y_1 = 1$ and $y_2 = 2$ become

$$z_1 = \ln(1 + 0) = 0, \quad z_2 = \ln(2 + \sqrt{3})\,,$$

and, noting that $\cosh^2 z - 1 = \sinh^2 z$,

$$A = \frac{9}{2}\int_{0}^{\ln(2+\sqrt{3})} \sinh^2 z\,\mathrm{d}z = \frac{9}{2}\left[\frac{1}{2}\sinh z\cosh z - \frac{1}{2}z\right]_{z=0}^{z=\ln(2+\sqrt{3})}$$

$z = \ln(2 + \sqrt{3}) = \operatorname{arcosh} 2 \quad \Longrightarrow \quad \cosh z = 2,\ \sinh z = \sqrt{2^2 - 1}$ and

$$\begin{aligned} A &= \frac{9}{2}\left(\left(\frac{1}{2}\sqrt{3}\cdot 2 - \frac{1}{2}\ln(2 + \sqrt{3})\right) - 0\right) \\ &= \frac{9}{2}\sqrt{3} - \frac{9}{4}\ln(2 + \sqrt{3}) \approx 4.8311 \end{aligned}$$

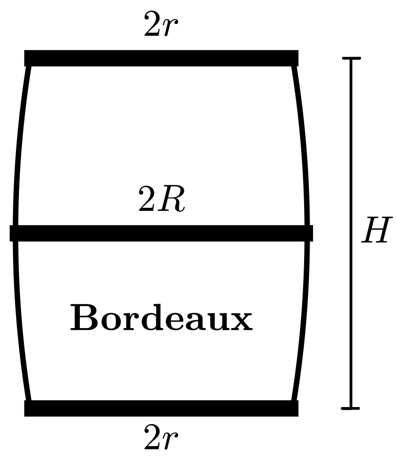
7.13 Construction of a 1000-Liter Barrel

The depicted barrel has a height of $H = 1.25\,\text{m}$, and the smallest and the largest (middle) radii r and R differ by $5\,\text{cm}$. Determine R for a volume V of 1000 liter ($1\,\text{m}^3$) with Kepler's rule[a]

$$V \approx \pi H \left(\frac{2}{3} R^2 + \frac{1}{3} r^2 \right) .$$

Compare with the exact volume for a parabola as profile. Moreover, compute the curved surface part of the barrel.

[a] identical to Simpson's rule for numerical integration



Resources:

Solution

Radius

```
# Kepler's approximation
V_Kepler := (H,R,r) -> Pi*H*(2*R^2/3+r^2/3)
```

$$V\_\text{Kepler} := (H,R,r) \longrightarrow \frac{\pi \cdot H \cdot (2 \cdot R^2 + r^2)}{3}$$

```
# assignment of the given values (5 cm = 1/20 m)
# solution of the quadratic equation V_Kepler = 1
H := 1.25: r := R-1/20: R := solve(V_Kepler(H,R,r)=1,R)
```

$$R := 0.5207, -0.4874$$

$R[1]$ contains the relevant positive solution.

Exact volume

profile curve: parabola p with maximum at $h = H/2$ with value R and value $r = R - 1/20$ at $h = 0$ and $h = H$

```
p := h -> R[1]-(2*h/H-1)^2/20;
```

$$p := h \longrightarrow R[1] - \frac{(2 \cdot h/H - 1)^2}{20}$$

formula for the volume of a solid of revolution, $V_{\text{exakt}} = \pi \int_0^H p(h)^2 \, \mathrm{d}h \quad \rightsquigarrow$

```
V_exakt := Pi*int(p(h)^2,h=0..H)
```

$$V\_exakt = 0.9987$$

error $V_{\text{Kepler}} - V_{\text{exakt}} = 0.0013$ ($\widehat{=}$ 1.3 liter), less than 1%

Surface

formula for a surface of revolution, $S = 2\pi \int_0^H p(h)\sqrt{1 + p'(h)^2} \, \mathrm{d}h \quad \rightsquigarrow$

```
S := 2*Pi*int(p(h)*sqrt(1+diff(p(h),h)^2),h=0..H)
```

$$S := 3.9754$$

7.14 Differentiation of Integrals

Differentiate the following integrals with respect to x for $x > 0$.

$$\text{a)} \int_0^{\sqrt{x}} e^{t^2}\, dt \qquad \text{b)} \int_0^1 \ln(x^2 + t^2)\, dt \qquad \text{c)} \int_0^{1/x} \frac{\sin(xt^2)}{t}\, dt$$

Resources: Fundamental Theorem of Calculus

Problem Variants

■

:

check

■

:

check

■

:

check

Solution

Differentiation with respect to an integration limit

fundamental theorem of calculus and chain rule $\implies$

$$\int_a^{y(x)} f(t)\,\mathrm{d}t = F(y(x)) - F(a), \quad f = F'$$

$$\frac{\mathrm{d}}{\mathrm{d}x}(F(y(x)) - F(a)) = F'(y(x))y'(x) = f(y(x))y'(x)$$

application to $y(x) = \sqrt{x}$, $f(t) = \mathrm{e}^{t^2}$ $\rightsquigarrow$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\sqrt{x}} \mathrm{e}^{t^2}\,\mathrm{d}t = \mathrm{e}^{(\sqrt{x})^2}\frac{\mathrm{d}}{dx}\sqrt{x} = \frac{\mathrm{e}^x}{2\sqrt{x}}$$

Differentiation of the integrand

differentiating under the integral $\rightsquigarrow$

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b f(t,x)\,\mathrm{d}t = \int_a^b \frac{\partial f(t,x)}{\partial x}\,\mathrm{d}t$$

application with $f(x,t) = \ln(x^2+t^2)$ $\rightsquigarrow$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 \ln(x^2+t^2)\,\mathrm{d}t &= \int_0^1 \frac{2x}{x^2+t^2}\,\mathrm{d}t = \int_0^1 \frac{1}{x}\frac{2}{1+(t/x)^2}\,\mathrm{d}t \\ &= [2\arctan(t/x)]_{t=0}^{t=1} = 2\arctan(1/x)\end{aligned}$$

Differentiation of the integration limit and the integrand

chain rule for partial derivatives:

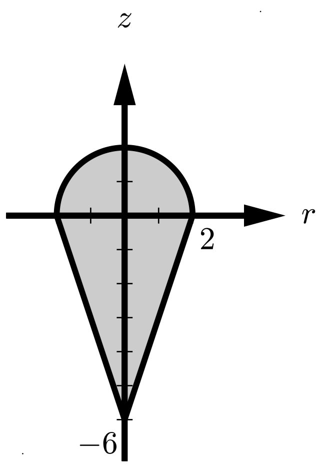
$$\frac{\mathrm{d}}{\mathrm{d}x}g(u(x),v(x)) = \partial_u g(u(x),v(x))u'(x) + \partial_v g(u(x),v(x))v'(x)$$

application with $g(u,v) = \int_a^u f(t,v)\,\mathrm{d}t$ and $u(x) = 1/x$, $v(x) = x$, $f(t,v) = \sin(vt^2)/t$, combining the previous examples $\rightsquigarrow$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{1/x} \frac{\sin(xt^2)}{t}\,\mathrm{d}t &= \frac{\sin(x/x^2)}{1/x}\left(-\frac{1}{x^2}\right) + \int_0^{1/x} t\cos(t^2x)\,\mathrm{d}t \\ &= -\frac{\sin(1/x)}{x} + \left[\frac{\sin(t^2x)}{2x}\right]_{t=0}^{t=1/x} = -\frac{\sin(1/x)}{2x}\end{aligned}$$

7.15 Center of Gravity of an Icecream Cone

Determine the center of gravity of the ice-cream cone with the depicted cross section, neglecting variations in the content, i.e., assuming constant density.



Resources: Center of Gravity of a Solid of Revolution

Solution

application of the formula

$$c_z = \pi \int_a^b z\, r(z)^2 \,\mathrm{d}z \Big/ \operatorname{vol} S, \quad \operatorname{vol} S = \pi \int_a^b r(z)^2 \,\mathrm{d}z$$

for the z-component of the center of gravity of a solid of revolution S with radius function $r(z)$, $a \le z \le b$

<u>Radius function</u>

For the semicircle, bounding the top part of the cross section, the theorem of Pythagoras implies

$$z^2 + r(z)^2 = 2^2, \quad \text{i.e., } r(z) = \sqrt{4 - z^2},\ 0 \le z \le 2\,.$$

For the straight line, bounding the bottom part,

$$r(z) = \frac{z+6}{3},\ -6 \le z \le 0\,.$$

<u>Volume</u>

applying the formulas for the volume of a hemisphere H (radius 2) and a cone C (radius 2, height 6) $\rightsquigarrow$

$$\operatorname{vol} S = \operatorname{vol} H + \operatorname{vol} C = \frac{2}{3}\,\pi\, 2^3 + \frac{1}{3}\,\pi\, 2^2 \cdot 6 = \frac{16}{3}\,\pi + 8\,\pi = \frac{40}{3}\,\pi$$

confirmation of the formulas with the formula for the volume of a solid of revolution:

- hemisphere

$$\operatorname{vol} H = \pi \int_0^2 (4 - z^2)\,\mathrm{d}z = \pi \left[4z - \frac{z^3}{3}\right]_{z=0}^{z=2} = \pi \left(8 - \frac{8}{3}\right) = \frac{16}{3}\,\pi \quad \checkmark$$

- cone

$$\operatorname{vol} C = \pi \int_{-6}^0 \left(\frac{z+6}{3}\right)^2 \mathrm{d}z = \pi \int_0^6 \frac{z^2}{9}\,\mathrm{d}z = \pi \left[\frac{z^3}{27}\right]_{z=0}^{z=6} = 8\,\pi \quad \checkmark$$

Center of gravity

contributions to $c_z = \pi \int_a^b z\, r(z)^2 \, \mathrm{d}z$:

- hemisphere

$$c_H = \pi \int_0^2 z\,(4 - z^2)\,\mathrm{d}z = \pi \left[2z^2 - \frac{z^4}{4}\right]_{z=0}^{z=2} = \pi\,(8-4) = 4\pi$$

- cone

$$\begin{aligned} c_C &= \pi \int_{-6}^0 z \left(\frac{z+6}{3}\right)^2 \mathrm{d}z = \pi \int_0^6 (z-6)\frac{z^2}{9}\,\mathrm{d}z = \pi \left[\frac{z^4}{36} - \frac{2z^3}{9}\right]_{z=0}^{z=6} \\ &= \pi\,(36-48) = -12\pi \end{aligned}$$

adding the contributions and dividing by the volume $\rightsquigarrow$

$$c_z = \frac{c_H + c_C}{\operatorname{vol} S} = \frac{4\pi - 12\pi}{(40/3)\pi} = -\frac{3}{5}$$

7.16 Existence of an Integral over $[\mathbf{0}, \infty)$ with a Singularity at 1

Prove the existence of the improper integral

$$\int_0^\infty \frac{\sin(\pi x)}{\ln x}\,\mathrm{d}x\,.$$

Resources: Improper Integral, Majorant

Solution

First, it is shown that integrand

$$f(x) = \frac{\sin(\pi x)}{\ln x}$$

is continuous by computing the limits for $x \to 0$ and $x \to 1$. Hence, the integral over any bounded interval exists. Then, the existence of the limit of $\int_{\mathrm{e}}^{b} f(x)\,\mathrm{d}x$ for $b \to \infty$ is proved[4].

Continuity of the integrand

- $x \to 0$: $\sin(\pi x) \to 0$ and $\ln x \to -\infty$ for $x \to 0$
  $\implies \quad \sin(\pi x)/\ln x \to 0$

- $x \to 1$: application of the *rule of L'Hôpital*,

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}, \quad \text{if } f(a) = g(a) = 0\,,$$

  with $f(x) = \sin(\pi x)$, $g(x) = \ln x \quad \rightsquigarrow$

$$\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x} = \lim_{x\to 1} \frac{\pi\cos(\pi x)}{1/x} = -\pi$$

Existence of the integral over the unbounded interval

integrating twice by parts $\quad \rightsquigarrow$

$$\int_{\mathrm{e}}^{\infty} \overbrace{\sin(\pi x)}^{u'(x)}\overbrace{(1/\ln x)}^{v(x)}\,\mathrm{d}x = \underbrace{\left[-\overbrace{\frac{\cos(\pi x)}{\pi}}^{u(x)}\overbrace{\frac{1}{\ln x}}^{v(x)}\right]_{x=\mathrm{e}}^{x=b}}_{I_1(b)} - \int_{\mathrm{e}}^{b} \overbrace{-\frac{\cos(\pi x)}{\pi}}^{u(x)}\overbrace{\left(-\frac{1}{x\ln^2 x}\right)}^{v'(x)}\mathrm{d}x$$

$$I_1(b) - \underbrace{\left[\frac{\sin(\pi x)}{\pi^2}\frac{1}{x\ln^2 x}\right]_{x=\mathrm{e}}^{x=b}}_{I_2(b)} + \underbrace{\int_{\mathrm{e}}^{b} \frac{\sin(\pi x)}{\pi^2}\left(-\frac{\ln^2 x + 2\ln x}{x^2\ln^4 x}\right)\mathrm{d}x}_{I_3(b)}$$

For $b \to \infty$,

$$I_1(b) \to -\frac{\cos(\pi \mathrm{e})}{\pi}, \quad I_2(b) \to -\frac{\sin(\pi\mathrm{e})}{\pi^2\mathrm{e}}\,,$$

[4] Instead of e, any lower limit could be used. Choosing e just makes the computations and estimates more convenient.

and $\lim_{b\to\infty} I_3(b)$ also exists, since the absolute value of the integrand is majorized by

$$\frac{1}{\pi^2}\frac{1+2}{x^2\ln^2 x} \le \frac{3}{x^2}, \quad x \ge \mathrm{e}\,.$$

7.17 Integral of a Product of a Polynomial with an Exponential Function over $\mathbb{R}$

Compute

$$\int_{-\infty}^{\infty} (x^2 + 3x)\, e^{1+4x-2x^2}\, dx$$

using that $\int_{-\infty}^{\infty} e^{-t^2}\, dt = \sqrt{\pi}$.

Resources: Improper Integral, Substitution, Integration by Parts

Solution

By completing the square in the exponent of the exponential function, followed by a linear substitution, the integral

$$\int_{-\infty}^{\infty} (x^2+3x)\,\mathrm{e}^{1+4x-2x^2}\,\mathrm{d}x$$

is transformed into a sum of integrals $\int_{-\infty}^{\infty} t^k\,\mathrm{e}^{-t^2}\,\mathrm{d}t$, which can be computed with integration by parts.

<u>Simplification by Substitution</u>

completing the square,

$$1+4x-2x^2 = -\left(\sqrt{2}\,x-\sqrt{2}\right)^2+3$$

$\rightsquigarrow$ substitution

$$t=\sqrt{2}\,x-\sqrt{2},\ x=t/\sqrt{2}+1, \quad \mathrm{d}t=\sqrt{2}\,\mathrm{d}x\,,$$

which simplifies the integral:

$$\begin{aligned} I &= \int_{-\infty}^{\infty} (x^2+3x)\,\mathrm{e}^{1+4x-2x^2}\,\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \left(\frac{1}{2}t^2+\frac{2}{\sqrt{2}}t+1+\frac{3}{\sqrt{2}}t+3\right)\,\mathrm{e}^{-t^2+3}\,\frac{1}{\sqrt{2}}\,\mathrm{d}t \\ &= \mathrm{e}^3\int_{-\infty}^{\infty}\left(\frac{\sqrt{2}}{4}t^2+\frac{5}{2}t+2\sqrt{2}\right)\,\mathrm{e}^{-t^2}\,\mathrm{d}t =: I_1+I_2+I_3 \end{aligned}$$

<u>Computation of the three integrals</u>

- $I_3=\mathrm{e}^3\cdot 2\sqrt{2}\cdot\sqrt{\pi}$
- $I_2=0$ because of symmetry (integral of an odd function over a symmetric interval)
- integration by parts with $u=t$, $v'=t\mathrm{e}^{-t^2}$ $\rightsquigarrow$

$$\begin{aligned} \frac{4}{\sqrt{2}}\,\mathrm{e}^{-3}\,I_1 &= \int_{-\infty}^{\infty} \underbrace{t}_{u(t)}\,\underbrace{t\,\mathrm{e}^{-t^2}}_{v'(t)}\,\mathrm{d}t \\ &= \overbrace{\left[t\,\underbrace{(-\mathrm{e}^{-t^2}/2)}_{v(t)}\right]_{-\infty}^{\infty}}^{=0} - \int_{\infty}^{\infty}\underbrace{1}_{u'(t)}\,(-\mathrm{e}^{-t^2}/2)\,\mathrm{d}t = \sqrt{\pi}/2\,, \end{aligned}$$

i.e., $I_1 = \frac{\sqrt{2}}{8}\, e^3 \sqrt{\pi}$

sum of the integrals:

$$I = I_1 + I_2 + I_3 = \frac{1}{8}\, e^3 \sqrt{2}\,\sqrt{\pi} + 0 + 2 \cdot e^3 \sqrt{2}\,\sqrt{\pi} = \frac{17}{8}\, e^3 \sqrt{2\pi} \approx 106.9873$$

7.18 Density Functions and Expected Values

For the functions

$$\text{a) } f(x) = c\,\mathrm{e}^{-x^2/4} \quad \text{b) } f(x) = c\,\mathrm{e}^{-3x} \quad \text{c) } f(x) = \frac{c}{(x+1)^5}$$

determine the constant c, so that $f(x)$ is a density function on the interval $[0, \infty)$, i.e., $\int_0^\infty f = 1$. Moreover, compute the expected value $\int_0^\infty x f(x)\,\mathrm{d}x$ in each case.

For part a) use that $\int_0^\infty \mathrm{e}^{-y^2}\,\mathrm{d}y = \sqrt{\pi}/2$.

Resources: Improper Integral, Integration by Parts

Solution

a) $f(x) = c\,\mathrm{e}^{-x^2/4}$

substituting $x = 2\,y$ with $\mathrm{d}x = 2\,\mathrm{d}y \quad \rightsquigarrow$

$$1 \overset{!}{=} \int_0^\infty f(x)\,\mathrm{d}x = c\int_0^\infty \mathrm{e}^{-y^2}\,2\,\mathrm{d}y = c\sqrt{\pi}\,,$$

i.e., $c = 1/\sqrt{\pi}$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(-2\,\mathrm{e}^{-x^2/4}\right) = x\,\mathrm{e}^{-x^2/4}$, $\lim_{x\to\infty}\mathrm{e}^{-x^2/2} = 0 \quad \rightsquigarrow \quad$ expected value

$$\frac{1}{\sqrt{\pi}}\int_0^\infty x\,\mathrm{e}^{-x^2/4}\,\mathrm{d}x = \left[-\frac{2}{\sqrt{\pi}}\,\mathrm{e}^{-x^2/4}\right]_{x=0}^{x=\infty} = -\frac{2}{\sqrt{\pi}}\,[0-(-1)] = \frac{2}{\sqrt{\pi}} \approx 1.1284$$

b) $f(x) = \mathrm{e}^{-3x}$

$$1 \overset{!}{=} c\int_0^\infty \mathrm{e}^{-3x}\,\mathrm{d}x = c\left[-\frac{1}{3}\,\mathrm{e}^{-3x}\right]_{x=0}^{x=\infty} = c\,(0-(-1/3)) = c/3$$

$\implies \quad c = 3$

integration by parts $\quad \rightsquigarrow \quad$ expected value

$$\begin{aligned}\int_0^\infty \underbrace{3x}_{u}\underbrace{\mathrm{e}^{-3x}}_{v'}\,\mathrm{d}x &= \Big[\underbrace{3x}_{u}\underbrace{\left(-\frac{1}{3}\mathrm{e}^{-3x}\right)}_{v}\Big]_{x=0}^{x=\infty} - \int_0^\infty \underbrace{3}_{u'}\underbrace{\left(-\frac{1}{3}\mathrm{e}^{-3x}\right)}_{v}\mathrm{d}x \\ &= 0 + \int_0^\infty \mathrm{e}^{-3x}\,\mathrm{d}x = \left[-\frac{1}{3}\,\mathrm{e}^{-3x}\right]_{x=0}^{x=\infty} = \frac{1}{3}\end{aligned}$$

c) $f(x) = c(x+1)^{-5}$

$$\int_0^\infty \frac{c\,\mathrm{d}x}{(x+1)^5} = \left[-\frac{c}{4(x+1)^4}\right]_{x=0}^{x=\infty} = \frac{1}{4} \quad \implies \quad c = 4$$

expected value

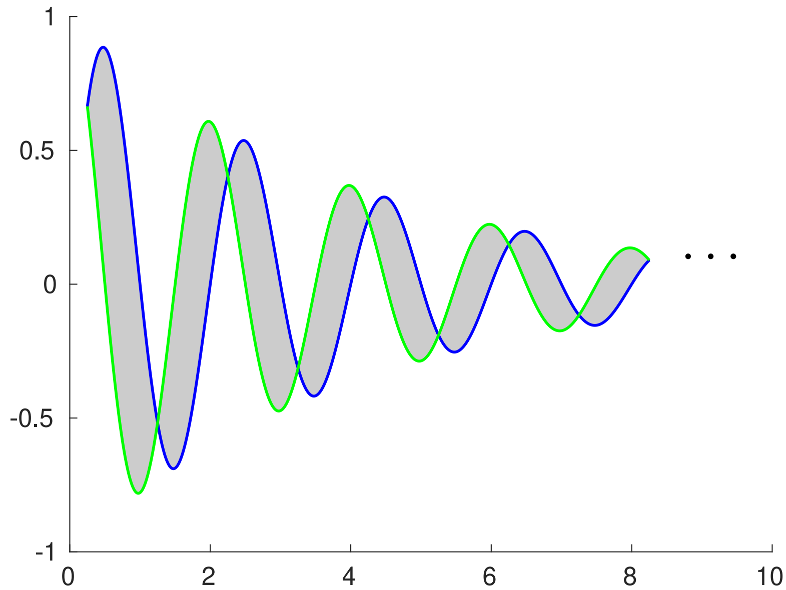
$$\int_0^\infty \frac{4x\,\mathrm{d}x}{(x+1)^5} = \left[-\frac{x}{(x+1)^4}\right]_{x=0}^{x=\infty} + \int_0^\infty \frac{\mathrm{d}x}{(x+1)^4} = \left[-\frac{1}{3(x+1)^3}\right]_{x=0}^{x=\infty} = \frac{1}{3}$$

7.19 Improper Integral and Geometric Series

Compute the (partially) depicted area, which is bounded by the graphs of the functions

$$f(x) = \mathrm{e}^{-x/4}\sin(\pi x), \quad g(x) = \mathrm{e}^{-x/4}\cos(\pi x)$$

with $x \in [1/4, \infty)$.



Resources: Improper Integral, Integration by Parts

Solution

It is shown that the subareas A_k, $k = 0, 1, \ldots$, with $1/4 + k \le x \le 5/4 + k$ decrease by a constant factor. Hence, it suffices to compute A_0, and to apply the formula for a geometric series.

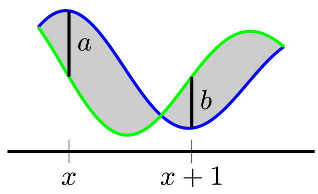
Decay of the areas

The subsets are separated by the intersections of the graphs of f and g which occur at the points where $\cos(\pi x) = \sin(\pi x)$, i.e., for

$$x_k = 1/4 + k, \quad k = 0, 1, \ldots .$$

comparing the vertical width of adjacent sets at corresponding points $\rightsquigarrow$

$$\begin{aligned} a &= e^{-x/4}\,|\sin(\pi x) - \cos(\pi x)|, \\ b &= e^{-(x+1)/4}\,|\sin(\pi x + \pi) - \cos(\pi x + \pi)| \end{aligned}$$



Hence, since $\cos(t+\pi) = -\cos(t)$ and $\sin(t+\pi) = -\sin(t)$,

$$b = e^{-1/4}\, a\,,$$

and the sub-ares A_k decrease by a factor $q = e^{-1/4}$.

applying the formula $\sum_{k=0}^{\infty} q^k = \dfrac{1}{1-q}$ for a geometric series $\rightsquigarrow$ total area

$$A = A_0 + A_1 + A_2 \cdots = A_0 + A_0 q + A_1 q^2 + \cdots = \frac{A_0}{1 - e^{-1/4}} \tag{1}$$

Total Area

integrating by parts $\rightsquigarrow$

$$A_0 = \int_{1/4}^{5/4} \underbrace{e^{-x/4}}_{u'(x)}\, \underbrace{(\sin(\pi x) - \cos(\pi x))}_{v(x)}\, dx =$$

$$\Big[\underbrace{(-4\,e^{-x/4})}_{u(x)}(\sin(\pi x) - \cos(\pi x))\Big]_{x=1/4}^{x=5/4} - \int_{1/4}^{5/4} (-4\,e^{-x/4})\, \underbrace{(\pi\cos(\pi x) + \pi\sin(\pi x))}_{v'(x)}\, dx$$

Since $\sin(\pi x) = \cos(\pi x)$ for $x = 1/4, 5/4$, the first term equals 0.

integrating the second term by parts $\rightsquigarrow$

$$A_0 = \int_{1/4}^{5/4} \underbrace{\left(4\,\mathrm{e}^{-x/4}\right)}_{u'(x)} \underbrace{\left(\pi\cos(\pi x) + \pi\sin(\pi x)\right)}_{v(x)}\,\mathrm{d}x =$$

$$\left[(-16\,\mathrm{e}^{-x/4})(\pi(\cos(\pi x) + \sin(\pi x)))\right]_{x=1/4}^{x=5/4} -$$

$$\underbrace{\int_{1/4}^{5/4} (-16\,\mathrm{e}^{-x/4})(\pi^2(-\sin(\pi x) + \cos(\pi x)))\,\mathrm{d}x}_{16\pi^2 A_0}$$

Since $-\cos(5\pi/4) = -\sin(5\pi/4) = \cos(\pi/4) = \sin(\pi/4) = \sqrt{2}/2$, the first term equals $16\pi\sqrt{2}\left(\mathrm{e}^{-5/16} + \mathrm{e}^{-1/16}\right)$, and solving for A_0 $\rightsquigarrow$

$$A_0 = \frac{16\pi\sqrt{2}\left(\mathrm{e}^{-5/16} + \mathrm{e}^{-1/16}\right)}{1 + 16\pi^2}\,.$$

substituting into (1) $\rightsquigarrow$

$$A = \frac{16\pi\sqrt{2}\left(\mathrm{e}^{-5/16} + \mathrm{e}^{-1/16}\right)}{(1 + 16\pi^2)(1 - \mathrm{e}^{-1/4})} \approx 3.3792$$

Chapter 8

Lexicon

8.1 Elementary Integrals

Riemann Integral

$$\int_a^b f(x)\,\mathrm{d}x = \underbrace{\int_a^b f}_{\text{shorthand notation}} = \lim_{|\Delta|\to 0} \sum_k f(\xi_k)\,\Delta x_k$$

with $\Delta x_k = x_k - x_{k-1}$, $|\Delta| = \max_k \Delta x_k$, $\xi_k \in (x_{k-1}, x_k)$

canonical choice of the partition Δ and the points ξ_k $\rightsquigarrow$ **midpoint rule**

$$\int_a^b f \approx h \sum_{k=0}^{n-1} f(a + kh + h/2), \quad h = (b-a)/n$$

Properties of the Integral

- linearity

$$\int_a^b f + g = \int_a^b f + \int_a^b g, \quad \int_a^b sf = s\int_a^b f$$

- estimate

$$\left|\int_a^b f\right| \le \int_a^b |f|$$

- scaling and translation

$$\int_a^b f(px+q)\,\mathrm{d}x = \frac{1}{p}\int_{pa+q}^{pb+q} f(y)\,\mathrm{d}y$$

Mean Value Theorem

$$\int_a^b f \underbrace{g}_{\ge 0} = f(c)\int_a^b g \quad \text{for some } c \in (a,b)$$

Fundamental Theorem of Calculus

If F is an antiderivative of the function f, i.e., if

$$F(x) = \int f(x)\,\mathrm{d}x \quad \Longleftrightarrow \quad f = F',$$

then

$$\int_a^b f = [F]_a^b = F(b) - F(a).$$

Table of Elementary Integrals

$f(x)$	$x^n,\ n \neq -1$	$\frac{1}{x}$	$\exp x$	$\ln x$	$\frac{1}{x^2+1}$
$F(x)$	$\frac{x^{n+1}}{n+1}$	$\ln\lvert x\rvert$	$\exp x$	$x\ln x - x$	$\arctan x$
$f(x)$	$\cos x$	$\sin x$	$\tan x$	$\cosh x$	$\sinh x$
$F(x)$	$\sin x$	$-\cos x$	$-\ln(\cos x)$	$\sinh x$	$\cosh x$

Trigonometric Polynomials

$$\int \underbrace{\sum_{|k|\le n} c_k \mathrm{e}^{\mathrm{i}kx}}_{p(x)}\,\mathrm{d}x = c + c_0 x + \sum_{0\neq|k|\le n} \frac{c_k}{\mathrm{i}k}\mathrm{e}^{\mathrm{i}kx}$$

definite integral over a periodicity interval:

$$\int_a^{a+2\pi} \mathrm{e}^{\mathrm{i}kx}\,\mathrm{d}x = 0\,\forall k \neq 0 \quad \Longrightarrow \quad \int_a^{a+2\pi} p = 2\pi c_0$$

Integration with Maple™

```
# f: expression, containing the variable x

# antiderivative
int(f,x)
```

```
# definite integral (symbolic)
int(f,x=a..b)

# definite integral (numeric)
Digits:=n   # optional specification of significant digits
int(f,x=a..b,'numeric')
```

⚠ Resist the temptation to use these commands **before** you have solved the problem without any (computer) assistance!

Integration with MATLAB®

```
% numeric
integral(f,a,b,'RelTol',tol_r,'AbsTol',tol_a)

% example, using default tolerances
f = @(x) exp(-x), integral(f,0,Inf)

% symbolic, antiderivative and definite integral
syms x, int(f,x), int(f,x,a,b)

% example:
syms x, int(sin(x),x), int(sin(x),x,0,pi)
```

8.2 Rational Integrands

Elementary Rational Integrands

- simple poles

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a}\ln|x+b/a|$$

$$\int \frac{c(x-a)+d}{(x-a)^2+b^2}\,\mathrm{d}x = \frac{c}{2}\ln((x-a)^2+b^2) + \frac{d}{b}\arctan\left(\frac{x-a}{b}\right)$$

- multiple poles

$$\int p^{-n-1} = -\frac{1}{n}p^{-n}$$

$$\int \frac{cp+d}{q^{n+1}} \underset{(*)}{=} -\frac{c}{2n\,q^n} + \frac{dp}{2b^2n\,q^n} + \frac{d(2n-1)}{2b^2n}\int \frac{1}{q^n}$$

with $p(x) = x-a$, $q(x) = (x-a)^2+b^2$
$(*)$ permits a recursive computation
special cases

$$\int \frac{x}{(x^2+1)^2}\,\mathrm{d}x = -\frac{1}{2(x^2+1)}, \quad \int \frac{\mathrm{d}x}{(x^2+1)^2} = \frac{x}{2(x^2+1)} + \int \frac{\mathrm{d}x}{2(x^2+1)}$$

transformation of the denominator $((x-a)^2+b^2)^{n+1}$ to the standard form $(y^2+1)^{n+1}$ with the substitution $x-a = by$

Partial Fraction Decomposition

$r = p/q$, degree $p = m$, degree $q = n$

- simple poles z_k, i.e., $q(x) = q_0(x-z_1)\cdots(x-z_n)$:

$$r(x) = f(x) + \sum_{k=1}^{n} \frac{c_k}{x-z_k}$$

with a polynomial f of degree $m-n$ ($f(x) = 0$ if $m<n$), which is the remainder if p is divided by q, and with

$$c_k = \lim_{x\to z_k}(x-z_k)\,r(x) = \frac{p(z_k)}{q_0\prod_{j\neq k}(z_k-z_j)}$$

special case

$$\frac{ax+b}{(x-z_1)(x-z_2)} = \frac{(az_1+b)/(z_1-z_2)}{x-z_1} + \frac{(az_2+b)/(z_2-z_1)}{x-z_2}$$

- multiplicity m_k of z_k:

$$r(x) = f(x) + \sum_k \sum_{j=1}^{m_k} \frac{c_{k,j}}{(x-z_k)^j} \tag{1}$$

  The coefficients of the polynomial f and the constants $c_{k,j}$ can be determined by comparing the coefficients of x^ℓ after multiplying with the common denominator.

Real Partial Fraction Decomposition
For a real rational function with complex conjugate poles $u_k \pm \mathrm{i}v_k$, alternatively to (1), the summands $\left(c^+_{k,j}/(x-u_k-\mathrm{i}v_k)^j + c^-_{k,j}/(x-u_k+\mathrm{i}v_k)^j\right)$ are replaced by a sum of the terms

$$\frac{d_{k,j}(x-u_k)+e_{k,j}}{((x-u_k)^2+v_k^2)^j}, \quad j = 1, \ldots, m_k\,.$$

Maple™
`convert(f,'parfrac','complex')`
`convert(f,'parfrac')` (real version)

8.3 Integration by Parts

Integration by Parts

$$\int f'g = fg - \int fg', \quad \int_a^b f'g = [fg]_a^b - \int_a^b fg'$$

$[fg]_a^b = 0$, if fg vanishes at a and b, or, if the product is periodic with period $(b-a)$.

Typical Applications

- $g(x) = x^k$, $f(x) = \mathrm{e}^x, \cos x, \sin x$ $\rightsquigarrow$ successive reduction of the polynomial degree
- $g(x) = \ln^k x$, $f(x) = x^\ell$ $\rightsquigarrow$ elimination of the logarithm after repeated integration by parts

8.4 Substitution

Substitution

$y = g(x)$, $x = g^{-1}(y)$, $\quad \mathrm{d}y = g'(x)\,\mathrm{d}x$

$$H(x) + C = \int \underbrace{f(g(x))g'(x)}_{h(x)}\,\mathrm{d}x = \int f(y)\,\mathrm{d}y = F(y) + \tilde{C}$$

or

$$H(b) - H(a) = \int_a^b \underbrace{(f \circ g)\, g'}_{h} = \int_{g(a)}^{g(b)} f = F(g(b)) - F(g(a))$$

application in both directions:

- An antiderivative H for integrands of the form $h = (f \circ g)g'$ is obtained by inserting $y = g(x)$ in an antiderivative F of f: $H(x) = F(y(x))$.
- An antiderivative for f can be constructed by substituting $y = g(x)$ and replacing $\mathrm{d}y$ by $g'(x)\,\mathrm{d}x$, then determining an antiderivative $H(x)$ of $h(x) = f(g(x))g'(x)$, and finally reversing the substitution, i.e., $F(y) = H(g^{-1}(y))$.

Definite integrals are handled analogously.

Special Substitutions

- $r(x, \sqrt{px+q}) \quad \to \quad \tilde{r}(y)$

$$y = \sqrt{px+q},\ \mathrm{d}x = 2y/p\,\mathrm{d}y$$

- $r(x, x^{1/m}, x^{1/n}) \quad \to \quad \tilde{r}(y)$

$$x = y^s,\ \mathrm{d}x = sy^{s-1}\,\mathrm{d}y, \quad s:\ \text{least common multiple of } m \text{ and } n$$

- $r(x, \sqrt{x^2-a^2}) \quad \to \quad \tilde{r}(\cosh y, \sinh y)$ or $\tilde{r}(\exp y)$

$$x = a\cosh y,\ \mathrm{d}x = a\ \sinh y\,\mathrm{d}y,\ \sqrt{x^2-a^2} = a\sinh y$$

- $r(x, \sqrt{x^2-a^2}) \quad \to \quad \tilde{r}(\cos y, \sin y)$

$$x = a/\cos y,\ \mathrm{d}x = a\sin y/\cos^2 y\,\mathrm{d}y,\ \sqrt{x^2-a^2} = a\tan y$$

- $r(x, \sqrt{a^2 - x^2}) \quad \rightarrow \quad \tilde{r}(\cos y, \sin y)$

$$x = a \sin y,\ \mathrm{d}x = a \cos y \, \mathrm{d}y,\ \sqrt{a^2 - x^2} = a \cos y$$

- $r(x, \sqrt{a^2 + x^2}) \quad \rightarrow \quad \tilde{r}(\cos y, \sin y)$

$$x = a \tan y,\ \mathrm{d}x = a / \cos^2 y \, \mathrm{d}y,\ \sqrt{a^2 + x^2} = a / \cos y$$

- $r(x, \sqrt{a^2 + x^2}) \quad \rightarrow \quad \tilde{r}(\cosh y, \sinh y)$ or $\tilde{r}(\exp y)$

$$x = a \sinh y,\ \mathrm{d}x = a \cosh y \, \mathrm{d}y,\ \sqrt{a^2 + x^2} = a \cosh y$$

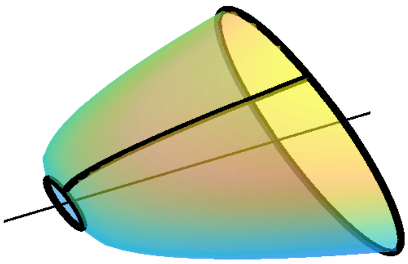
- $r(\cos x, \sin x) \quad \rightarrow \quad \tilde{r}(y)$

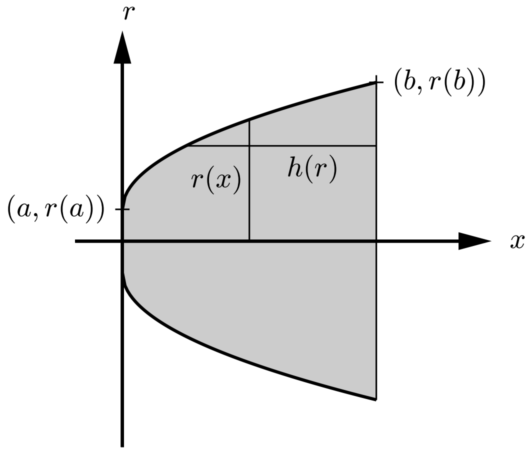
$$y = \tan(x/2),\ \mathrm{d}x = 2 \, \mathrm{d}y / (1 + y^2),\ \cos x = \frac{1 - y^2}{1 + y^2},\ \sin x = \frac{2y}{1 + y^2}$$

Too many formulas? **No**, this is just the tip of the iceberg! Determining indefinite integrals can require rather ingenious manipulations.

8.5 Solids of Revolution

Volume and Surface of a Solid of Revolution





rotation about the x-axis with radius $r(x)$, $a \le x \le b$ (cross section of the solid depicted on the right)

- **Volume:** $\pi \int_a^b r(x)^2 \, \mathrm{d}x$
- **Surface:** $2\pi \int_a^b r(x) \sqrt{1 + r'(x)^2} \, \mathrm{d}x$

volume of a solid bounded by an interior and an exterior radius function r_- and r_+:

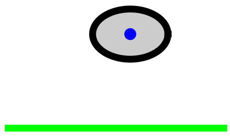
$$\pi \int_a^b r_+(x)^2 - r_-(x)^2 \, \mathrm{d}x$$

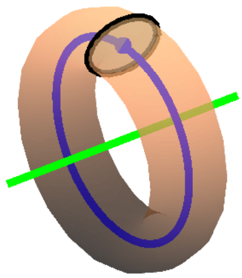
alternative computation of the volume for a monoton radius function:

$$2\pi \int_0^{r_{\max}} r h(r) \, \mathrm{d}r = \pi(b-a) r_{\min}^2 + 2\pi \int_{r_{\min}}^{r_{\max}} r h(r) \, \mathrm{d}r \,,$$

with $h(r)$ the height of the cylinder with radius r contained in the solid

Guldin's Rules





- The surface of a solid of revolution is equal to the product of the length of the generating curve C (black) and the length of the circle (blue), traversed by the center of gravity of C under the rotation about the symmetry axis (green).
- The volume of a solid of revolution is equal to the product of the area of the generating set A (gray) and the length of the circle (blue), traversed by the center of gravity of A under the rotation about the symmetry axis (green).

In both cases, the radius of the circle equals the distance of the center of gravity from the rotation axis.

⚠ Note that, in general, the centers of gravity of C and A will not coincide.

Center of Gravity of a Solid of Revolution

radius function $r(x)$, $a \leq x \leq b$, and constant density $\rightsquigarrow$ $C = (c_x, 0, 0)$ with

$$c_x = \int_a^b x r(x)^2 \, \mathrm{d}x \Big/ \int_a^b r(x)^2 \, \mathrm{d}x$$

8.6 Improper Integrals

Improper Integral

definite integral $\int_a^b f$ with unbounded integration interval or with $|f(x)| \to \infty$ for $x \to a^+$ or/and $x \to b^-$

$$\int_a^b f = \lim_{c\to a^+} \int_c^p f + \lim_{d\to b^-} \int_p^d f, \quad p \in (a,b)$$

Sufficient for the existence of the limits and, hence, for the existence of the improper integral, is that f is absolutely integrable, i.e.,

$$\int_c^d |f| \le \text{const}\, \forall\, [c,d] \subset (a,b)\,.$$

Majorant

$$|f(x)| \le |g(x)|,\ g \text{ absolutely integrable} \quad \Longrightarrow \quad f \text{ absolutely integrable}$$

commonly used majorants:

- $c\,x^r$, $r > -1$ for $\int_0^a f(x)\,\mathrm{d}x$
- $c\,x^r$, $r < -1$ for $\int_a^\infty f(x)\,\mathrm{d}x$

Minorant

$$0 < g(x) \le |f(x)|,\ g \text{ not integrable} \quad \Longrightarrow \quad f \text{ not absolutely integrable}$$

($\Longrightarrow$ not integrable, if f does not change sign)

commonly used minorants:

- $c\,x^r$, $r \le -1$ for $\int_0^a f(x)\,\mathrm{d}x$
- $c\,x^r$, $r \ge -1$ for $\int_a^\infty f(x)\,\mathrm{d}x$

Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1}\mathrm{e}^{-t}\,\mathrm{d}t, \quad x \in (0,\infty)$$

functional equation: $\Gamma(x+1) = x\Gamma(x)$, $\Gamma(n+1) = n!$