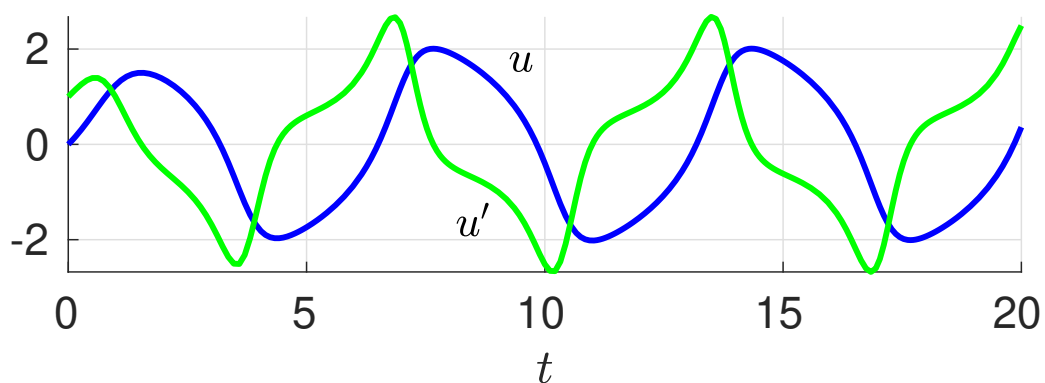


# MathTraining

## Problems and Solutions for Differential Equations



Klaus Höllig

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Aufgaben und Lösungen zur Höheren Mathematik 3, 4. Auflage  
by Klaus Höllig and Jörg Hörner  
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# Preface

The problem set of the book covers all basic topics of a first course on differential equations. It can be used to practice for exams, to facilitate the completion of homework assignments, and to review course material. Interactive variants to model problems with detailed solutions permit the student reader to test his comprehension of the relevant techniques. In addition to the collection of problems, a small mathematics lexicon contains brief descriptions of the relevant theorems, methods, and definitions.

There exists also a sportive aspect of mathematics - challenging problems requiring ideas beyond the standard techniques. The problems in the chapter *Calculus Highlights* are perhaps too difficult for undergraduates. They are included to initiate or strengthen fascination for mathematics. It is definitely not a mistake to practice substantially harder than necessary ...!

The book is partially translated from

## [Aufgaben und Lösungen zur Höheren Mathematik 3](#)

by Jörg Hörner and the author. It supplements this textbook by providing detailed solutions to tests for the chapters on *Differential Equations*. Moreover, the book includes additional problems, in particular problem variants for the topics of the tests.

The author wishes the readers success in their studies **and** hopes that mathematics will become one of their favorite subjects!

Klaus Höllig

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# Introduction

The book contains problems with detailed solutions, problem variants with interactive result verification, and a mathematics lexicon for the principal topics which are usually subject of a first course on differential equations:

- First Order Differential Equations,
- Second Order Differential Equations,
- Systems of Differential Equations,
- Laplace Transforms.

The problem set can be used to practice for exams, to facilitate the completion of homework assignments, and to deepen the comprehension of course material. How is this accomplished most effectively? Remembering his own student days, the author makes the following recommendations to a student reader.

Consider, as an example, a problem from the chapter on *First Order Differential Equations*:

## 1.9 Linear Substitution

Determine the solution  $y(x)$  of the initial value problem

$$y' = \frac{1}{x+y} - 1, \quad y(1) = 1.$$

**Resources:** [Linear Substitution](#)

Before looking at the solution of the problem, review the relevant theory or methods (resources). Clicking on the link *Linear Substitution* leads to



the following brief description of the solution technique from the *Lexicon* in chapter 6.

### Linear Substitution

$$y' = f(a + bx + cy) \xrightarrow{z=a+bx+cy} \frac{z'}{b + cf(z)} = 1 \quad (\text{separable})$$

If the right-hand side of the differential equation is a function  $f$  of a linear expression  $z = a + bx + cy$  ( $a = 0$ ,  $b = c = 1$  in the sample problem),  $z$  should be introduced as a new dependent variable. This results in a separable differential equation which can be solved more easily.

Try to solve the problem with these instructions. Then compare your computations with the solution given in the book:

### Solution

#### Substitution

replacing the linear expression  $x + y(x)$  by  $z(x)$   $\rightsquigarrow$

$$y'(x) = \frac{1}{x + y(x)} - 1 \iff z' - 1 = \frac{1}{z} - 1 \iff zz' = 1 \quad (1)$$

#### Solution of the separable differential equation

integrating (1), using that  $\frac{d}{dx}g(z(x)) = g'(z(x))z'(x)$  with  $g'(z) = z$ , and undoing the substitution  $\rightsquigarrow$

$$z^2/2 = x + c \iff (x + y)^2 = 2x + 2c$$

#### Initial value

$$y(1) = 1 \implies (1 + 1)^2 = 2 + 2c, \text{ i.e. } c = 1 \text{ and hence}$$

$$y(x) = \sqrt{2x + 2} - x$$

The solutions are written in a keyword-like style, as you would employ when you comment your solutions in an exam or for homework problems. For example, the phrase “integrating (1)  $\rightsquigarrow$  ...” stands for “By integrating

equation (1) we obtain ...". Other typical phrases are "simplifying  $\rightsquigarrow$  ...", "chain rule  $\implies$  ...", "characteristic polynomial: ...".

There is just as much detail included as is necessary for the mathematical arguments. Complete sentences are rarely used.

Following common practice, the dependence on a variable is often omitted. For example,

$$y' = f(x, y) = x + \sin y$$

is more "readable" than the formally correct equation

$$y'(x) = f(x, y(x)) = x + \sin(y(x)).$$

Unlike a computer, a human can deal with a slightly incorrect syntax.

To gain more practice with the solution technique, it is highly recommended to solve some (preferably all ...) of the problem variants, following the principal model problem for each topic. For *Linear Substitution* the variants are:

### Problem Variants

■  $y' = (1 - 2x + 3y)^2 + 2/3, y(1) = 1/3$

---

$y(0) = -?.??:$

check

■  $y' = \exp(2x + y) + 2, y(1) = 2$

---

$y(0) = ?.??:$

check

■  $y' = \frac{x}{x + 2y} - \frac{1}{2}, y(0) = 1$

---

$y(1) = ?.??:$

check

You can check your solution by typing your answer in the field adjacent to the  - box, replacing every question mark by a character (digit or

letter). Convert your result to a decimal, truncated to the number of digits indicated. For example,

$$2/3 \rightarrow 0.6666 \dots \xrightarrow{?.??} 0.66, \quad \text{answer : } \boxed{066}.$$

Note that the period is omitted; only the characters corresponding to the question marks are typed.

The solutions for the three problem variants are

$$(1) \quad y(x) = (2x - 1)/3, \quad y(0) = -1/3 = -0.3333 \dots,$$

$$(2) \quad y(x) = 2x + \ln x, \quad y(2) = 4.6931 \dots,$$

$$(3) \quad y(x) = \sqrt{x^2/2 + 1} - x/2, \quad y(1) = 0.7247 \dots$$

Hence, the correct input is

$$(1) \quad -?.?? \rightarrow \boxed{033} \qquad (2) \quad ?.?? \rightarrow \boxed{469} \qquad (3) \quad ?.?? \rightarrow \boxed{072}.$$

As mentioned in the beginning, the problem set can also assist you in completing homework assignments. Just look for a similar problem and study its solution. Similarly, for methods and examples presented in class, practice with the relevant problems.

The above remarks pertain to the first four chapters which exclusively discuss the solution of standard problems. Usually, such problems constitute the major portion of an exam or homework assignment. Hence, to review the basic techniques involved is of primary importance. Applying these techniques to more advanced problems is a natural next step. The chapter *Calculus Highlights* contains examples of rather challenging applications. You do not have to be disappointed, if you cannot solve any of these problems; they are definitely very difficult. It is legitimate to immediately look at the solutions and learn how the methods from the previous chapters are applied in an advanced setting. Also, as mentioned in the Preface, it is not a mistake to practice substantially harder than necessary ...!

You have solved some of the problems in chapter 5 without resorting to the solutions. Then ...

**... you can take pride in your mastery of the principal techniques for solving differential equations!**

With the previous explanations aimed at student readers, instructors could (obviously) also benefit from the interactive problem collection. The solutions of the model problems can be used as examples in class and some of the variants assigned as homework problems. Students will welcome the possibility of checking results before submitting or presenting their solutions in the exercise sections.

**Disclaimer:** Although the solutions and answers to the variants have been thoroughly checked, mistakes can always occur<sup>1</sup>. Please, write to the author (Klaus.Hoellig@gmail.com) if you find any errors.

---

<sup>1</sup>A statement by a teaching assistant to encourage students, which the author will always remember: “This year, the final exam is not too difficult - your professor could check the results without committing any errors!”.

# Chapter 1

## First Order Differential Equations

## 1.1 Initial Value Problem for the Exponential Function

Solve the initial value problem

$$y' = 2y, \quad y(3) = 4.$$

**Resources:** [Differential Equation for the Exponential Function](#)

### Problem Variants

■  $y' = y/2, y(2) = -1/2$

---

$y(0) = ?$ ???:

check

■  $y' = -y, y(1) = 2$

---

$y(0) = ?$ ???:

check

■  $y' = y, y(1) = y(2) + 3$

---

$y(0) = -?$ ???:

check

**Solution**

$$\frac{d}{dx}(c e^{px}) = p (c e^{px}) \implies y(x) = c e^{px} \text{ solves } y' = py$$

$$p = 2 \quad \rightsquigarrow$$

$$y(x) = c e^{2x}$$

substituting the initial condition  $y(3) = 4 \implies$

$$c e^{2 \cdot 3} = 4, \quad \text{i.e. } c = 4 e^{-6}$$

and

$$y(x) = 4 e^{-6} e^{2x} = 4 e^{2(x-3)}$$

## 1.2 Direction Field for a Differential Equation

Sketch the direction field of the differential equation

$$y' = (3 - x)(y^2 - 1)/4$$

as well as a few solutions. Determine the regions where solutions are increasing and decreasing, respectively. Do exist constant solutions?

**Resources:** [First Order Differential Equation](#)

### Problem Variants

■  $y' = 2y^3 + y^2 - y$

---

constant solutions for  $y(0) = -?, ?, ?.?:$

check

■  $y' = y^2 - 4x^2$

---

increasing for  $|y| > ?|x|:$

check

■  $2yy' = -x^3$

---

increasing for  $x < ?:$

check



## Solution

### Direction field

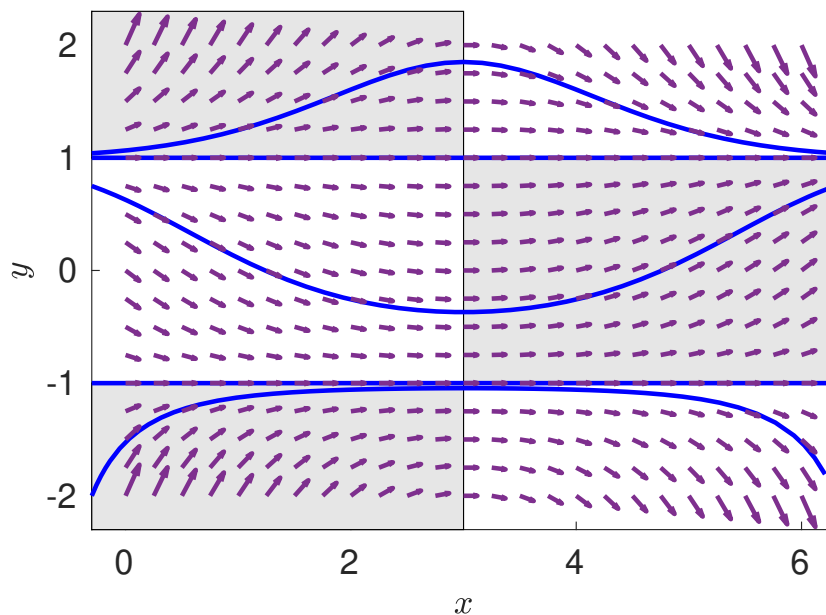
The direction field of a differential equation assigns to each point  $P = (x, y)$  the tangent direction of the solution passing through  $P$ :

$$(x, y) \mapsto (1, y'(x)) = (1, (3-x)(y(x)^2 - 1)/4)$$

a few directions:

$$(1, 2) \mapsto (1, (3-1) \cdot (2^2 - 1)/4) = (1, 3/2), \quad (2, 1) \mapsto (1, 0), \quad (4, 2) \mapsto (1, -3/4)$$

plot of the direction field and of solutions for several initial values:



### Monotonicity of Solutions

- increasing (shaded regions)  $\iff (3-x)(y^2-1) > 0$ , i.e.  
 $0 \leq x < 3 \wedge |y| > 1$  or  $x > 3 \wedge |y| < 1$
- decreasing  $\iff (3-x)(y^2-1) < 0$ , i.e.  
 $0 \leq x < 3 \wedge |y| < 1$  or  $x > 3 \wedge |y| > 1$
- constant solutions for  $|y(0)| = 1$ , since  $y(x)^2 - 1 = 0 \forall x \implies y'(x) \equiv 0$

The qualitative behavior is illustrated with graphs of the solutions for several initial values.

MATLAB<sup>®</sup> script

The direction field can be plotted with the following MATLAB<sup>®</sup> commands.

```
h = 0.25; % grid spacing
[x,y] = meshgrid(0:h:6,-2:h:2); % grid points
f = @(x,y) (3-x).*(y.^2-1)/4; % slopes
% tangent direction at (x,y): (1,f(x,y))
quiver(x,y,ones(size(f(x,y))),f(x,y))
```

## 1.3 Linear First Order Initial Value Problem

Determine the solution  $y(x)$  of the initial value problem

$$y' = \frac{2y}{x+1} + 3, \quad y(4) = 0.$$

**Resources:** [First Order Linear Differential Equation](#)

### Problem Variants

■  $y' = \frac{x+1}{x}y + e^x, y(1) = 0$

---

$y(2) = ??.??:$

check

■  $y' = \frac{2y}{2-x} + 3, y(1) = 1$

---

$y(0) = -?.??:$

check

■  $y' = -\tan x y + \cos x, y(0) = 0$

---

$y(1) = ??.??:$

check

## Solution

### Structure of the solution

The solution of a linear differential equation  $y' = py + f$  can be written in the form

$$y = y_h + y_p,$$

where  $y_p$  is a particular solution and  $y_h$  is the general solution of the homogeneous differential equation  $y'_h = py_h$ .

### Construction of $y_h$

$$\frac{d}{dx} (c e^{P(x)}) = P'(x) (c e^{P(x)}) \implies y_h(x) = c e^{P(x)} \text{ solves } y'_h = py_h \text{ with } p = P'$$

substituting  $p(x) = 2/(x+1) \rightsquigarrow P(x) = \int p(x) dx = 2 \ln|x+1| + c'$  and hence

$$y_h(x) = c'' e^{P(x)} = c'' e^{2 \ln|x+1| + c'} = c(x+1)^2, \quad c = c'' e^{c'}$$

### Construction of $y_p$

application of the method of variation of constants, i.e. replacing  $c$  by  $C(x)$  in the expression for  $y_h$  to obtain a particular solution  $y_p$

substituting the ansatz  $y_p(x) = C(x)(x+1)^2$  into the differential equation  $y' = \frac{2}{x+1}y + 3 \rightsquigarrow$

$$C'(x)(x+1)^2 + 2C(x)(x+1) = \frac{2C(x)(x+1)^2}{x+1} + 3$$

cancelling the common term  $2C(x)(x+1)$  on both sides  $\rightsquigarrow$

$$C'(x) = \frac{3}{(x+1)^2}$$

integrating  $\rightsquigarrow$

$$C(x) = -\frac{3}{x+1} + \tilde{c}$$

choosing  $\tilde{c} = 0 \rightsquigarrow$

$$y_p(x) = C(x)(x+1)^2 = -3(x+1)$$

Initial value

$$y(4) = 0 \quad \implies$$

$$y_h(4) + y_p(4) = c(4 + 1)^2 - 3(4 + 1) = 0,$$

i.e.  $c = 3/5$ , and

$$y(x) = \frac{3}{5}(x + 1)^2 - 3(x + 1) = \left(\frac{3}{5}x - \frac{12}{5}\right)(x + 1)$$

## 1.4 First Order Differential Equation with Exponentials

Determine the general solution  $y(x)$  of the differential equation

$$y' = 2y + 2e^x + e^{2x} .$$

**Resources:** [Method of Undetermined Coefficients](#)

### Problem Variants

■  $y' = -y + e^{3x}$

---

$c \exp(??) + \exp(??)/?:$

check

■  $y' = y/2 - e^{x/2}$

---

$(c??) \exp(??):$

check

■  $y' = -2y + \cos(4x)$

---

$c \exp(???) + \cos(??)/?? + \sin(??)/?:$

check

## Solution

### Structure of the solution

The general solution of the differential equation  $y' = py + \sum_k f_k$  with  $p \in \mathbb{R}$  has the form

$$y(x) = c e^{px} + \sum_k y_k(x), \quad ,$$

where  $y_k$  is a particular solution of  $y'_k = py_k + f_k$  (principle of superposition).

### Construction of the particular solutions $y_k$

differential equation

$$y' = 2y + 2e^x + e^{2x}$$

- ansatz for  $f_1(x) = 2e^x$ :  $y_1(x) = a e^x$   
substituting into  $y'_1 = 2y_1 + f_1 \rightsquigarrow$

$$a e^x = 2a e^x + 2e^x, \quad \text{i.e. } a = -2$$

- ansatz for  $f_2(x) = e^{2x}$ :  $y_2(x) = bx e^{2x}$   
The additional factor  $x$  is necessary since  $f_2$  solves the homogeneous differential equation  $y' = 2y$  (resonance).  
substituting into  $y'_2 = 2y_2 + f_2 \rightsquigarrow$

$$b e^{2x} + 2bx e^{2x} = 2bx e^{2x} + e^{2x}, \quad \text{i.e. } b = 1$$

combining the solutions  $\rightsquigarrow$

$$y(x) = c e^{2x} + y_1(x) + y_2(x) = c e^{2x} - 2e^x + x e^{2x}$$

## 1.5 Separable Differential Equation

Determine the general solution  $y(x)$  of

$$y' = \frac{\sin(x+y)}{\cos y} - \sin x.$$

**Resources:** [Separable Differential Equation](#)

### Problem Variants

■  $y' = \sqrt{xy}, x > 0$

---

$$y(x) = (\sqrt{x^2} + c)^2:$$

check

■  $y' = x^2y^2 + y^2$

---

$$y(x) = 3/(c - x^2):$$

check

■  $y' = xy + x/y$

---

$$y(x) = \pm \sqrt{c \exp(x^2)}:$$

check



## Solution

### Separation of variables

rewriting the differential equation in the form

$$g(y)y' = f(x) \quad (1)$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y \quad \rightsquigarrow$$

$$\begin{aligned} y' &= \frac{\sin(x + y)}{\cos y} - \sin x \\ &= \frac{\sin x \cos y + \cos x \sin y}{\cos y} - \frac{\sin x \cos y}{\cos y} = \frac{\cos x \sin y}{\cos y} \end{aligned}$$

bringing  $x$  and  $y$  to different sides  $\rightsquigarrow$

$$\underbrace{\frac{\cos y}{\sin y}}_{\cot y} y' = \cos x, \quad (2)$$

i.e.  $g(y) = \cot y$ ,  $f(x) = \cos x$  in equation (1)

### Integration

chain rule  $\implies$

$$\frac{d}{dx}G(y(x)) = g(y(x))y'(x), \quad \int g(y(x))y'(x) dx = G(y(x))$$

with  $g = G'$

remembering the derivation of the indefinite integral  $G(y) = \ln |\sin y|$  of  $g(y) = \cot y$ , shown below, or invoking the Maple<sup>TM</sup> command `int(cot(x), x)`, integrating equation (2)  $\rightsquigarrow$

$$\ln |\sin y(x)| = \sin x + c$$

solving for  $y$ <sup>1</sup>  $\rightsquigarrow$

$$y(x) = \arcsin(C \exp(\sin x)), \quad C = \pm e^c$$

### Indefinite integral of $\cot y$

substituting  $z = \sin y$   $\rightsquigarrow$

$$\int \frac{\cos y}{\sin y} dy = \int \frac{\cos y}{z} \underbrace{\frac{dy}{dz}}_{1/\cos y} dz = \int \frac{1}{z} dz = \ln |z| + c = \ln |\sin y| + c$$

---

<sup>1</sup>It is not always possible to obtain an explicit formula for  $y(x)$ , as in this case.

## 1.6 Logistic Population Model with Harvesting

The differential equation

$$u' = u(4 - u) - h$$

models the number  $u(t) \cdot 1000$  of fishes in a lake after  $t$  years for a fishing rate  $h$ . Sketch the direction field for  $h = 3$  and determine the solution  $u(t)$  for  $u(0) = 4$ . Which initial values lead to extinction of the fish population?

**Resources:** [Separable Differential Equation](#)

### Problem Variants

■  $h = 0, u(0) = 2$

---

$u(1) = ?$ :

check

■  $h = 4, u(0) = 3$

---

$u(1) = ?$ :

check

■  $h = 5, u(0) = 2$

---

$u(\pi/4) = ?$ :

check

## Solution

Solution of the initial value problem

separable differential equation:

$$u' = u(4 - u) - 3 \iff -\frac{u'}{(u-3)(u-1)} = 1$$

partial fraction decomposition  $\rightsquigarrow$

$$\frac{1}{2} \left( \frac{1}{u-1} - \frac{1}{u-3} \right) u' = 1$$

integrating, using  $\int \frac{1}{u(t)-p} u'(t) dt = \ln |u(t)-p|$  and  $\ln a - \ln b = \ln(a/b)$   $\rightsquigarrow$

$$\frac{1}{2} \ln \left| \frac{u-1}{u-3} \right| = t + c$$

initial value  $u(0) = 4 \implies c = \frac{1}{2} \ln 3$   
solving for  $u(t)$  with  $u(t) > 3$   $\rightsquigarrow$

$$u(t) - 1 = (u(t) - 3) e^{2t + \ln 3}, \quad u(t) = \frac{9e^{2t} - 1}{3e^{2t} - 1}$$

## Direction field

The graphs of solutions  $u(t)$  are tangential to the vector field

$$\begin{pmatrix} t \\ u \end{pmatrix} \mapsto \vec{F} = \begin{pmatrix} 1 \\ u(t)(4 - u(t)) - 3 \end{pmatrix}$$

MATLAB<sup>®</sup> -script for plotting  $\vec{F}$  as well the graphs of constant solutions and the solution with initial value  $u(0) = 4$

```
% direction field
[t,u] = meshgrid(0:0.2:6,0.2:0.2:4);
du = u.*(4-u)-3;
quiver(t,u,ones(size(du)),du)

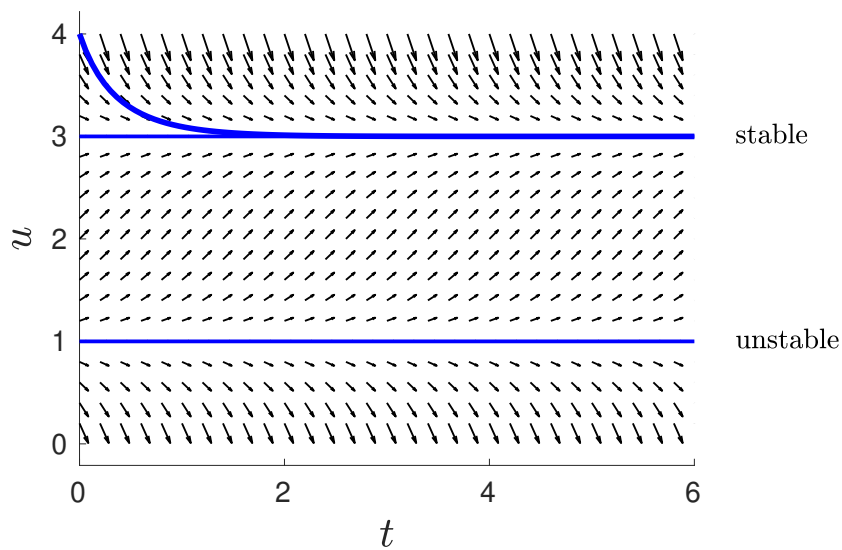
% constant solutions u=1 and u=3
plot([0 0; 6 6],[1 3; 1 3],'-b')
```

```

% solution with initial value u(0)=4
t = linspace(0,6);
u = (9*exp(2*t)-1)./(3*exp(2*t)-1);
plot(t,u,'-b')

% labels and text
...

```



### Asymptotic behavior

- $u(0) > 3$ : solutions decrease to 3
- $u(t) = 3$ : stable constant solution (nearby solutions converge to 3)
- $1 < u(0) < 3$ : solutions increase to 3
- $u(t) = 1$ : unstable constant solution (nearby solutions diverge from 1)
- $0 < u(0) < 1$ : solutions decrease to 0 (extinction of the fish population)

## 1.7 Separable Initial Value Problem

Determine the solution  $y(x)$  of the initial value problem

$$y' = \exp(x + y), \quad y(0) = 0.$$

**Resources:** [Separable Differential Equation](#)

### Problem Variants

■  $xyy' = 1, y(1) = 1$

---

$y(0) = ?$ :

check

■  $y' = \sin(2x)/\cos y, y(0) = 0$

---

$y(1) = ?$ :

check

■  $y' = \frac{x+1}{1-y}, y(2) = 2$

---

$y(0) = ?$ :

check

## Solution

### Separation of variables

multiplying the differential equation  $y' = \exp(x + y)$  by  $\exp(-y)$   $\rightsquigarrow$

$$\exp(-y)y' = \exp(x), \quad (1)$$

i.e. the standard form of a separable differential equation with the variables  $x$  and  $y$  on different sides

### Integration

integrating (1), using that, by the chain rule,  $\frac{d}{dx}(-\exp(-y(x))) = \exp(-y(x))y'(x)$   
 $\rightsquigarrow$

$$-\exp(-y) = \exp(x) + c \quad (2)$$

### Initial condition

$$y(0) = 0 \quad \implies$$

$$-\exp(-0) = \exp(0) + c \quad \iff \quad -1 = 1 + c,$$

i.e.  $c = -2$

solving (2) for  $y(x)$   $\rightsquigarrow$

$$y(x) = -\ln(2 - \exp(x))$$

## 1.8 General Solution of a Bernoulli Differential Equation

Determine the general solution  $y(x)$  of the differential equation

$$y' = y + xy^2.$$

**Resources:** [Bernoulli Differential Equation](#)

### Problem Variants

■  $y' = y + x/y$

---

$$y(x) = \pm \sqrt{c \exp(x^2) - x^2/2}:$$

check

■  $y' = y/x + y^3$

---

$$y(x) = \pm 1/\sqrt{c - 2x^2/3}:$$

check

■  $y' = y + \sqrt{y}$

---

$$\sqrt{y(x)} = c \exp(x^2) - x^2/2:$$

check

## Solution

### Substitution

substituting  $z = y^{-1} \iff y = 1/z$  in the differential equation  $y' = y + xy^2$   
 $\rightsquigarrow$  linear differential equation

$$\underbrace{-\frac{z'}{z^2}}_{y'} = \frac{1}{z} + x \frac{1}{z^2} \iff z' = -z - x \quad (1)$$

### Solution of the linear differential equation

substituting the ansatz  $z(x) = a + bx$  into (1)  $\rightsquigarrow$

$$b = -a - bx - x, \quad \text{i.e. } b = -1, a = 1,$$

and hence  $z_p(x) = 1 - x$  is a particular solution

adding the general solution  $z_h(x) = ce^{-x}$  of the homogeneous differential equation  $z' = -z$   $\rightsquigarrow$

$$z(x) = z_h(x) + z_p(x) = ce^{-x} + 1 - x$$

### Backsubstitution

undoing the transformation  $\rightsquigarrow$

$$y(x) = \frac{1}{z(x)} = \frac{1}{ce^{-x} + 1 - x}$$



## 1.9 Linear Substitution

Determine the solution  $y(x)$  of the initial value problem

$$y' = \frac{1}{x+y} - 1, \quad y(1) = 1.$$

**Resources:** [Linear Substitution](#)

### Problem Variants

■  $y' = (1 - 2x + 3y)^2 + 2/3, y(1) = 1/3$

---

$y(0) = -?.??:$

check

■  $y' = \exp(2x + y) + 2, y(1) = 2$

---

$y(0) = ?.??:$

check

■  $y' = \frac{x}{x+2y} - \frac{1}{2}, y(0) = 1$

---

$y(1) = ?.??:$

check

## Solution

### Substitution

replacing the linear expression  $x + y(x)$  by  $z(x)$   $\rightsquigarrow$  separable differential equation:

$$y'(x) = \frac{1}{x + y(x)} - 1 \iff z' - 1 = \frac{1}{z} - 1 \iff zz' = 1 \quad (1)$$

### Solution of the separable differential equation

integrating (1), using that  $\frac{d}{dx}g(z(x)) = g'(z(x))z'(x)$  with  $g'(z) = z$ , and undoing the substitution  $\rightsquigarrow$

$$\underbrace{z(x)^2/2}_{g(z(x))} = x + c \iff (x + y(x))^2 = 2x + 2c \quad (2)$$

### Initial value

$y(1) = 1 \implies (1 + 1)^2 = 2 + 2c$ , i.e.  $c = 1$ , and, solving (2) for  $y(x)$ ,

$$y(x) = \sqrt{2x + 2} - x$$

## 1.10 Initial Value Problem for a Differential Equation with Homogeneous Right-hand Side

Solve the initial value problem

$$y' = x/y + y/x, \quad y(1) = 2.$$

**Resources:** [Differential Equation with Homogeneous Right-hand Side](#), [Separable Differential Equation](#)

### Problem Variants

■  $y' = \frac{xy + y^2}{x^2}, y(2) = 1$

---

$y(1) = -?.??:$

check

■  $y' = y/x - (x/y)^2, y(1) = 1$

---

$y(1/2) = ?.??:$

check

■  $y' = \frac{y}{x + y}, y(1) = 1$

---

$x = ? \ln? + y:$

check

## Solution

### Substitution

substituting  $y(x) = xz(x)$  into the differential equation  $y' = x/y + y/x \rightsquigarrow$

$$z + xz' = 1/z + z \iff z z' = \frac{1}{x} \quad (1)$$

### Solution of the separable differential equation

integrating (1), using that, by the chain rule,  $\frac{d}{dx}z(x)^2 = 2z(x)z'(x) \rightsquigarrow$

$$z^2/2 = \ln|x| + c \iff z(x) = \pm\sqrt{2\ln|x| + 2c}$$

undoing the transformation  $\rightsquigarrow$

$$y(x) = xz(x) = \pm x\sqrt{2\ln|x| + 2c}$$

### Initial value

$y(1) = 2 \implies 2 = 1 \cdot \sqrt{2\ln 1 + 2c}$ , i.e.  $c = 2$  and

$$y(x) = x\sqrt{2\ln x + 4}$$

## 1.11 Exact Initial Value Problem

Solve the initial value problem

$$(x + 2y) dx + 2x dy = 0, \quad y(2) = 2.$$

**Resources:** [Exact Differential Equation](#)

### Problem Variants

■  $(x - y)dx + (2y - x)dy = 0, y(-1) = 1$

---

$y(0) = ?$ :

check

■  $(3 + y)dx + (x - 2)dy = 0, y(1) = 3$

---

$y(0) = ?$ :

check

■  $(ye^{xy} - 1)dx + xe^{xy}dy = 0, y(0) = 0$

---

$y(2) = ?$ :

check

## Solution

### Verification of exactness

check the necessary condition for exactness,  $p_y = q_x$  (which is also sufficient, if  $p$  and  $q$  are globally defined, as is the case for this problem), for the differential equation

$$\underbrace{(x + 2y)}_p dx + \underbrace{2x}_q dy = 0 \quad (1)$$

$$p_y = \partial_y(x + 2y) = 2, \quad q_x = \partial_x(2x) = 2 \quad \checkmark$$

### Construction of a Potential

construct  $F$  with  $\text{grad } F = (p, q)^t$ :

- $\partial_x F = p = x + 2y \quad \implies \quad F = x^2/2 + 2xy + c(y)$
- $\partial_y F = 2x + c'(y) = 2x \quad \implies \quad c(y) = C$

$\rightsquigarrow$  implicit representation for the solution of (1)

$$F(x, y) = x^2/2 + 2xy + C = 0 \quad (2)$$

### Initial condition

$y(2) = 2 \quad \implies \quad F(2, 2) = 2 + 8 + C = 0$ , i.e.  $C = -10$ , and, solving (2) for  $y$ ,

$$y(x) = \frac{10 - x^2/2}{2x} = \frac{5}{x} - \frac{x}{4}$$

## 1.12 Integrating Factor

Determine the general solution  $y(x)$  of the differential equation

$$y dx + (1 + 2x + 3y) dy = 0,$$

with the aid of an integrating factor  $a(y)$ .

**Resources:** [Integrating Factor](#)

### Problem Variants

■  $(1 + xy) dx + x^2 dy, a(x)$

---

$c = ??(x) + ?y:$

check

■  $(y + xy + y^3/3) dx + (x + y^2) dy = 0, a(x)$

---

$c = e^?(?x + y^?)?:$

check

■  $(2xy - y^3) dx + (3x^2 - 5xy^2) dy = 0, a(y)$

---

$c = x^?y^? - xy^?:$

check

## Solution

### Integrating Factor

find  $a(y)$  such that

$$\underbrace{ay}_{p} dx + \underbrace{a(1 + 2x + 3y)}_q dy = 0$$

is exact

necessary condition for exactness,  $\partial_y p = \partial_x q$  (also sufficient since  $p$  and  $q$  are globally defined)  $\implies$

$$a'(y)y + a(y) = 2a(y) \iff a'(y) = a(y)/y,$$

choosing  $a(y) = y \rightsquigarrow$

$$y^2 dx + (y + 2xy + 3y^2) dy = 0 \tag{1}$$

### Solution of the exact differential equation

construct  $F$  with  $\text{grad } F = (p, q)^t$ , to obtain an implicit representation  $F(x, y) = 0$  for the solution  $y(x)$  of (1)

$$\partial_x F = p = y^2 \implies F = xy^2 + c(y)$$

$$\partial_y F = 2xy + c'(y) = q = y + 2xy + 3y^2 \implies c(y) = y^2/2 + y^3 + C, \text{ i.e.}$$

$$F(x, y) = xy^2 + y^2/2 + y^3 + C$$



## 1.13 Discretization Error of Difference Methods

The differential equation  $y' = f(x, y)$  can be approximated with the trapezoidal rule:

$$y(x+h) = y(x) + \frac{h}{2}(f(x, y(x)) + f(x+h, y(x+h))) + h\Delta(x, h).$$

Show that the discretization error satisfies  $\Delta(x, h) = ch^2 + O(h^3)$ .

**Resources:** [Discretization Error](#)

### Problem Variants

■ Euler's method:  $y(x+h) = y(x) + hf(x, y(x)) + h\Delta(x, h)$

---

$\Delta(1, h) = (-?.?)h + O(h^2)$  for  $y(x) = \ln x$ :

check

■ midpoint rule:  $y(x+h) = y(x-h) + 2hf(x, y(x)) + h\Delta(x, h)$

---

$\Delta(1, h) = (-?)h^2 + O(h^3)$  for  $y(x) = 1/x$ :

check

■ Runge-Kutta method:  $y(x+h) = y(x) + hf(x+h/2, y(x)+(h/2)f(x, y(x))) + h\Delta(x, h)$

---

$\Delta(0, h) = (?.??)h^2 + O(h^3)$  for  $y(x) = e^x$  ( $f(x, y) = e^x$ ):

check

## Solution

Taylor expansion of  $y$  and  $y'$

$$\begin{aligned}y(x+h) &= y_0 + y_1 h + \frac{1}{2} y_2 h^2 + \frac{1}{6} y_3 h^3 + \dots, \quad y_k = y^{(k)}(x) \\y'(x+h) &= f(x+h, y(x+h)) = y_1 + y_2 h + \frac{1}{2} y_3 h^2 + \dots\end{aligned}$$

Discretization error

substituting the Taylor expansions of the solution of the differential equation  $y'(x) = f(x, y(x))$  into the trapezoidal approximation

$$y(x+h) = y(x) + \frac{h}{2}(f(x, y(x)) + f(x+h, y(x+h))) + h\Delta(x, h)$$

$\rightsquigarrow$  expansion of the truncation error

$$\begin{aligned}\Delta(x, h) &= \frac{y(x+h) - y(x)}{h} - \frac{y'(x) + y'(x+h)}{2} \\&= y_1 + \frac{1}{2} y_2 h + \frac{1}{6} y_3 h^2 - \frac{y_1 + y_1 + y_2 h + \frac{1}{2} y_3 h^2}{2} + O(h^3) \\&= \underbrace{-\frac{1}{12} y_3}_{c} h^2 + O(h^3)\end{aligned}$$

## 1.14 First Order Differential Equation with Maple™

Determine the solution  $y(x)$  of the initial value problem

$$(x - 3) y' = y^2 - 1, \quad y(4) = 0.$$

**Resources:** [Differential Equations with Maple™](#)

### Problem Variants

■  $y' = y + x/y, y(1) = 1$

---

$y(2) = ?$ :

check

■  $(x - 2y) dx = (2x + y) dy, y(0) = 1$

---

$y(1) = ?$ :

check

■  $y' = y + x \sin x \exp x, y(0) = 0$

---

$y(1) = ?$ :

check

## Solution

General solution of the differential equation

```
# load the differential equation package
with(DEtools)
# specify the differential equation
de := (x-3)*diff(y(x),x) = y(x)^2-1
# determine the type of the differential equation (optional)
odeadvisor(de)
```

[\_separable]

```
# obtain the general solution of the differential equation
gen_sol := dsolve(de)
```

$$y(x) = -\tanh(\ln(x-3) + c)$$

This looks complicated. The Maple™ command `simplify(gen_sol)` does not change this expression. Using  $\tanh t = \frac{\sinh t}{\cosh t} = \frac{e^t - e^{-t}}{e^t + e^{-t}}$ , a **human** might prefer the following manipulation:

$$\begin{aligned} -\tanh(\ln(x-3) + c) &= -\frac{(x-3)e^c - (x-3)^{-1}e^{-c}}{(x-3)e^c + (x-3)^{-1}e^{-c}} \\ &= \frac{1 - C^2(x-3)^2}{1 + C^2(x-3)^2} \end{aligned} \quad (1)$$

where, for the last equality, the fraction was expanded with  $(x-3)e^c$  and  $e^c$  was abbreviated with  $C$ .

Solution of the initial value problem

```
# specify the initial condition
ic := y(4)=0
# solve the initial value problem
sol := dsolve({de,ic})
```

$$y(x) = \frac{-x^2 + 6x - 8}{x^2 - 6x + 10}$$

cf. the expression (1) with  $C = 1$

## 1.15 First Order Boundary Value Problem with MATLAB<sup>®</sup>

Determine the positive  $\pi$ -periodic solution  $y(x)$  of the differential equation

$$y' = y(1 - y) + \sin^2 x.$$

**Resources:** [Differential Equations with MATLAB<sup>®</sup>](#)

### Problem Variants

■  $y' = \ln(x + y), y(4) = y(3) + 2$

---

$y(3) = ?$ :

check

■  $y' = -y^2 - e^{-x}, y(0) = 1, y(z) = 0$

---

$z = ?$ :

check

■  $y' = e^{-y} + ax, y(0) = 0, y(1) = 1$

---

$a = ?$ :

check

## Solution

Function for computing  $y(\pi)$  for an initial value  $y(0)$

```
function y_pi = ivp(y0)
% solves y' = y*(1-y)+sin(x)^2, y(0)=y0 and returns y(pi)
f = @(x,y) y*(1-y)+sin(x)^2;
[X,Y] = ode45(f,[0 pi],y0);
y_pi = Y(end);
```

Solution of the equation  $y(0) = y(\pi)$

```
err = @(y0) ivp(y0)-y0; % periodicity condition
y_pi = fzero(err,1) % computes a zero of err near 1
1.2363
```

## 1.16 First Order Initial Value Problem with MATLAB<sup>®</sup>

Plot the solution  $y(x)$  of the initial value problem

$$y' = \cos y + \sin x, \quad y(0) = 0,$$

on the interval  $[0, 10]$  and determine  $y(10)$ .

**Resources:** [Differential Equations with Matlab](#)

### Problem Variants

■  $y' = \cos(xy)$

---

$y(10) = ?$ ??:

check

■  $y' = (1 + y)e^{-x}$  (in view of the linearity of the differential equation, use the MATLAB<sup>®</sup> function `integral`)

---

$y(10) = ?$ ??:

check

■  $y' = \ln(2 + x - y)$

---

$y(10) = ?$ ??:

check

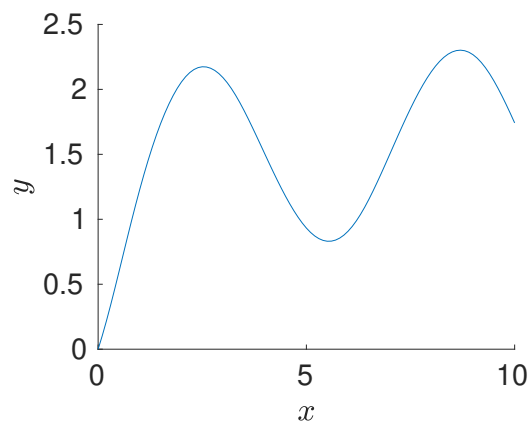
## Solution

MATLAB<sup>®</sup> script for computing  $y(x)$  for  $x \in [0, 10]$

```
dy = @(x,y) cos(y)+sin(x) % function (x,y) -> y'(x)
y0 = 0 % initial value y(x0), x0 = X(1)
% solution of the initial value problem on [0,10]
[X,Y] = ode45(dy,[0,10],y0)
% X: evaluation points
% Y(k): y(X(k))
```

Plot of  $y(x)$ ,  $0 \leq x \leq 10$

```
plot(X,Y)
Y(end) % y(10)
ans = 1.7419
```





## Chapter 2

# Second Order Differential Equations

## 2.1 Initial Value Problem for a Linear Oscillator

Determine the solution  $u(t)$  of the initial value problem

$$u'' + 4u = \sin(2t), \quad u(0) = 1, \quad u'(0) = 0.$$

**Resources:** [Linear Oscillator](#), [Method of Undetermined Coefficients](#)

### Problem Variants

■  $u'' + 9u = \cos(2t), \quad u(0) = 1, \quad u'(0) = 3$

---

$u(\pi) = -?.??:$

check

■  $u'' + 2u = 1, \quad u(0) = 0, \quad u'(0) = 0$

---

$u(1) = ?.???:$

check

■  $u'' + u = \sin(t), \quad u(0) = 0, \quad u'(0) = 1$

---

$u(\pi) = ?.???:$

check

## Solution

### Structure of the solution

linearity of the differential equation

$$u'' - 4u = \sin(2t)$$

$\implies u = u_h + u_p$  with  $u_h$  the solution of the homogeneous differential equation  $u'' + 4u = 0$ , i.e.

$$u_h(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

and  $u_p$  a particular solution of the inhomogeneous differential equation  $u'' + 4u = \sin(2t)$

### Particular solution

resonance (The right-hand side  $\sin(2t)$  solves the homogeneous differential equation.)  $\rightsquigarrow$  ansatz

$$u_p(t) = t\tilde{u}(t), \quad \tilde{u}(t) = \tilde{c}_1 \cos(2t) + \tilde{c}_2 \sin(2t)$$

substituting this expression into the differential equation and computing the second derivative of  $t\tilde{u}''(t)$  with Leibniz' rule ( $(fg)'' = f''g + 2f'g' + fg''$ )  
 $\rightsquigarrow$

$$\sin(2t) \stackrel{!}{=} u_p'' + 4u_p = 2\tilde{u}' + \underbrace{t\tilde{u}'' + 4t\tilde{u}}_{=0} = -4\tilde{c}_1 \sin(2t) + 4\tilde{c}_2 \cos(2t),$$

i.e.  $\tilde{c}_1 = -1/4$ ,  $\tilde{c}_2 = 0$  and  $u_p(t) = -\frac{1}{4}t \cos(2t)$

adding  $u_p$  to the general solution of the homogeneous differential equation  
 $\rightsquigarrow$

$$u(t) = u_h(t) + u_p(t) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4}t \cos(2t)$$

### Initial conditions

$$\begin{aligned} 1 &\stackrel{!}{=} u(0) = c_1 \cos(2t) + c_2 \sin(2t) - \frac{1}{4}t \cos(2t) \Big|_{t=0} = c_1 \\ 0 &\stackrel{!}{=} u'(0) = -2 \sin(2t) + 2c_2 \cos(2t) - \frac{1}{4} \cos(2t) + \frac{1}{2}t \sin(2t) \Big|_{t=0} \\ &= 2c_2 - 1/4 \end{aligned}$$

$\rightsquigarrow c_1 = 1, c_2 = 1/8$  and

$$u(t) = \cos(2t) + \frac{1}{8} \sin(2t) - \frac{1}{4}t \cos(2t)$$

## 2.2 General Solution of a Homogeneous Second Order Differential Equation with Constant Coefficients

Determine the general solution  $u(t)$  of the differential equation

$$u'' + u' - 2u = 0.$$

**Resources:** [Homogeneous Second Order Differential Equation with Constant Coefficients](#)

### Problem Variants

■  $2u'' + 3u' - 2u = 0$

---

$$u(t) = c_1 \exp(t/?) + c_2 \exp(-?t):$$

check

■  $u'' + 2u' + u = 0$

---

$$u(t) = (c_1 + c_2?) \exp(?t):$$

check

■  $u'' + 4u' + 8u = 0$

---

$$u(t) = (c_1 \cos(?t) + c_2 \sin(?t)) \exp(??t):$$

check

### Solution

substituting the ansatz  $u(t) = e^{\lambda t}$  into the differential equation  $u'' + u' - 2u = 0$   $\rightsquigarrow$

$$\lambda^2 e^{\lambda t} + \lambda e^{\lambda t} - 2e^{\lambda t} = 0 \iff p(\lambda) = \lambda^2 + \lambda - 2 = 0$$

two real zeros  $\lambda_1 = 1$ ,  $\lambda_2 = -2$  of the characteristic polynomial  $p$   $\rightsquigarrow$   
general solution

$$u(t) = \sum_k c_k e^{\lambda_k t} = c_1 e^t + c_2 e^{-2t}$$

## 2.3 Second Order Linear Inhomogeneous Initial Value Problem

Determine the solution  $u(t)$  of the initial value problem

$$u'' - 2u' + 5u = 4e^t, \quad u(0) = u'(0) = 0.$$

**Resources:** [Homogeneous Second Order Differential Equation with Constant Coefficients](#), [Method of Undetermined Coefficients](#)

### Problem Variants

■  $u'' - u' - 2u = 4t, u(0) = -1, u'(0) = 3$

---

$u(1) = ?$ ???:

check

■  $u'' + 2u' + 2u = 5 \sin(2t), u(0) = -1, u'(0) = 1$

---

$u(1) = ?$ ???:

check

■  $u'' - u = 2e^t, u(0) = 2, u'(0) = 2$

---

$u(1) = ?$ ???:

check

## Solution

### Structure of the solution

linearity of the differential equation

$$u'' - 2u' + 5u = 4e^t$$

$\implies u = u_h + u_p$  with  $u_p$  a particular solution and  $u_h$  a solution of the homogeneous differential equation

$$u'' - 2u' + 5u = 0 \quad (1)$$

with integration constants determined by the initial conditions

### Construction of $u_h$

characteristic polynomial of (1):

$$\lambda^2 - 2\lambda + 5$$

complex conjugate zeros  $\lambda = r \pm si = 1 \pm \sqrt{1^2 - 5} = 1 \pm 2i \rightsquigarrow$

$$u_h(t) = e^{rt}(c_1 \cos(st) + c_2 \sin(st)) = e^t(c_1 \cos(2t) + c_2 \sin(2t))$$

### Construction of $u_p$

substituting the ansatz  $u_p(t) = ce^t$  into the differential equation  $u'' - 2u' + 5u = 4e^t \rightsquigarrow$

$$ce^t - 2ce^t + 5ce^t = 4e^t \iff c = 1$$

i.e.  $u_p(t) = e^t$

### Initial values

$u(0) = 0, u'(0) = 0$  for

$$u(t) = u_h(t) + u_p(t) = e^t(c_1 \cos(2t) + c_2 \sin(2t)) + e^t$$

$\rightsquigarrow$  linear system for  $c_1$  and  $c_2$ :

$$\begin{aligned} 0 &\stackrel{!}{=} u_h(0) + u_p(0) \\ &= e^t(c_1 \cos(2t) + c_2 \sin(2t)) + e^t \Big|_{t=0} \\ &= c_1 + 1 \implies c_1 = -1 \end{aligned}$$

$$\begin{aligned} 0 &\stackrel{!}{=} u'_h(0) + u'_p(0) \\ &= e^t(-\cos(2t) + c_2 \sin(2t)) + e^t(2 \sin(2t) + 2c_2 \cos(2t)) + e^t \Big|_{t=0} \\ &= -1 + 2c_2 + 1 \implies c_2 = 0, \end{aligned}$$

i.e.  $u(t) = e^t(1 - \cos(2t))$



## 2.4 Linear Second Order Boundary Value Problem

Solve the boundary value problem

$$u'' + 2u' + 2u = 0, \quad u(0) = 1, \quad u'(\pi) = 0.$$

**Resources:** [Homogeneous Second Order Differential Equation with Constant Coefficients](#)

### Problem Variants

■  $u'' = 4u, u(0) = 1, u(1) = 0$

---

$u(1/2) = ?$ :

check

■  $u'' = -u, u(\pi) = u(0) + 1, u'(\pi) = u'(0) - 1$

---

$u(0) = -?$ :

check

■  $u'' + 3u' + 2u = 0, u'(0) = 0, u'(1) = 1$

---

$u(0) = -?$ :

check

## Solution

### General solution

characteristic polynomial of the differential equation  $u'' + 2u' + 2u = 0$ :

$$\lambda^2 + 2\lambda + 2$$

zeros  $\lambda_{1,2} = -1 \pm i \rightsquigarrow$

$$u(t) = e^{-t}(c_1 \cos t + c_2 \sin t) \quad (1)$$

### Boundary conditions

$u(0) = 1, u'(\pi) = 0 \rightsquigarrow$  linear equations for  $c_k$ :

- $1 = u(0) = e^{-t}(c_1 \cos t + c_2 \sin t)|_{t=0} \implies c_1 = 1$
- $0 = u'(\pi) = -e^{-t}(\cos t + c_2 \sin t) + e^{-t}(-\sin t + c_2 \cos t)|_{t=\pi}$   
 $= -e^{-\pi}(-1) + e^{-\pi}(-c_2)$   
 $\implies c_2 = 1$

substituting the values  $c_k$  into (1)  $\rightsquigarrow$

$$u(t) = e^{-t}(\cos t + \sin t)$$

## 2.5 Initial Value Problem for Euler's Differential Equation

Solve the initial value problem

$$t^2 u'' - 2u = 0, \quad u(1) = 3, \quad u'(1) = 3.$$

**Resources:** [Euler's Differential Equation](#)

### Problem Variants

■  $t^2 u'' + tu' - u = 1, \quad u(1) = -1, \quad u'(1) = 2$

---

$u(2) = ?$ :

check

■  $t^2 u'' + tu' = -4u, \quad u(1) = 2, \quad u'(1) = 0$

---

$u(2) = ?$ :

check

■  $4t^2 u'' = -u, \quad u(1) = 2, \quad u'(1) = 3$

---

$u(4) = ?$ :

check

## Solution

### General solution

substituting the ansatz  $u(t) = t^\lambda$  into the differential equation  $t^2 u'' - 2u = 0$

$\rightsquigarrow$

$$t^2 \underbrace{\lambda(\lambda - 1)t^{\lambda-2}}_{u''} - 2t^\lambda = 0 \iff \lambda^2 - \lambda - 2 = 0$$

possible exponents  $\lambda_1 = -1, \lambda_2 = 2 \rightsquigarrow$

$$u(t) = c_1 t^{\lambda_1} + c_2 t^{\lambda_2} = c_1 t^{-1} + c_2 t^2$$

### Initial values

$u(1) = 3, u'(1) = 3 \rightsquigarrow$  linear system:

$$3 \stackrel{!}{=} u(1) = c_1 t^{-1} + c_2 t^2 \Big|_{t=1} = c_1 + c_2,$$

$$3 \stackrel{!}{=} u'(t) = -c_1 t^{-2} + 2c_2 t \Big|_{t=1} = -c_1 + 2c_2$$

solution:  $c_1 = 1, c_2 = 2$ , i.e.

$$u(t) = t^{-1} + 2t^2$$

## 2.6 Autonomous Initial Value Problem

Solve the initial value problem

$$3u''u' = 2u, \quad u(1) = 1, \quad u'(1) = 1.$$

**Resources:** [Phase Plane](#)

### Problem Variants

■  $2u''u' = e^u, \quad u(0) = 0, \quad u'(0) = 1$

---

$u(4) = ?$ :

check

■  $2u'' = u'u, \quad u(1) = 2, \quad u'(1) = 2$

---

$u(4) = -?$ :

check

■  $uu'' = (u')^2, \quad u(2) = 1, \quad u'(2) = 1$

---

$u(3) = ?$ ??:

check

## Solution

### Reduction to two first order differential equations

substituting  $u'(t) = v(u)$  into the differential equation  $3u''u' = 2u \rightsquigarrow$

$$\begin{aligned}3u''(t)u'(t) &= 3 \left( \frac{d}{dt}v(u) \right) v(u) = 3 \left( \frac{d}{du}v(u) \frac{du}{dt} \right) v(u) \\ &= 3(v'(u)v(u))v(u) = 2u\end{aligned}$$

### Solving the first order differential equations

- integrating  $3v'v^2 = 2u$  with respect to  $u \rightsquigarrow$

$$v^3 = u^2 + c$$

with  $c = 0$  in view of the initial conditions  $u(1) = 1, v(u(1)) = u'(1) = 1$

- solving  $v^3 = u^2$  for  $v = u'$  and integrating the resulting differential equation

$$u'(t) = v(u(t)) = u(t)^{2/3} \iff u^{-2/3}u' = 1$$

with respect to  $t \rightsquigarrow$

$$3u^{1/3} = t + C$$

with  $C = 2$  in view of the initial condition  $u(1) = 1$ , i.e.

$$u(t) = (t + 2)^3/27$$

## 2.7 Damped Forced Oscillations

Determine the periodic solution  $u_*(t)$  of the differential equation

$$u'' + u' + u = \cos(\omega t),$$

as well as the resonance frequency  $\omega_*$ .

**Resources:** [Forced Damped Oscillations](#)

### Problem Variants

■  $u'' + 2u' + 3u = 4 \cos(\omega t)$ , resonance frequency

---

$\omega_* = ?$ :

check

■  $u'' + 3u' + u = 5 \sin(2t)$ , periodic solution

---

$u_*(0) = -?.??$ :

check

■  $u'' + u' + 2u = e^{it}$ , periodic solution

---

$u_*(t) = ?.??e^{i(t-?.??)}$ :

check

## Solution

### Periodic solution

substituting the ansatz  $u(t) = a \cos(\omega t) + b \sin(\omega t)$  for the periodic solution into the differential equation  $u'' + u' + u = \cos(\omega t)$   $\rightsquigarrow$

$$(-a\omega^2 C - b\omega^2 S) + (-a\omega S + b\omega C) + (aC + bS) = C$$

with  $C = \cos(\omega t)$  and  $S = \sin(\omega t)$

comparing the coefficients of  $C$  and  $S$   $\rightsquigarrow$  linear system for  $a$  and  $b$ :

$$\begin{aligned} (1 - \omega^2) a + \omega b &= 1 \\ -\omega a + (1 - \omega^2) b &= 0 \end{aligned}$$

Cramer's rule  $\rightsquigarrow$

$$\begin{aligned} a &= \frac{1}{d} \begin{vmatrix} 1 & \omega \\ 0 & 1 - \omega^2 \end{vmatrix} = \frac{1 - \omega^2}{d}, & b &= \frac{1}{d} \begin{vmatrix} 1 - \omega^2 & 1 \\ -\omega & 0 \end{vmatrix} = \frac{\omega}{d} \\ d &= \begin{vmatrix} 1 - \omega^2 & \omega \\ -\omega & 1 - \omega^2 \end{vmatrix} = (1 - \omega^2)^2 + \omega^2 \end{aligned}$$

and

$$u(t) = \frac{1}{(1 - \omega^2)^2 + \omega^2} ((1 - \omega^2) \cos(\omega t) + \omega \sin(\omega t))$$

### Resonance frequency

maximizing the amplitude

$$c = \sqrt{a^2 + b^2} = \sqrt{(1 - \omega^2)^2 + \omega^2}/d = 1/\sqrt{d},$$

or, equivalently, minimizing  $d(\Omega) = (1 - \Omega)^2 + \Omega$ ,  $\Omega = \omega^2$   $\rightsquigarrow$

$$0 \stackrel{!}{=} d'(\Omega) = -2(1 - \Omega) + 1 \iff \Omega = 1/2,$$

i.e. the resonance frequency  $\omega_* = 1/\sqrt{2}$



## 2.8 Motion, Governed by a Potential

Determine the maximal velocity  $v_{\max} = \max_t |u'(t)|$  for the solution  $u(t)$  of the initial value problem

$$u'' = -2u^3, \quad u(0) = 2, \quad u'(0) = 0.$$

**Resources:** [Conservation of Energy](#)

### Problem Variants

■  $u'' = -u \exp(u^2), \quad u(1) = 2, \quad u'(1) = 2$

---

$v_{\max} = ?$ :

check

■  $u'' = \sin u, \quad u(0) = 0, \quad u'(0) = 0$

---

$v_{\max} = ?$ :

check

■  $u'' = -u^3 + u, \quad u(0) = 0, \quad u'(0) = 2$

---

$v_{\max} = ?$ :

check

## Solution

Curves of constant energy in the phase plane

multiplying the differential equation  $u'' = -2u^3$  with  $u'$  and integrating  $\rightsquigarrow$

$$u''u' + 2u^3u' = 0 \iff \frac{d}{dt}E(t) = 0, \quad E = \frac{1}{2}v^2 + \frac{1}{2}u^4, \quad v = u',$$

i.e. constant energy  $E$ , yielding an implicit representation of the solution in the  $(u, v)$ -plane

Maximal velocity

$$u(0) = 2, \quad v(0) = u'(0) = 0 \implies$$

$$E = \frac{1}{2}v^2 + \frac{1}{2}u^4 = 0 + \frac{1}{2}2^4 = 8,$$

and  $|v| = \sqrt{2E - u^4}$  is maximal for  $u = 0$ , i.e.

$$v_{\max} = \sqrt{2E} = \sqrt{2 \cdot 8} = 4$$

## 2.9 Phase Plane Trajectories for an Autonomous Second Order Differential Equation

Solve the pendulum equation

$$u'' = -\pi \sin(\pi u)$$

and plot typical trajectories in the  $(u, v)$ -plane ( $v = u'$ ) near the equilibrium points.

**Resources:** [Phase Plane](#), [Conservation of Energy](#)

### Problem Variants

■  $u'' = 3u^2 - 4u^3$

---

constant energy  $E = v^2/2 + u^3 - u^4$ :

check

■  $u'' = -\frac{1}{1+u^2}$

---

constant energy  $E = v^2/2 + \arctan u$ :

check

■  $u'' = \frac{2u}{1+u^2}$

---

constant energy  $E = v^2/2 - \ln(1+u^2)$ :

check

## Solution

### Equilibrium points

points  $(u, v) = (u, u')$  with  $u'(t) = u''(t) = 0$

$$u''(t) = -\pi \sin(\pi u(t)) \implies u(t) = \pi k, k \in \mathbb{Z},$$

i.e.,  $(u, v) = (\pi k, 0)$

### Solution of the differential equation

multiplying the differential equation  $u'' + \pi \sin(\pi u) = 0$  with  $v = u'$   $\rightsquigarrow$

$$\underbrace{v'(t)}_{u''} v(t) + \pi \sin(\pi u(t)) u'(t) = 0$$

applying the chain rule and integrating  $\rightsquigarrow$

$$v(t)^2/2 - \cos(\pi u(t)) = E \quad \forall t,$$

an implicit representation of the solutions in the phase plane (curves of constant energy  $E$ )

### Alternative solution

dividing the differential equations  $u'(t) = v(t)$ ,  $v'(t) = -\pi \sin(\pi u)$   $\rightsquigarrow$

$$\frac{dv}{du} = \frac{dv}{dt} / \frac{du}{dt} = -\frac{\pi \sin(\pi u)}{v},$$

a separable differential equation for  $v(u)$  which is easily integrated

### Phase plane trajectories with MATLAB<sup>®</sup>

```
hold on % overlay of plots
```

```
% evaluation grid
```

```
umin = -2.5; umax = 2.5; vmin = -2.5; vmax = 2.5;
```

```
h = 0.01; % grid spacing
```

```
[u,v] = meshgrid(umin:h:umax,vmin:h:vmax);
```

```
% level curves of constant energy
```

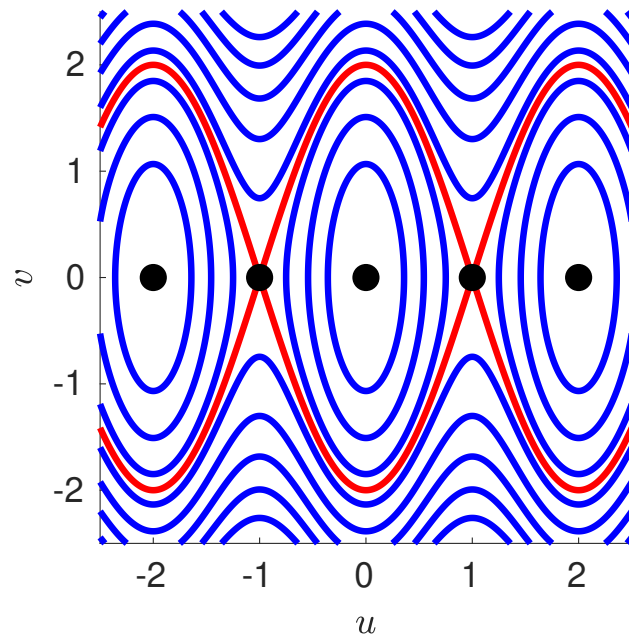
```
E = v.^2/2-cos(pi*u);
```

```
contour(u,v,E,8,'color','b')
```

```
% trajectory, separating periodic and unbounded solutions
```

```
contour(u,v,E,[1 1], 'color', 'r') % energy E=1
% equilibrium points
plot([-2 -1 0 1 2],[0 0 0 0 0], '.k');

hold off
```



## 2.10 Forced Oscillations with Maple™

Plot the solution  $u(t)$  of the initial value problem

$$u'' + 4u' + 3u = \cos t, \quad u(0) = 1, \quad u'(0) = 0,$$

on the interval  $[0, 4\pi]$ .

**Resources:** [Differential Equations with Maple™](#)

### Problem Variants

■  $u'' + u' + u = \sin(2t), \quad u(0) = 0, \quad u'(0) = 0$

---

$u(2\pi) = -?.??:$

check

■  $u'' + 4u = \cos(2t), \quad u(0) = 1, \quad u'(0) = 0$

---

$u(2\pi) = ?:$

check

■  $u'' + 100u = \sin(9t), \quad u(0) = 0, \quad u'(0) = 1$

---

$u(2\pi) = ?:$

check

Notice the different qualitative behavior of the solutions for the three variants.

## Solution

### General solution of the differential equation

```
# load the differential equation package
with(DEtools)
# specify the differential equation
de := D^(2)(u)(t)+4*D(u)(t)+3*u(t) = cos(t)
# determine the general solution
> gen_sol := dsolve(de)
```

$$gen\_sol := u(t) = e^{-3t} \_C2 + e^{-t} \_C1 + \frac{1}{10} \cos t + \frac{1}{5} \sin t$$

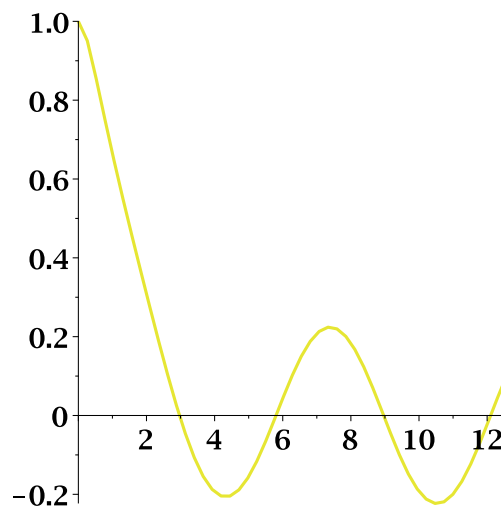
### Solution of the initial value problem

```
# specify the initial conditions
ic := {u(0)=1,D(u)(0)=0}
# solve the initial value problem
sol := IVPsol(ic,gen_sol)
```

$$sol := u(t) = -\frac{7e^{-3t}}{20} + \frac{5e^{-t}}{4} + \frac{\cos(t)}{10} + \frac{\sin(t)}{5}$$

```
# plot the solution
DEplot(de,u(t),t=0..4*Pi,{ic})
```

For DEplot, the Maple™ syntax seems to require an additional pair of curly brackets unlike for the command IVPsol.



strong damping → fast decay of the nonperiodic component

## 2.11 Sturm-Liouville Problem with Maple™

Solve the boundary value problem

$$(pu')' = qu, \quad u(0) = 0, u(1) = 1,$$

for  $p(t) = e^t$ ,  $q(t) = e^{-t}$  and for  $p(t) = e^{u(t)}$ ,  $q(t) = e^{-u(t)}$ .

**Resources:** [Differential Equations with Maple™](#)

### Problem Variants

■  $p(t) = 1 + t$ ,  $q(t) = 1$ ,  $u'(0) = 0$ ,  $u(1) = 1$

---

$u(1/2) = ?$ :

check

■  $p(t) = 1$ ,  $q(t) = t + u(t)$ ,  $u(0) = 0$ ,  $u(1) = 1$

---

$u'(0) = ?$ :

check

■  $p(t) = e^{-t}$ ,  $q(t) = 2e^{-t}$ ,  $u'(0) = 1$ ,  $u'(1) = 0$

---

$u(1/2) = -?$ :

check



## Solution

Analytic solution:  $p(t) = e^t, q(t) = e^{-t}$

```
# differential equation
de := diff(exp(t)*diff(u(t),t),t) = exp(-t)*u(t)
# boundary conditions
bc := u(0)=0, u(1)=1
# solution
sol := simplify(dsolve({de,bc}))
```

$$sol := u(t) = \frac{\cosh(1) \sinh(e^{-t}) - \sinh(1) \cosh(e^{-t})}{\cosh(1) \sinh(e^{-1}) - \sinh(1) \cosh(e^{-1})}$$

## An impressive Maple™ result!

Numerical solution:  $p = e^{u(t)}, q(t) = e^{-u(t)}$

```
de := diff(exp(u(t))*diff(u(t),t),t) = exp(-u(t))*u(t)
bc := u(0)=0, u(1)=1
sol := dsolve({de,bc})
```

$sol := ()$

Maple™ cannot obtain an analytic solution (neither can a human?)  $\rightsquigarrow$   
numerical method

```
sol := dsolve({de,bc},numeric)
# dsolve returns a procedure, which evaluates the solution
```

$sol := \mathbf{proc}(x\_bvp) \dots \mathbf{end\ proc}$

```
# value and derivative at t=1/2
sol(1/2)
```

$$\left[ t = 0.5, u(t) = 0.5990, \frac{d}{dt}u(t) = 0.9381 \right]$$

## 2.12 Second Order Differential Equation with MATLAB<sup>®</sup>

Determine and plot the solution  $u(t)$  of the initial value problem

$$u'' = -tu^3, \quad u(0) = 0, \quad u'(0) = 1,$$

on the interval  $[0, 10]$ .

**Resources:** [Differential Equations with Matlab](#)

### Problem Variants

■  $u'' = -u + \cos(t^2), \quad u(0) = 0, \quad u'(0) = 0$

---

$u(10) = -?.??:$

check

■  $u'' = -u'/2 - 2 \sin u, \quad u(0) = 1, \quad u'(0) = 0$

---

$u(10) = ?.???:$

check

■  $u'' = u' \cos t - u, \quad u(0) = 1, \quad u'(1) = 1$

---

$u(10) = -?.??:$

check

## Solution

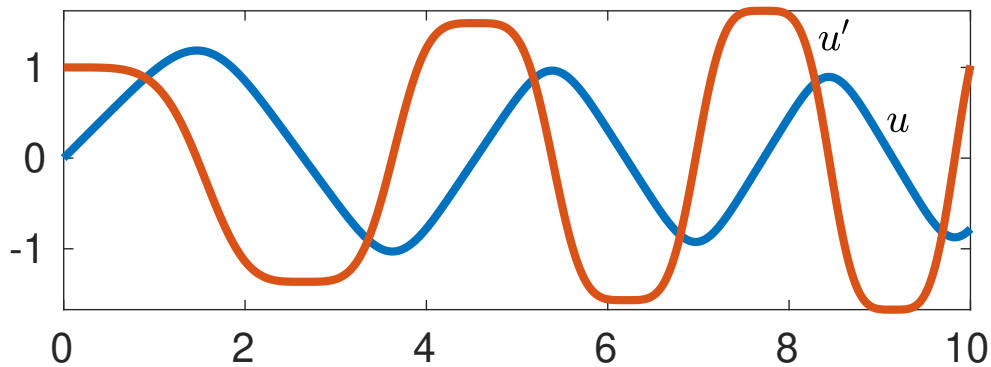
### Transformation to standard form

introducing variables for the function  $u$  and its first derivative,  $v_1 := u$ ,  $v_2 := u'$   $\rightsquigarrow$  equivalent first order system for the differential equation  $u'' = -tu^3$ :

$$v' = \begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} v_2 \\ -tv_1^3 \end{pmatrix} =: f(t, v)$$

### MATLAB<sup>®</sup> script

```
% right-hand side of the differential equation
f = @(t,v) [v(2); -t*v(1)^3];
% solution of the initial value problem with moderate accuracy
% on the interval [0,10] with initial condition v(0)=[0;1]
[t,v] = ode45(f,[0,10],[0;1]);
% v(k,1:2): solution at time t(k)
% simultaneous plot the two components of u and u' of v
plot(t,v)
```



## 2.13 Newton's Method for a Boundary Value Problem

Solve the boundary value problem

$$u'' = \exp(tu), \quad u(0) = 0, \quad u(1) = 1,$$

by determining the slope  $s = u'(0)$  (shooting method).

**Resources:** [Differential Equations with Matlab](#), [Differentiation with Respect to Initial Conditions](#)

### Problem Variants

■  $u'' = \cos(t + u), \quad u(0) = 0, \quad u(1) = 1$

---

$u'(0) = ?$ :

check

■  $u'' = tuu', \quad u(0) = 1, \quad u(2) = 3$

---

$u'(0) = ?$ :

check

■  $u'' = t + u^2, \quad u(0) = 0, \quad u'(1) = 1$

---

$u'(0) = ?$ :

check

## Solution

### Newton's method

equation to be solved:

$$f(s) = u(1, s) - 1 = 0$$

with  $u(t, s)$  (The slope  $s$  is added as an additional parameter.) the solution of the initial value problem

$$u_{tt}(t, s) = \exp(tu(t, s)), \quad u(0, s) = 0, \quad u_t(0, s) = s \quad (1)$$

computation of the derivative  $f'(s) = u_s(1, s)$  by differentiating (1) with respect to  $s$ :

$$u_{tts}(t, s) = \exp(tu(t, s))tu_s(t, s), \quad u_s(0, s) = 0, \quad u_{ts}(0, s) = 1 \quad (2)$$

simultaneous solution of the initial value problems (1,2) as a first order system for  $v = (u, u_t, u_s, u_{st})^t$ :

$$v' = F(t, v) = \begin{pmatrix} v_2 \\ \exp(tv_1) \\ v_4 \\ \exp(tv_1)tv_3 \end{pmatrix}, \quad v(0) = \begin{pmatrix} 0 \\ s \\ 0 \\ 1 \end{pmatrix}$$

### MATLAB<sup>®</sup> implementation of the Newton iteration

```
% linear interpolation of the boundary values 0,1
% -> initial slope
s = 1
% implementation assuming convergence
% (no "safety checks")
ds = inf; tol = 1.0e-10 % tolerance
while abs(ds) > tol
    % Newton step: s <- s-f/df
    [f,df] = newton(s); ds = f/df; s = s-ds
end

1.0000000000000000
0.436197527177781
0.424933421261371
0.424929983699617
```

0.424929983699298

```
function [f,df] = newton(s)
F = @(t,v) [v(2); exp(t*v(1)); v(4); exp(t*v(1))*t*v(3)];
% solution of the system
[t,vt] = ode45(F,[0;1],[0;s;0;1]); % vt(:,k) = v_k(t(:))
f = vt(end,1)-1; df = vt(end,3); % u(1,s)-1; u_s(1,s)
↪ slope  $s \approx 0.4249299836$ 
```

## Chapter 3

# Systems of Differential Equations

### 3.1 Taylor Approximation for a System of Differential Equations

Approximate  $x(0.1)$  and  $y(0.1)$  for the initial value problem

$$x' = y^3, x(0) = 1, \quad y' = x^4, y(0) = 2,$$

using quadratic Taylor polynomials.

**Resources:** [System of Differential Equations](#)

#### Problem Variants

■  $x' = \cos(x + y), y' = \sin(xy), x(0) = 0, y(0) = \pi$

---

$x(0.1) \approx -??.?, y(0, 1) \approx ??.??:$

check

■  $u' = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} u, u(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

---

$u(0.1)^t \approx (?.??, ?.??):$

check

■  $x' = x x^t x, x(0) = (1, 2)^t$

---

$x(0.1)^t \approx (?.??, ?.??):$

check



## Solution

### Second derivatives

differentiating the differential equations  $x' = y^3$ ,  $y' = x^4$   $\rightsquigarrow$

$$x''(t) = \frac{d}{dt}y(t)^3 = 3y(t)^2y'(t) = 3y(t)^2x(t)^4$$

$$y''(t) = \frac{d}{dt}x(t)^4 = 4x(t)^3x'(t) = 4x(t)^3y(t)^3$$

initial conditions  $x(0) = 1$ ,  $y(0) = 2$   $\rightsquigarrow$

$$x'(0) = y(0)^3 = 8, \quad x''(0) = 3y(0)^2x(0)^4 = 12, \quad y'(0) = 1, \quad y''(0) = 32$$

### Quadratic Taylor approximation

evaluating the Taylor polynomial at 0.1  $\rightsquigarrow$

$$\begin{aligned}x(0.1) &\approx x(0) + x'(0) \cdot 0.1 + \frac{1}{2}x''(0) \cdot 0.1^2 \\ &= 1 + 8 \cdot 0.1 + 6 \cdot 0.01 = 1.86 \\ y(0.1) &\approx 2 + 1 \cdot 0.1 + 16 \cdot 0.01 = 2.26\end{aligned}$$

## 3.2 Initial Value Problem for a Linear Homogeneous System

Solve the initial value problem

$$\begin{aligned}u_1' &= 3u_1 - u_2, & u_1(0) &= 0, \\u_2' &= 4u_1 - 2u_2, & u_2(0) &= 3.\end{aligned}$$

**Resources:** [Diagonalization of a Linear System of Differential Equations](#)

### Problem Variants

■ 
$$\begin{aligned}u_1' &= 2u_1 + u_2, & u_1(0) &= 1, \\u_2' &= u_1 + 2u_2, & u_2(0) &= 2\end{aligned}$$

---

$u(\ln 2)^t = (??, ??):$

check

■ 
$$\begin{aligned}u_1' &= u_2, & u_1(0) &= 1, \\u_2' &= 2u_1 + u_2, & u_2(0) &= 5\end{aligned}$$

---

$u(\ln 2)^t = (?.?, ???.?):$

check

■ 
$$\begin{aligned}u_1' &= u_2, & u_1(0) &= 2, \\u_2' &= -u_1, & u_2(0) &= 2\end{aligned}$$

---

$u(\pi)^t = (-?, -?):$

check

## Solution

Matrix form of the linear system of differential equations

$$u'_1 = 3u_1 - u_2, u'_2 = 4u_1 - 2u_2 \quad \rightsquigarrow$$

$$\begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}}_A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

## Eigenvalues

characteristic polynomial

$$p(\lambda) = \det \underbrace{\begin{pmatrix} 3 - \lambda & -1 \\ 4 & -2 - \lambda \end{pmatrix}}_{A - \lambda E} = (3 - \lambda)(-2 - \lambda) - 4(-1) = \lambda^2 - \lambda - 2$$

zeros of  $p \rightsquigarrow$  eigenvalues  $\lambda_1 = -1, \lambda_2 = 2$  of  $A$

## Eigenvectors

determine the eigenvectors  $v$  by solving  $(A - \lambda E)v = (0, 0)^t$ :

$$A - \lambda_1 E = \begin{pmatrix} 3 - (-1) & -1 \\ 4 & -2 - (-1) \end{pmatrix} = \begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \implies v_1 \parallel \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$
$$A - \lambda_2 E = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix} \implies v_2 \parallel \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

## General solution

sum of eigensolutions  $\rightsquigarrow$

$$u(t) = \sum_{k=1}^2 c_k v_k e^{\lambda_k t} = c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t}$$

## Initial values

$u(0) = (0, 3)^t \rightsquigarrow$  linear system for the constants  $c_k$ :

$$c_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

$\implies c_1 = 1, c_2 = -1$  and

$$u(t) = \begin{pmatrix} e^{-t} - e^{2t} \\ 4e^{-t} - e^{2t} \end{pmatrix}$$

### 3.3 Initial Value Problem for a Three-Dimensional System

Determine the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

and solve the initial value problem  $u' = Au$ ,  $u(0) = (0, 1, 0)^t$ .

**Resources:** [Diagonalization of a Linear System of Differential Equations](#)

#### Problem Variants

■  $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ ,  $u(0) = (1, 1, 0)^t$

---

$u_2(1) = ?$ ??:

check

■  $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ ,  $u(0) = (0, 1, 1)^t$

---

$u_2(1) = ?$ ??:

check

■  $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ ,  $u(0) = (1, 0, 1)^t$

---

$u_2(1) = ?$ ??:

check

## Solution

### Eigenvalues

characteristic polynomial

$$\begin{aligned}\det(A - \lambda E) &= \begin{vmatrix} 1 - \lambda & 1 & 0 \\ 1 & 0 - \lambda & 1 \\ 0 & 1 & 1 - \lambda \end{vmatrix} \\ &= (1 - \lambda)(-\lambda)(1 - \lambda) - (1 - \lambda) - (1 - \lambda) = (1 - \lambda)(-\lambda + \lambda^2 - 2)\end{aligned}$$

zeros  $\rightsquigarrow$  eigenvalues  $\lambda = 2, \varrho = 1, \sigma = -1$

### Eigenvectors

- Since all rows of  $A$  sum to 2,  $\xi = (1, 1, 1)^t$  is an eigenvector to  $\lambda = 2$ .
- eigenvector to  $\varrho = 1$ : nontrivial solution  $\eta$  of

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = (A - \varrho E)\eta = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}$$

$\rightsquigarrow \eta = (1, 0, -1)^t$  as a possible choice

- eigenvector to  $\sigma = -1$ : orthogonality of the eigenvectors for symmetric matrices  $\implies \zeta = (1, -2, 1)^t$  up to normalization

### General solution of $u' = Au$

sum of eigensolutions:

$$u(t) = a\xi e^{\lambda t} + b\eta e^{\varrho t} + c\zeta e^{\sigma t} = a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} + b \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} e^t + c \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t}$$

### Initial value

$$u(0) = (0, 1, 0)^t \iff$$

$$a \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

formula for the coefficients of a vector with respect to an orthogonal basis  
(scalar product with the normalized basis vectors)  $\implies$

$$a = \frac{u(0)^t \xi}{\xi^t \xi} = (0 \ 1 \ 0) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} / (1 \ 1 \ 1) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{3}$$

and, analogously,  $b = 0/2 = 0$ ,  $c = -2/6 = -1/3$

$\rightsquigarrow$  solution

$$u(t) = \frac{1}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{2t} - \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} e^{-t} = \frac{1}{3} \begin{pmatrix} e^{2t} - e^{-t} \\ e^{2t} + 2e^{-t} \\ e^{2t} - e^{-t} \end{pmatrix}$$

## 3.4 Linear Homogeneous System of Differential Equations

Determine the general solution of

$$u_1' = u_1 + u_2, \quad u_2' = -u_1 + u_2.$$

**Resources:** [Diagonalization of a Linear System of Differential Equations](#)

### Problem Variants

■  $u_1' = 2u_2, u_2' = 2u_1 + 3u_2$

---

$$u(t) = (c_1 e^{4t} + c_2 e^{2t}, c_1 e^{2t} + c_2 e^{2t}):$$

check

■  $u_1' = u_2, u_2' = -4u_1$

---

$$u(t) = (c_1 \cos(2t) + c_2 \sin(2t), -c_1 \sin(2t) + c_2 \cos(2t)):$$

check

■  $u_1' = 3u_1 + 3u_2, u_2' = 2u_1 + 4u_2$

---

$$u(t) = (c_1 e^{2t} + 3c_2 e^t, c_1 e^{2t} - c_2 e^t):$$

check

## Solution

Matrix form of the system of differential equations

$$u'_1 = u_1 + u_2, \quad u'_2 = -u_1 + u_2 \quad \rightsquigarrow$$

$$\begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}}_A \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \iff u' = Au$$

Eigensolutions

characteristic polynomial

$$p(\lambda) = \det(A - \lambda E) = \det \begin{pmatrix} 1 - \lambda & 1 \\ -1 & 1 - \lambda \end{pmatrix} = (1 - \lambda)^2 + 1$$

zeros of  $p$   $\rightsquigarrow$  eigenvalues  $\lambda_{\pm} = 1 \pm i$  of  $A$

corresponding eigenvectors (solutions of  $(A - \lambda E)v = (0, 0)^t$ ):

$$v_+ = (1, i)^t, \quad v_- = (1, -i)^t$$

eigensolutions:

$$u_{\pm}(t) = v_{\pm} e^{(1 \pm i)t} = \begin{pmatrix} 1 \\ \pm i \end{pmatrix} e^t (\cos t \pm i \sin t)$$

Real general solution

real and imaginary parts of  $u_+$   $\rightsquigarrow$  real solutions

$$\tilde{u}_+(t) = \operatorname{Re} u_+(t) = \operatorname{Re} \begin{pmatrix} e^t (\cos t + i \sin t) \\ e^t (i \cos t - \sin t) \end{pmatrix} = e^t \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$$

$$\tilde{u}_-(t) = \operatorname{Im} u_+(t) = e^t \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$

alternatively:  $\tilde{u}_+ = (u_+ + u_-)/2$ ,  $\tilde{u}_- = (u_+ - u_-)/(2i)$

linear combination  $\rightsquigarrow$  general solution

$$u(t) = c_+ \tilde{u}_+(t) + c_- \tilde{u}_-(t) = e^t \begin{pmatrix} c_+ \cos t + c_- \sin t \\ -c_+ \sin t + c_- \cos t \end{pmatrix}$$



### 3.5 Elimination for a System of Linear Differential Equations

Determine the general solution of

$$u_1' + 3u_1 - 2u_2 = 1, \quad 2u_1 - u_2' = 0$$

using the elimination method.

**Resources:** [Elimination for a Linear System of Differential Equations](#)

#### Problem Variants

■  $2u_1' + u_2 = 2, u_2' - 2u_1 = 2$

---

$$u(t) = (c_2 \cos(t) + c_1 \sin(t) - \frac{1}{2}, c_1 \sin(t) - c_2 \cos(t) + \frac{1}{2}):$$

check

■  $u_1' + 2u_1 + u_2 = 0, u_2' - u_1 = 1$

---

$$u(t) = (e^{-t}(c_1 + c_2 t), e^{-t}(c_1 t + c_2)):$$

check

■  $u_1' - u_1 - 4u_2 = 3, u_2' - 2u_1 + u_2 = 0$

---

$$u(t) = (c_1 e^{-t} + c_2 e^{2t} - \frac{1}{7}, c_1 e^{2t} + \frac{1}{7} c_2 e^{2t} - \frac{2}{7}):$$

check

## Solution

### Elimination of $u_1$

denoting the differential equations

$$u_1' + 3u_1 - 2u_2 = 1, \quad 2u_1 - u_2' = 0$$

by  $E_1$  and  $E_2$ , forming the difference  $\frac{d}{dt}E_1 - 2E_2 \rightsquigarrow$

$$(u_1'' + 3u_1' - 2u_2') - (4u_1 - 2u_2') = u_1'' + 3u_1' - 4u_1 = \frac{d}{dt}1 - 0 = 0$$

### Solution of the second order equation

characteristic polynomial of  $u_1'' + 3u_1' - 4u_1 = 0$ :

$$\lambda^2 + 3\lambda - 4$$

zeros  $\lambda_1 = 1, \lambda_2 = -4 \rightsquigarrow$  general solution

$$u_1(t) = c_1 e^t + c_2 e^{-4t}$$

### Computation of $u_2$

solving  $E_1 : u_1' + 3u_1 - 2u_2 = 1$  for  $u_2$  and substituting the expression for  $u_1$

$\rightsquigarrow$

$$\begin{aligned} u_2(t) &= -\frac{1}{2} + \frac{3}{2}u_1(t) + \frac{1}{2}u_1'(t) \\ &= -\frac{1}{2} + \frac{3}{2}\underbrace{(c_1 e^t + c_2 e^{-4t})}_{u_1(t)} + \frac{1}{2}\underbrace{(c_1 e^t - 4c_2 e^{-4t})}_{u_1'(t)} \\ &= -\frac{1}{2} + 2c_1 e^t - \frac{1}{2}c_2 e^{-4t} \end{aligned}$$

## 3.6 Elimination for a System of Initial Value Problems

Determine the solution  $(x(t), y(t))^t$  of the initial value problem

$$\begin{aligned}x' &= x + 2y + e^{2t}, & x(0) &= 3 \\y' &= 2x + y - 2e^{2t}, & y(0) &= 0\end{aligned} \cdot$$

**Resources:** [Elimination for a Linear System of Differential Equations](#)

### Problem Variants

■  $x' = y + 1, x(0) = 0, \quad y' = -x + t, y(0) = 1$

---

$x(\pi) + y(\pi) = ?$ ???:

check

■  $x' = x + y + 1, x(0) = 2, \quad y' = x + y - 1, y(0) = 0$

---

$x(1) + y(1) = ?$ ???:

check

■  $x' = 3x + 2y + e^t, x(0) = 0, \quad y' = -2x - y, y(0) = 0$

---

$x(1) + y(1) = ?$ ???:

check

## Solution

### Elimination of $x(t)$

solving the differential equation  $y' = 2x + y - 2e^{2t}$  for  $x(t)$  and substituting the resulting expression

$$x = y'/2 - y/2 + e^{2t} \quad (1)$$

into the differential equation  $x' = x + 2y + e^{2t} \rightsquigarrow$

$$y''/2 - y'/2 + 2e^{2t} = y'/2 - y/2 + e^{2t} + 2y + e^{2t}$$

and, after simplification,

$$y'' - 2y' - 3y = 0, \quad y(0) = 0 \quad (2)$$

### Solution of the second order initial value problem

characteristic polynomial of (2)

$$\lambda^2 - 2\lambda - 3 = (\lambda + 1)(\lambda - 3)$$

zeros  $\lambda_1 = -1$ ,  $\lambda_2 = 3 \rightsquigarrow$  solution

$$y(t) = c_1 e^{-t} + c_2 e^{3t} = ce^{-t} - ce^{3t}$$

with  $c := c_1 = -c_2$  in view of the initial condition  $y(0) = 0$

### Computation of $x(t)$

(1)  $\implies$

$$\begin{aligned} x(t) &= \underbrace{(-ce^{-t} - 3ce^{3t})/2}_{y'(t)} - \underbrace{(ce^{-t} - ce^{3t})/2}_{y(t)} + e^{2t} \\ &= -ce^{-t} - ce^{3t} + e^{2t} \end{aligned}$$

initial condition  $x(0) = 3 \implies c = -1$ , i.e.

$$x(t) = e^{-t} + e^{3t} + e^{2t}, \quad y(t) = -e^{-t} + e^{3t}$$

## 3.7 Field Lines of a Vector Field

Determine the field lines of the vector field  $\vec{F}$  with the potential  $U = x^2y$  and illustrate the orthogonality to the level curves of  $U$ .

**Resources:** [Separable Differential Equation](#)

### Problem Variants

■  $U = xy$

---

implicit representation  $c = y^2/x^2$ :

check

■  $U = x^2 + 2y^2$

---

implicit representation  $y = cx^2$ :

check

■  $U = x + y^2$

---

implicit representation  $y = c^{1/2} - (1/2)x$ :

check

## Solution

System of differential equations for the field lines

vector field corresponding to the potential  $U = x^2y$ :

$$\vec{F} = \text{grad } U = \begin{pmatrix} 2xy \\ x^2 \end{pmatrix}$$

Since the tangents of the field lines  $L : t \mapsto (x(t), y(t))$  are parallel to  $\vec{F}$ ,

$$x' = c(2xy), \quad y' = cx^2. \quad (1)$$

Implicit representation of the field lines

dividing the differential equations (1)  $\rightsquigarrow$

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = x^2 / (2xy) = x/2y \quad \iff \quad 2y(x)y'(x) = x$$

integrating this separable differential equation  $\rightsquigarrow$  implicit representation of the field lines:

$$L(x, y) = y^2 - x^2/2 = c \quad (\text{hyperbolas})$$

Level curves of  $U$  and field lines

MATLAB<sup>®</sup> graphic

```
hold on % overlay of plots
```

```
% graphic window
```

```
xmin = -4; xmax = 4; ymin = -4; ymax = 4;
```

```
% grid for evaluating U and L
```

```
h = 0.1;
```

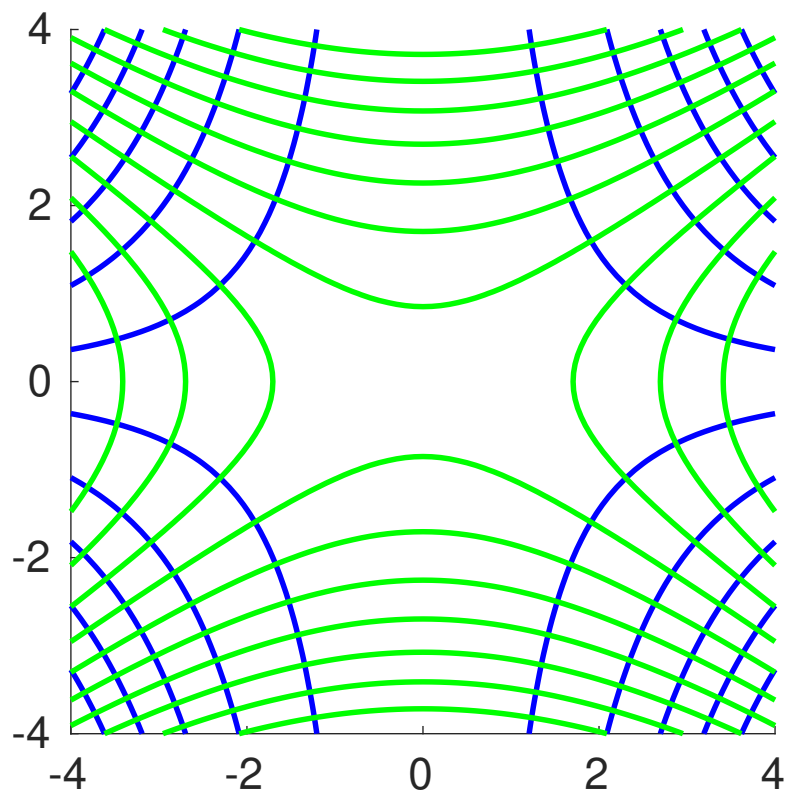
```
[x,y] = meshgrid(xmin:h:xmax,ymin:h:ymax);
```

```
% level curves  $x^2 y = C$ 
```

```
U = x.^2.*y;
```

```
contour(x,y,U,10,'Color','b') % 10 blue curves
```

```
% field lines  $y^2 - x^2/2 = C$   
L =  $y.^2 - x.^2/2$ ;  
contour(x,y,L,10,'Color','g') % 10 green lines  
  
axis equal % equal scaling of coordinates  
  
hold off
```



## 3.8 Circular Motion

Solve the initial value problem

$$u' = \underbrace{a \times u}_{\text{cross product}} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad u(0) = p,$$

for  $a = (2, 1, 2)^t$ ,  $p = (0, 3, 0)^t$ .

**Resources:** [Linear System of Differential Equations](#)

### Problem Variants

■  $a = (1, 1, 0)^t$ ,  $p = (0, 1, 1)^t$

---

$u_2(\pi) = ?$ :

check

■  $a = (3, 0, 4)^t$ ,  $p = (4, 2, -3)^t$

---

$u_2(\pi) = -?$ :

check

■  $a = (1, 2, -2)^t$ ,  $p = (-2, 2, 1)^t$

---

$u_2(\pi) = -?$ :

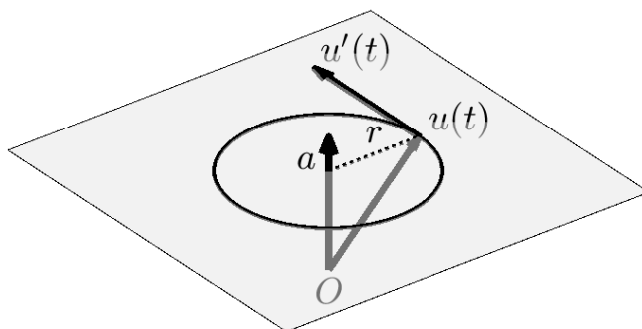
check



## Solution

### General considerations

$u' = a \times u, u(0) = p \implies$  The curve  $t \mapsto u(t)$  traverses a circle  $C : |u(t) - m| = r$  in a plane with normal vector  $a$  as is shown below.



$$(i) \quad \frac{d}{dt} (a \cdot u) = a \cdot u' = a \cdot (a \times u) = 0 \implies$$

$$u(t) \in E : a \cdot u(t) = a \cdot p$$

$$(ii) \quad \frac{d}{dt} (u \cdot u) = 2u \cdot u' = 2u \cdot (a \times u) = 0 \implies$$

$$|u(t)| = |p|, \quad \varphi = \angle(a, u(t)) = \arccos \frac{a \cdot u(t)}{|a||u(t)|} = \arccos \frac{a \cdot p}{|a||p|}$$

$$(i), (ii) \implies m = |p| \cos \varphi \frac{a}{|a|}, \quad r = |p| \sin \varphi$$

### General solution of the differential equation

choose a unit vector  $\hat{b} \perp \hat{a}$ ,  $\hat{a} = a/|a|$

express  $u$  in terms of the orthonormal basis  $\hat{a}, \hat{b}, \hat{c} = \hat{a} \times \hat{b}$ :

$$u(t) = \alpha(t)\hat{a} + \beta(t)\hat{b} + \gamma(t)\hat{c}$$

substituting into the differential equation  $u' = a \times u$  and noting that the cross product of parallel vectors vanishes  $\rightsquigarrow$

$$\alpha' \hat{a} + \beta' \hat{b} + \gamma' \hat{c} = \underbrace{|a| \hat{a}}_a \times (\beta \hat{b} + \gamma \hat{c}) = |a|(\beta \hat{c} - \gamma \hat{b})$$

comparing coefficients of the basis vectors  $\implies$

$$\alpha' = 0, \quad \beta' = -|a|\gamma, \quad \gamma' = |a|\beta$$

combining the last two equations,  $\beta'' = -|a|^2\beta$ , and hence

$$\alpha(t) = d_0, \quad \beta(t) = d_1 \cos(|a|t) + d_2 \sin(|a|t), \quad \gamma(t) = d_1 \sin(|a|t) - d_2 \cos(|a|t),$$

with constants  $d_k$  determined from the initial conditions

Concrete data

$a = (2, 1, 2)^t \rightsquigarrow$  possible choice of an orthonormal basis

$$\hat{a} = \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \quad \hat{b} = \frac{1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \quad \hat{c} = \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

Substituting into the general solution

$$u(t) = d_0 \hat{a} + (d_1 \cos(|a|t) + d_2 \sin(|a|t)) \hat{b} + (d_1 \sin(|a|t) - d_2 \cos(|a|t)) \hat{c}$$

with  $t = 0$ , the initial condition  $p = (0, 3, 0)^t = u(0)$  implies

$$p = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} = \frac{d_0}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \frac{d_1}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} - \frac{d_2}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}.$$

formula for the coefficients with respect to an orthonormal basis  $\implies$

$$d_0 = p \cdot \hat{a} = \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 2/3 \\ 1/3 \\ 2/3 \end{pmatrix} = 1, \quad d_1 = p \cdot \hat{b} = 2, \quad -d_2 = p \cdot \hat{c} = 2$$

substituting into the general solution  $\rightsquigarrow$

$$\begin{aligned} u(t) &= \frac{1}{3} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \frac{2 \cos(3t) - 2 \sin(3t)}{3} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + \frac{2 \sin(3t) + 2 \cos(3t)}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix} \\ &= \frac{1}{3} \begin{pmatrix} 2 - 2 \cos(3t) - 6 \sin(3t) \\ 1 + 8 \cos(3t) \\ 2 - 2 \cos(3t) + 6 \sin(3t) \end{pmatrix} \end{aligned}$$

### 3.9 Initial Value Problem in Jordan Form

Determine the solution  $u(t)$  of the initial value problem

$$u' = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} u, \quad u(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

**Resources:** [Jordan Form of a System of Linear Differential Equations, Method of Undetermined Coefficients](#)

#### Problem Variants

■  $u' = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} u, \quad u(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

---

$u_1(1) = ?$ :

check

■  $u' = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

---

$u_1(1) = ?$ :

check

■  $u' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} u + \begin{pmatrix} 3t \\ 0 \end{pmatrix}, \quad u(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

---

$u_1(1) = ?$ :

check

## Solution

The triangular form of the system of differential equations permits to successively determine  $u_3$ ,  $u_2$ , and  $u_1$ .

### Computation of $u_3$

$$u_3' = u_3, \quad u_3(0) = 1$$

$$\implies u_3(t) = e^t$$

### Computation of $u_2$

$$u_2' = u_2 + \underbrace{e^t}_{u_3(t)}, \quad u_2(0) = 1 \tag{1}$$

ansatz (sum of the general solution of the homogeneous differential equation  $u_2' = u_2$  and a particular solution):

$$u_2(t) = ae^t + bte^t$$

with  $a = 1$  in view of the initial condition and with the factor  $t$  in the second term since  $e^t$  is a solution of the homogeneous differential equation substituting this ansatz into (1)  $\rightsquigarrow$

$$(1 + b + bt)e^t = (e^t + bte^t) + e^t,$$

i.e.  $b = 1$

### Computation of $u_1$

$$u_1' = u_1 + \underbrace{e^t + te^t}_{u_2(t)}, \quad u_1(0) = 1$$

ansatz

$$u_1(t) = ae^t + bte^t + ct^2e^t$$

with  $a = 1$  in view of the initial condition  $\rightsquigarrow$

$$(1 + b + bt + 2ct + ct^2)e^t = (e^t + bte^t + ct^2e^t) + (e^t + te^t),$$

i.e.  $b = 1, c = 1/2$

summarizing:

$$u(t) = \begin{pmatrix} 1 + t + t^2/2 \\ 1 + t \\ 1 \end{pmatrix} e^t$$

### 3.10 Trajectories of a Nonlinear Autonomous System

Determine an implicit representation of the solutions of the system

$$x' = x - y^3, \quad y' = x^3 - y$$

in the  $xy$ -plane, draw a few trajectories, and mark the equilibrium points.

**Resources:** [Exact Differential Equation](#), [Separable Differential Equation](#)

#### Problem Variants

■  $x' = 1 - y, y' = 2 - x$

---

$$x^2 - y^2 - x + y = c:$$

check

■  $x' = \cos y, y' = \sin x$

---

$$\cos(x) + \sin(y) = c:$$

check

■  $x' = 1 - 2y, y' = 3x^2$

---

$$x^2 + y^2 - y = c:$$

check

## Solution

### Equilibrium points

solutions of

$$x' = y' = 0 \iff x - y^3 = 0 \wedge x^3 - y = 0$$

solving the second equation for  $y$  and substituting  $y = x^3$  into the first equation  $\rightsquigarrow x - x^9 = 0$  with the solutions  $x = 0, \pm 1$

3 equilibrium points:  $(-1, -1), (0, 0), (1, 1)$

### Elimination of $t$

dividing the differential equations  $x'(t) = x(t) - y(t)^3$  and  $y'(t) = x(t)^3 - y(t)$   
 $\rightsquigarrow$

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{x^3 - y}{x - y^3}$$

rearranging terms  $\rightsquigarrow$

$$(x^3 - y) dx + (y^3 - x) dy = 0 \tag{1}$$

exact differential equation since  $\partial_y(x^3 - y) = \partial_x(y^3 - x)$

Implicit representation  $F(x, y) = C$  of the trajectories

exactness of (1)  $\implies$

$$\partial_x F(x, y) = x^3 - y, \quad \partial_y F(x, y) = y^3 - x$$

integrating the first equation with respect to  $x \implies F(x, y) = x^4/4 - xy + c(y)$

substituting into the second equation  $\implies -x + c'(y) \stackrel{!}{=} y^3 - x$ , i.e.

$$c(y) = y^4/4 + \tilde{c}$$

choosing<sup>1</sup>  $\tilde{c} = 0$ ,

$$F(x, y) = x^4/4 - xy + y^4/4$$

### Plot of trajectories with Maple™

```
# equilibrium points
```

```
pts := [[-1, -1], [0, 0], [1, 1]]
```

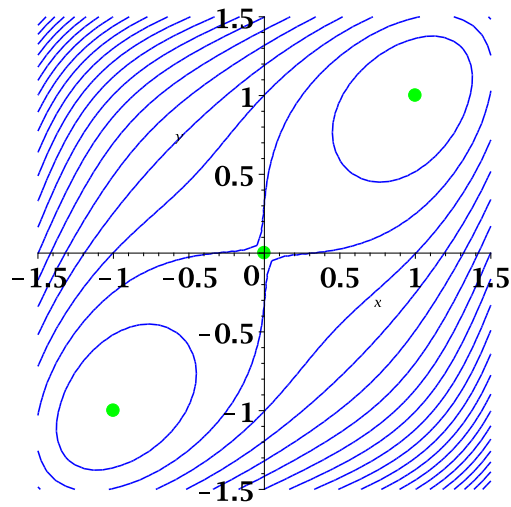
```
points := pointplot(pts, symbol=solidcircle,  
    symbolsize=20, color=green)
```

---

<sup>1</sup>The integration constant  $\tilde{c}$  is irrelevant for the implicit representation  $F(x, y) = C$ .

```
# trajectories
F := x^4/4-x*y+y^4/4
contourplot(F, x=-1.5..1.5, y=-1.5..1.5, contours=20,
  color=blue)

# simultaneous plot with font specification
display(points, curves, axesfont=[times,bold,20])
```





### 3.11 Type of a Critical Point of a System of Differential Equations and General Solution

Determine the type of the critical point  $(0, 0)$  for the system of differential equations

$$x' - x - y = 0, \quad y' + 4x + 3y = 0,$$

as well as the general solution.

**Resources:** [Stability of Linear Systems of Differential Equations](#), [Elimination for a Linear System of Differential Equations](#)

#### Problem Variants

■  $x' = 2x + 3y, y' = 3x + y$

---

stable (s) or unstable (u), knot (k) or saddle (s) ?, ?:

check

■  $x' = -2x + y, y' = -x - 2y$

---

stable (s) or unstable (u), knot (k) or spiral (s) ?, ?:

check

■  $x' = -5y, y' = 2x + 2y$

---

stable (s) or unstable (u), knot (k) or spiral (s) ?, ?:

check

## Solution

Matrix form of the system of differential equations

$$x' - x - y = 0, y' + 4x + 3y = 0 \quad \rightsquigarrow$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ -4 & -3 \end{pmatrix}}_A \begin{pmatrix} x \\ y \end{pmatrix}$$

Classification of the critical point (0,0)

- $\det A = 1 > 0$ ,  $\text{trace } A = -2 < 0 \implies$  stable
- $\det A \leq ((\text{trace } A)/2)^2 \implies$  knot (degenerate since the two terms are equal)

Construction of the general solution with the elimination technique

$$\left(\frac{d}{dt} - 1\right) \underbrace{(y' + 4x + 3y = 0)}_{\text{equation 2}} - 4 \underbrace{(x' - x - y = 0)}_{\text{equation 1}} \rightsquigarrow$$

$$(y'' - y' + 4x' - 4x + 3y' - 3y) - (4x' - 4x - 4y) = y'' + 2y' + y = 0$$

characteristic polynomial:  $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2$

double zero  $\lambda = -1 \rightsquigarrow$

$$y(t) = (c_1 + c_2 t)e^{-t}$$

substituting into equation 2:  $y' + 4x + 3y = 0 \rightsquigarrow$

$$\begin{aligned} x(t) &= -\frac{1}{4}(y' + 3y) = -\frac{1}{4}((-c_1 + c_2 - c_2 t)e^{-t} + 3(c_1 + c_2 t)e^{-t}) \\ &= (-(c_2/2)t - c_1/2 - c_2/4)e^{-t} \end{aligned}$$

## 3.12 Stable Critical Points of a Nonlinear System of Differential Equations

Determine the stable critical points of

$$u' = -u^3 + v, v' = -v^3 + w, w' = -w^3 + u.$$

**Resources:** [Stability of Autonomous Systems of Differential Equations](#)

### Problem Variants

■  $u' = u(v - 1), v' = v(u - 1)$

---

stable critical point  $(?, ?)$ :

check

■  $u' = u(3 - v), v' = v(1 + u - v)$

---

stable critical point  $(?, ?)$ :

check

■  $u' = v, v' = u - u^3 + v$

---

stable critical point  $(?, ?)$ :

check

## Solution

### Critical Points

nonlinear system of differential equations

$$\begin{pmatrix} u' \\ v' \\ w' \end{pmatrix} = F(u, v, w), \quad F(u, v, w) = \begin{pmatrix} -u^3 + v \\ -v^3 + w \\ -w^3 + u \end{pmatrix}$$

$$F(u, v, w) = (0, 0, 0)^t \text{ for a critical point } p = (u, v, w) \iff \\ v = u^3, w = v^3 = u^9, u = w^3 = u^{27},$$

i.e.  $u \in \{-1, 0, 1\}$ , corresponding to three critical points:  $p_0 = (0, 0, 0)$ ,  $p_+ = (1, 1, 1)$ ,  $p_- = (-1, -1, -1)$

### Stability criteria

characterization of stability for a point  $p = (u, v, w)^t$  in terms of the eigenvalues  $\lambda_k$  of the Jacobi matrix

$$F'(p) = \begin{pmatrix} -3u^2 & 1 & 0 \\ 0 & -3v^2 & 1 \\ 1 & 0 & -3w^2 \end{pmatrix}$$

sufficient criteria:

- stable, if  $\operatorname{Re} \lambda_k < 0 \forall k$
- unstable, if  $\exists \lambda_k : \operatorname{Re} \lambda_k > 0$

### Application to the three critical points

- $p_0 = (0, 0, 0)$

$$F'(0, 0, 0) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

has eigenvalue 1 corresponding to the eigenvector  $(1, 1, 1)^t$  and hence  $p_0$  is unstable

- $p_{\pm} = \pm(1, 1, 1)$

$$F'(u, v, w) = \begin{pmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 1 & 0 & -3 \end{pmatrix}$$

criterion of Gerschgorin for the eigenvalues  $\lambda \implies |\lambda - (-3)| \leq 1 \\ \implies \operatorname{Re} \lambda < 0 \implies \text{stability of } p_{\pm}$

### 3.13 Linear Systems of Differential Equations with MATLAB<sup>®</sup>

Solve the initial value problem  $u' = Au$ ,  $u(0) = b$  with

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -3 & 2 \\ -4 & -4 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix},$$

and plot the projections of the curve  $t \mapsto u(t)$ ,  $t \in [0, 10]$ , on the  $u_1u_2$ -,  $u_1u_3$ -, and  $u_2u_3$ -plane.

**Resources:** [Diagonalization of a Linear System of Differential Equations](#)

#### Problem Variants

■  $A = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$ ,  $b = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ , curve in the  $u_1u_3$ -plane

---

$10^4 u_2(10) = ??:$

check

■  $A = \begin{pmatrix} -1 & -3 \\ 3 & -1 \end{pmatrix}$ ,  $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ , curve in the  $u_1u_2$ -plane

---

$10^4 u_1(10) = ?..??:$

check

■  $A = \begin{pmatrix} -1 & 1 & 3 & 0 \\ 1 & -2 & 0 & 4 \\ -3 & 0 & -1 & 0 \\ 0 & -4 & 0 & 2 \end{pmatrix}$ ,  $b = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ , curve in the  $u_1u_4$ -plane

---

$10^4 u_1(10) = ?..??:$

check

## Solution

### Diagonalization of the initial value problem

particular solution of  $u' = Au$ :

$$u(t) = ve^{\lambda t}$$

with  $v$  an eigenvector for an eigenvalue  $\lambda$  of  $A$

For a basis of eigenvectors  $v_k$ , a general solution of the system of differential equations can be written as linear combination of eigensolutions:

$$u(t) = \sum_{k=1}^3 c_k v_k e^{\lambda_k t}.$$

initial condition  $b = u(0) = \sum_k c_k v_k \implies$

$$c = V^{-1}b, \quad V = (v_1, v_2, v_3) =: \begin{pmatrix} v_{1,1} & \cdots & v_{1,3} \\ \vdots & & \vdots \\ v_{3,1} & \cdots & v_{3,3} \end{pmatrix}$$

values of the solution components  $u_j$  at time  $t_\ell$

$$u_j(t_\ell) = \sum_k c_k \underbrace{v_{j,k}}_{w_{j,k}} \underbrace{e^{\lambda_k t_\ell}}_{e_{k,\ell}} =: (WE)_{j,\ell}$$

### MATLAB<sup>®</sup> script

```
A = [1 2 2; -2 -3 2; -4 -4 -1]; b = [1;-1;4];
% eigenvectors and diagonal matrix of eigenvalues
[V,Lambda] = eig(A)
V = -0.3536-0.3536i -0.3536+0.3536i 0.7071+0.0000i
    0.3536-0.3536i 0.3536+0.3536i -0.7071+0.0000i
    0.7071+0.0000i 0.7071+0.0000i 0.0000+0.0000i

Lambda = -1.0+4.0i 0.0 0.0
         0.0 -1.0-4.0i 0.0
         0.0 0.0 -1.0

% solution of the linear system resulting from the initial value
% scaling of the columns of V -> W = (w^1,w^2,w^3)
```

```

c = V\b, W = V*diag(c)
c = 2.8284    W = -1.0-1.0i -1.0+1.0i  3
    2.8284         1.0-1.0i  1.0+1.0i -3
    4.2426         2.0         2.0     0

```

solution  $u_j(t) = \sum_{k=1}^3 w_{j,k} e^{\lambda_k t}$ , i.e.

$$\begin{aligned}
u(t) &= \begin{pmatrix} -1-i \\ 1-i \\ 2 \end{pmatrix} e^{(-1+4i)t} + \begin{pmatrix} -1+i \\ 1+i \\ 2 \end{pmatrix} e^{(-1-4i)t} + \begin{pmatrix} 3 \\ -3 \\ 0 \end{pmatrix} e^{-t} \\
&= e^{-t} \begin{pmatrix} -2 \cos(4t) + 2 \sin(4t) + 3 \\ 2 \cos(4t) + 2 \sin(4t) - 3 \\ 4 \cos(4t) \end{pmatrix}
\end{aligned}$$

Plot of the projections on the coordinate planes

```

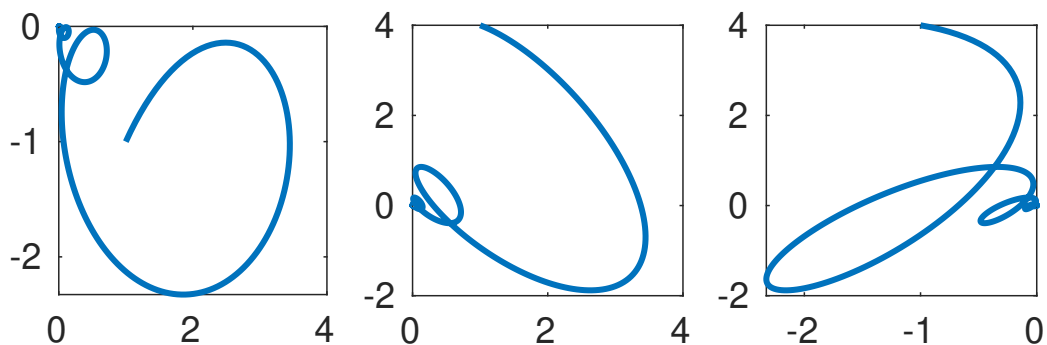
dt = 0.01; T = [0:dt:10];
E = exp(diag(Lambda)*T);
uT = W*E;
% suppress imaginary parts due to rounding errors
uT = real(uT);

```

```

subplot(1,3,1) % left image
plot(uT(1,:),uT(2,:))
subplot(1,3,2) % middle image
plot(uT(1,:),uT(3,:))
subplot(1,3,3) % right image
plot(uT(2,:),uT(3,:))

```



# Chapter 4

## Laplace Transform



## 4.1 Laplace Transform of Exponential, Sine, and Cosine

Determine the Laplace transform  $U(s)$  of the function  $u(t) = e^{2t} \cos^2 t$ .

**Resources:** [Laplace Transform of Exponential Functions](#)

### Problem Variants

■  $u(t) = e^{-3t} \cos t \sin t$

---

$U(1) = ?$ :

check

■  $u(t) = \sinh(2t) \cos(3t)$

---

$U(3) = -?$ :

check

■  $u(t) = \sin(t)^2 + 2^t$

---

$U(2) = ?$ :

check

## Solution

### Simplification

$$\cos(2t) = \cos^2 t - \sin^2 t = 2 \cos^2 t - 1 \quad \rightsquigarrow$$

$$\cos^2 t = \frac{1}{2} + \frac{1}{2} \cos(2t)$$

### Laplace transform of the elementary terms

using the formula

$$e^{\lambda t} \cos(\omega t) \xrightarrow{\mathcal{L}} \frac{s - \lambda}{(s - \lambda)^2 + \omega^2}$$

with  $\lambda = 2, \omega = 0$  and  $\lambda = 2, \omega = 2 \quad \rightsquigarrow$

$$u(t) = e^{2t} \cos^2 t = \frac{1}{2} e^{2t} + \frac{1}{2} e^{2t} \cos(2t)$$

$$U(s) \xrightarrow{\mathcal{L}} = \frac{1}{2(s-2)} + \frac{s-2}{2((s-2)^2+4)} = \frac{s^2-4s+6}{(s-2)((s-2)^2+4)}$$

### Verification with Maple™

```
# load the package for integral transforms
with(inttrans)
# compute the Laplace transform
u := exp(2*t)*cos(t)^2
U := laplace(u,t,s)
```

## 4.2 Laplace Transform of a Polynomial and an Exponential

Determine the Laplace transform  $U(s)$  of the function  $u(t) = (t + e^{-3t})^2$ .

**Resources:** [Laplace Transform of Exponential Functions](#)

### Problem Variants

■  $u(t) = t^2 \cosh(3t)$

---

$U(4) = ?$ ???:

check

■  $u(t) = (2 + t \exp(t/2))^2$

---

$U(2) = ?$ ???:

check

■  $u(t) = (t + \sinh t)^3$

---

$U(4) = ?$ ???:

check

## Solution

### Simplification

binomial theorem  $\rightsquigarrow$

$$u(t) = (t + e^{-3t})^2 = t^2 + 2te^{-3t} + e^{-6t}$$

### Laplace transform of the elementary terms

applying the formula

$$t^n e^{\lambda t} \xrightarrow{\mathcal{L}} \frac{n!}{(s - \lambda)^{n+1}}$$

with  $n = 2, \lambda = 0$ ,  $n = 1, \lambda = -3$  and  $n = 0, \lambda = -6$   $\rightsquigarrow$

$$U(s) = \frac{2}{s^3} + \frac{2}{(s + 3)^2} + \frac{1}{s + 6}$$

### 4.3 Rules for Laplace Transforms

Determine the Laplace transform  $U(s)$  of the function  $u(t) = t \sin(2t)e^{3t}$ .

**Resources:** [Laplace Transform of Exponential Functions](#), [Differentiation and Integration of Laplace Transforms](#)

#### Problem Variants

■  $u(t) = t^2 \cos(t)^2$

---

$U(1) = ?$ :

check

■  $u(t) = \frac{d}{dt}(\exp(-3t) \sin(2t))$

---

$U(1) = ?$ :

check

■  $u(t) = t \frac{d}{dt}(\sin t \cosh t)$

---

$U(2) = ?$ :

check

## Solution

Laplace transform of the elementary term

applying the formula

$$e^{\lambda t} \sin(\omega t) \xrightarrow{\mathcal{L}} \frac{\omega}{(s - \lambda)^2 + \omega^2}$$

with  $\lambda = 2$ ,  $\omega = 3 \rightsquigarrow$

$$v(t) = e^{2t} \sin(3t) \xrightarrow{\mathcal{L}} \frac{3}{(s - 2)^2 + 9} = V(t)$$

Multiplication with  $t$

applying the transformation rule

$$tv(t) \xrightarrow{\mathcal{L}} -\frac{d}{ds}V(s)$$

to  $u(t) = t \underbrace{e^{2t} \sin(3t)}_{v(t)} \rightsquigarrow$

$$U(s) = -\frac{d}{ds} \frac{3}{(s - 2)^2 + 9} = \frac{6(s - 2)}{((s - 2)^2 + 9)^2} = \frac{6s - 12}{((s - 2)^2 + 9)^2}$$

## 4.4 Inverse Laplace Transform of a Rational Function

Determine the inverse Laplace transform  $u(t)$  of the function

$$U(s) = \frac{s^2 + s + 1}{s^3 - s^2 + s - 1}.$$

**Resources:** [Laplace Transform of Exponential Functions](#)

### Problem Variants

■  $U(s) = \frac{1 + 3s}{s^2 - 4}$

---

$u(1) = ??.??:$

check

■  $U(s) = \frac{1}{s^3 - 4s^2}$

---

$u(1) = ?..??:$

check

■  $U(s) = \frac{1 + s^2}{4s - s^3}$

---

$u(1) = -?..??:$

check

## Solution

### Partial fraction decomposition

factoring the denominator of  $U(s)$ ,  $s^3 - s^2 + s - 1 = (s^2 + 1)(s - 1) \rightsquigarrow$   
ansatz

$$U(s) = \frac{s^2 + s + 1}{s^3 - s^2 + s - 1} = \frac{c}{s - 1} + \frac{as + b}{s^2 + 1}$$

for a partial fraction decomposition of  $U$

- multiplying with  $s - 1$  and setting  $s = 1 \implies c = 3/2$
- subtracting the term  $(3/2)/(s - 1) \rightsquigarrow$

$$\begin{aligned} \frac{as + b}{s^2 + 1} &= \frac{s^2 + s + 1}{(s^2 + 1)(s - 1)} - \frac{(3/2)(s^2 + 1)}{(s^2 + 1)(s - 1)} \\ &= \frac{-(s^2 - 2s + 1)/2}{(s^2 + 1)(s - 1)} = \frac{-s/2 + 1/2}{s^2 + 1}, \end{aligned}$$

i.e.  $a = -1/2$ ,  $b = 1/2$

### Inverse Laplace transform of the elementary terms

applying the formulas

$$\frac{n!}{(s - \lambda)^{n+1}} \xrightarrow{\mathcal{L}^{-1}} t^n e^{\lambda t}, \quad \frac{a(s - \lambda) + b\omega}{(s - \lambda)^2 + \omega^2} \xrightarrow{\mathcal{L}^{-1}} e^{\lambda t}(a \cos(\omega t) + b \sin(\omega t))$$

with  $n = 0$ ,  $\lambda = 1$  and  $\lambda = 0$ ,  $\omega = 1 \rightsquigarrow$

$$U(s) = \frac{3/2}{s - 1} + \frac{-s/2 + 1/2}{s^2 + 1} \xrightarrow{\mathcal{L}^{-1}} u(t) = \frac{3}{2}e^t - \frac{1}{2}\cos t + \frac{1}{2}\sin t$$

### Verification with Maple™

```
# loading the package for integral transforms
with(inttrans)
# computing the inverse transform
U := (s^2+s+1)/(s^3-s^2+s-1)
u := invlaplace(U,s,t)
```



## 4.5 Laplace Transform of a Periodic Function

Determine the Laplace transform  $U(s)$  of the function  $u(t) = |\sin t|$ .

**Resources:** [Laplace Transform of Periodic Functions](#)

### Problem Variants

■ 1-periodic extension of  $u(t) = t, 0 \leq t < 1$

---

$U(1) = ?$ :

check

■ 2-periodic extension of  $u(t) = t(2 - t), 0 \leq t < 2$

---

$U(1) = ?$ :

check

■  $2\pi$ -periodic extension of  $u(t) = \text{sign} \sin t, 0 \leq t < 2\pi$

---

$U(1) = ?$ :

check

## Solution

### Laplace transform of the truncated function

Laplace transform of  $v = u\chi$  with  $\chi$  the characteristic function of  $[0, \pi]$  and  $u(t) = |\sin t| \rightsquigarrow$

$$\begin{aligned} V(s) &= \int_0^\pi \sin t e^{-st} dt = \underbrace{\left[ -\sin t \frac{e^{-st}}{s} \right]_{t=0}^{t=\pi}}_{=0} + \int_0^\pi \cos t \frac{e^{-st}}{s} dt \\ &= \left[ -\cos t \frac{e^{-st}}{s^2} \right]_{t=0}^{t=\pi} - \int_0^\pi \sin t \frac{e^{-st}}{s^2} dt = \frac{e^{-s\pi} + 1}{s^2} - V(s)/s^2 \end{aligned}$$

solving for  $V(s) \rightsquigarrow$

$$V(s) = \frac{e^{-s\pi} + 1}{s^2} / (1 + 1/s^2) = \frac{e^{-s\pi} + 1}{s^2 + 1}$$

### Laplace transform of the $\pi$ -periodic extension $u$ of $v$

substituting the expression for the Laplace transform  $V$  into the formula

$$U(s) = \frac{1}{1 - e^{-Ts}} V(s)$$

for  $T$ -periodic functions with  $T = \pi \rightsquigarrow$

$$U(s) = \frac{1 + e^{-\pi s}}{(1 - e^{-\pi s})(s^2 + 1)}$$

## 4.6 Laplace Transform of an Integral Equation

Determine the solution  $u(t)$  of the integral equation

$$u(t) + \int_0^t e^{2(t-\tau)} u(\tau) \, d\tau = t.$$

**Resources:** [Convolution and Laplace Transforms](#), [Laplace Transform of Exponential Functions](#)

### Problem Variants

■  $u(t) - \int_0^t (t - \tau)u(\tau) \, d\tau = 1$

---

$u(0) = ?$ :

check

■  $u(t) - \int_0^t 2 \cos(t - \tau)u(\tau) \, d\tau = \sin t$

---

$u(1) = ?$ ??:

check

■  $u(t) - \int_0^t u(\tau) \, d\tau = \exp(3t)$

---

$u(0) = ?$ :

check

## Solution

Laplace transform of the integral equation

$$v(t) = \underbrace{\int_0^t f(t-\tau)u(\tau) d\tau}_{(f \star u)(t)} \xrightarrow{\mathcal{L}} V(s) = F(s)U(s)$$

with  $f(t) = e^{2t}$  and  $t^n e^{\lambda t} \xrightarrow{\mathcal{L}} \frac{n!}{(s-\lambda)^{n+1}} \rightsquigarrow$

$$u(t) + \int_0^t e^{2(t-\tau)}u(\tau) d\tau = t \xrightarrow{\mathcal{L}} U(s) + \frac{1}{s-2}U(s) = \frac{1}{s^2},$$

i.e.  $U(s) = \frac{s-2}{s^2(s-1)}$

Partial fraction decomposition

ansatz

$$U(s) = \frac{s-2}{s^2(s-1)} = \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s-1}$$

- multiplying with  $s^2$  and setting  $s = 0 \implies a = 2$
- multiplying with  $s-1$  and setting  $s = 1 \implies c = -1$
- setting  $s = -1 \implies 3/2 = 2 - b + 1/2$ , i.e.  $b = 1$

Inverse Laplace transform

applying the formula

$$\frac{n!}{(s-\lambda)^{n+1}} \xrightarrow{\mathcal{L}^{-1}} t^n e^{\lambda t}$$

with  $n = 1, \lambda = 0$ ,  $n = 0, \lambda = 0$ , and  $n = 0, \lambda = 1 \rightsquigarrow$

$$U(s) = \frac{2}{s^2} + \frac{1}{s} - \frac{1}{s-1} \xrightarrow{\mathcal{L}^{-1}} u(t) = 2t + 1 - e^t$$

## 4.7 Laplace Transform of a First Order Initial Value Problem

Determine the solution  $u(t)$  of the initial value problem

$$u' + 3u = e^{-t} \sin(2t), \quad u(0) = 4,$$

using Laplace transforms.

**Resources:** [Laplace Transform of Linear First Order Differential Equations](#)

### Problem Variants

■  $u' - 2u = \sin(t)^2, u(0) = 2$

---

$u(1) = ???.??:$

check

■  $u' + u = \exp(-t), u(0) = 1$

---

$u(1) = ?.??:$

check

■  $u' - 3u = 3t^2, u(0) = 3$

---

$u(1) = ???.??:$

check

## Solution

Laplace transform of the differential equation

applying the formulas

$$u'(t) \xrightarrow{\mathcal{L}} sU(s) - u(0), \quad e^{\lambda t} \sin(\omega t) \xrightarrow{\mathcal{L}} \frac{\omega}{(s - \lambda)^2 + \omega^2}$$

with  $u(0) = 4$ ,  $\lambda = -1$ ,  $\omega = 2 \rightsquigarrow$

$$u' + 3u = e^{-t} \sin(2t) \xrightarrow{\mathcal{L}} sU(s) - 4 + 3U(s) = \frac{2}{(s + 1)^2 + 4}$$

i.e.

$$U(s) = \frac{4}{s + 3} + \underbrace{\frac{2}{((s + 1)^2 + 4)(s + 3)}}_{V(s)}$$

Partial fraction decomposition

ansatz

$$V(s) = \frac{2}{((s + 1)^2 + 4)(s + 3)} = \frac{a}{s + 3} + \frac{bs + c}{(s + 1)^2 + 4}$$

- multiplying with  $s + 3$  and setting  $s = -3 \implies a = 1/4$
- subtracting the term  $(1/4)/(s + 3) \rightsquigarrow$

$$\begin{aligned} & \frac{2}{((s + 1)^2 + 4)(s + 3)} - \frac{((s + 1)^2 + 4)/4}{((s + 1)^2 + 4)(s + 3)} \\ &= \frac{-s^2/4 - s/2 + 3/4}{((s + 1)^2 + 4)(s + 3)} = \frac{-s/4 + 1/4}{(s + 1)^2 + 4} \end{aligned}$$

$$\rightsquigarrow U(s) = \frac{4}{s + 3} + V(s) = \frac{17/4}{s + 3} + \frac{-(s + 1)/4 + 2/4}{(s + 1)^2 + 4}$$

Inverse Laplace transform

applying the formulas

$$\frac{1}{s - \lambda} \xrightarrow{\mathcal{L}^{-1}} e^{\lambda t}, \quad \frac{s - \lambda}{(s - \lambda)^2 + \omega^2} \xrightarrow{\mathcal{L}^{-1}} e^{\lambda t} \cos(\omega t), \quad \frac{\omega}{(s - \lambda)^2 + \omega^2} \xrightarrow{\mathcal{L}^{-1}} e^{\lambda t} \sin(\omega t)$$

with  $\lambda = -3$  and  $\lambda = -1$ ,  $\omega = 2 \rightsquigarrow$

$$u(t) = \frac{17}{4}e^{-3t} - \frac{1}{4}e^{-t} \cos(2t) + \frac{1}{4}e^{-t} \sin(2t)$$

Verification with Maple™

```
dsolve({diff(u(t),t)+3*u(t)=exp(-t)*sin(2*t),u(0)=4})
```

## 4.8 Laplace Transform of a Second Order Initial Value Problem

Determine the solution  $u(t)$  of the initial value problem

$$u'' - 6u' + 9u = 0, \quad u(0) = 2, \quad u'(0) = -3$$

using Laplace transforms.

**Resources:** [Laplace Transform of Linear Second Order Differential Equations](#)

### Problem Variants

■  $u'' + 4u = \cos t, u(0) = 2, u'(0) = 2$

---

$u(1) = ?$ :

check

■  $u'' + 2u' + u = \sin t, u(0) = 0, u'(0) = 0$

---

$u(1) = ?$ :

check

■  $u'' + u' - 2u = 3t \exp t, u(0) = 0, u'(0) = 1$

---

$u(1) = ?$ :

check



## Solution

Laplace transform of the differential equation

applying the formulas

$$u'(t) \xrightarrow{\mathcal{L}} sU(s) - u(0), \quad u''(t) \xrightarrow{\mathcal{L}} s^2U(s) - su(0) - u'(0)$$

to the initial value problem  $u'' - 6u' + 9u = 0$ ,  $u(0) = 2$ ,  $u'(0) = -3 \rightsquigarrow$

$$(s^2U(s) - 2s + 3) - 6(sU(s) - 2) + 9U(s) = 0,$$

i.e.  $U(s) = \frac{2s - 15}{s^2 - 6s + 9}$

Partial fraction decomposition

$$U(s) = \frac{2s - 15}{s^2 - 6s + 9} = \frac{2(s - 3) - 9}{(s - 3)^2} = \frac{2}{s - 3} - \frac{9}{(s - 3)^2}$$

Inverse Laplace transform

application of the formula

$$t^n e^{\lambda t} \xrightarrow{\mathcal{L}} \frac{n!}{(s - \lambda)^{n+1}}$$

with  $\lambda = 3$  and  $n = 0, 1 \rightsquigarrow$  inverse Laplace transform

$$u(t) = 2e^{3t} - 9te^{3t}$$

Verification with Maple™

```
de := diff(u(t),t$2)-6*diff(u(t),t)+9*u(t)=0
ic := u(0)=2, D(u)(0)=-3
dsolve({de,ic})
```

# Chapter 5

## Calculus Highlights

## 5.1 Chasing Pirates

Continuously heading for the target, a navy speedboat is in pursuit of a freighter, which is sailing north at 20 knots under a black flag<sup>1</sup>. After how many minutes does the speedboat catch up with the freighter, if it sails twice as fast and has started 10 kilometers east of the freighter?

**Resources:** [Separable Differential Equation](#)

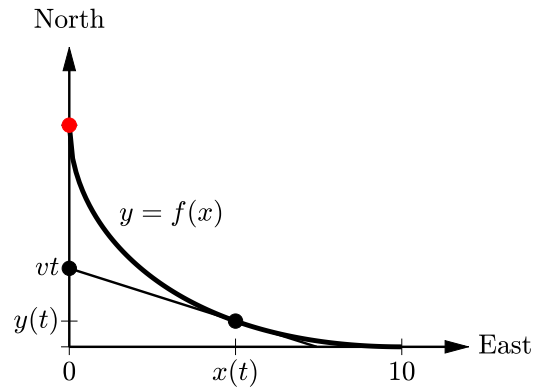
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<sup>1</sup>a classical pursuit problem, studied already by Pierre Bouguer in the eighteenth century

## Solution

Differential equation for the path of the speedboat

parametrization of the path:  $t \mapsto (x(t), y(t))$ ,  $y(t) = f(x(t))$



$(x'(t), y'(t))$  points towards the position  $(0, vt)$  of the freighter  $\implies$

$$f'(x) = \frac{y - vt}{x - 0} \quad \text{i.e.} \quad f'(x(t))x(t) + vt = f(x(t))$$

differentiating with respect to  $t$ , using the “dot-notation”  $\implies$

$$f''(x)\dot{x}x + f'(x)\dot{x} + v = f'(x)\dot{x} \quad \text{bzw.} \quad f''(x)\dot{x}x = -v$$

constant velocity  $2v$  of the speedboat,  $y(t) = f(x(t))$   $\implies$

$$2v = \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{\dot{x}^2 + (f'(x)\dot{x})^2} = -\dot{x}\sqrt{1 + f'(x)^2}$$

The minus sign is necessary in view of the north-east route of the speedboat. solving for  $\dot{x}$  and substituting into the differential equation  $f''(x)\dot{x}x = -v$

$\rightsquigarrow$

$$\frac{f''(x)}{\sqrt{1 + f'(x)^2}} = \frac{1}{2x}$$

with the initial values

- $f(10) = 0$  (starting position 10 kilometers east of the freighter)
- $f'(10) = 0$  (route of the speedboat initially west)

### Solution of the initial value problem

substituting  $g = f'$   $\rightsquigarrow$  separable differential equation for  $g(x)$ ,

$$\frac{dg}{\sqrt{1+g^2}} = \frac{dx}{2x}, \quad dg = g'(x) dx,$$

which can be integrated explicitly:

$$\int \frac{dg}{\sqrt{1+g^2}} = \int \frac{dx}{2x} \rightsquigarrow \operatorname{arcsinh}(g) = \ln(g + \sqrt{1+g^2}) = \frac{1}{2} \ln x + C$$

initial value  $g(10) = f'(10) = 0 \implies$

$$\underbrace{\ln(0 + \sqrt{1+0})}_{=0} = \frac{1}{2} \ln 10 + C, \quad \text{i.e. } C = -\frac{1}{2} \ln 10$$

solving for  $g$ ,  $\frac{1}{2} \ln x - \frac{1}{2} \ln 10 = \ln(\sqrt{x}/\sqrt{10}) \rightsquigarrow$

$$\begin{aligned} g + \sqrt{1+g^2} &= \sqrt{x}/\sqrt{10} \iff (1+g^2) = (\sqrt{x}/\sqrt{10} - g)^2 \\ &\iff g = \frac{1}{2\sqrt{10}}x^{1/2} - \frac{\sqrt{10}}{2}x^{-1/2} \end{aligned}$$

integrating  $\rightsquigarrow$

$$f(x) = \int g(x) dx = \frac{1}{3\sqrt{10}}x^{3/2} - \sqrt{10}x^{1/2} + \tilde{C}$$

initial value  $f(10) = 0 \implies$

$$\frac{10}{3} - 10 + \tilde{C} = 0, \quad \text{i.e. } \tilde{C} = 20/3$$

### Time of pursuit $T$

The speedboat reaches the freighter at position  $(0, f(0)) = (0, 20/3)$ , i.e. after a distance of 6.666 km, travelled by the freighter.

20 knots  $\hat{=}$  37.04 km/h  $\implies$

$$T = \frac{6.666\text{km}}{37.04\text{km/h}} = 0.1800 \text{ h} \approx 11 \text{ minutes}$$

## 5.2 Acceleration of a Sports Car

One of the newer models of the classic Porsche 911 reaches the velocity  $v_* = 300$  km/h in  $t_* \approx 27$ s. Neglecting friction, determine the average accelerating force  $F$  by solving the differential equation

$$mv' = F - cv^2, \quad v(0) = 0,$$

with  $m = 1715$  kg the weight of the car and a driver,  $v(t)$  its velocity, and  $c \approx 0.50$  the coefficient in a simplified model of air resistance<sup>2</sup>.

**Resources:** [Separable Differential Equation](#)

---

<sup>2</sup> $c = \frac{1}{2}c_dA\rho$  with  $c_d = 0.33$  the drag coefficient of the car,  $A = 1.303 \cdot 1.900$  m<sup>2</sup> its cross sectional area, and  $\rho = 1.225$  kg/m<sup>3</sup> the air density

## Solution

### Solution of the initial value problem

The differential equation is separable:

$$\frac{m}{F - cv^2} v' = 1.$$

partial fraction decomposition  $\rightsquigarrow$

$$\frac{m}{2\sqrt{F}} \left( \frac{1}{\sqrt{F} + \sqrt{c}v} + \frac{1}{\sqrt{F} - \sqrt{c}v} \right) v' = 1$$

integrating, using  $\int \frac{dv}{p+qv} = \frac{\ln|p+qv|}{q}$   $\rightsquigarrow$

$$\frac{m}{2\sqrt{F}} \left( \frac{\ln|\sqrt{F} + \sqrt{c}v|}{\sqrt{c}} - \frac{\ln|\sqrt{F} - \sqrt{c}v|}{\sqrt{c}} \right) = t + C$$

and, after simplification,

$$\frac{\sqrt{F} + \sqrt{c}v}{\sqrt{F} - \sqrt{c}v} = e^{2\sqrt{cF}(t+C)/m}, \quad F > cv^2$$

substituting the initial value  $v(0) = 0 \implies C = 0$

abbreviating  $E := e^{2\sqrt{cF}t/m}$  and solving for  $v$   $\rightsquigarrow$

$$\sqrt{F} + \sqrt{c}v = E\sqrt{F} - E\sqrt{c}v \iff v = \frac{E\sqrt{F} - \sqrt{F}}{E\sqrt{c} + \sqrt{c}} = \frac{E - 1}{E + 1} \frac{\sqrt{F}}{\sqrt{c}},$$

i.e.

$$\begin{aligned} v(t) &= \frac{e^{2\sqrt{cF}t/m} - 1}{e^{2\sqrt{cF}t/m} + 1} \frac{\sqrt{F}}{\sqrt{c}} = \frac{e^{t\sqrt{cF}/m} - e^{-t\sqrt{cF}/m}}{e^{t\sqrt{cF}/m} + e^{-t\sqrt{cF}/m}} \sqrt{F/c} \\ &= \tanh(t\sqrt{cF}/m) \sqrt{F/c} \end{aligned}$$

### Computation of the accelerating force

converting km/h to m/s and using Maple<sup>TM</sup> to solve the equation  $v(t_\star) = v_\star$

$\rightsquigarrow$

```
c := 0.50; m := 1715; t300 := 27; v300 := 300*1000/3600;
F := fsolve(v300 = tanh(t300*sqrt(c*F)/m)*sqrt(F/c), F)
```

$$F \approx 6678 \text{ Newton}$$

### 5.3 Periodic Orbit or Journey of no Return?

After an “almost collision” of two space ships, which both are out of fuel, the crews, rightfully extremely worried, are asking: “What will be our future flight path? Will we leave our solar system?” To give a reliable answer, solve the differential equation<sup>3</sup>

$$\frac{d^2}{d\varphi^2} \frac{1}{r(\varphi)} + \frac{1}{r(\varphi)} = C := GM/|r(0) \cdot v_y|^2$$

for the distance  $r$  from the sun as a function of the trajectory angle  $\varphi$  (polar coordinates) and the flight data

	distance $r(0)$ to the sun	velocity $(v_x, v_y)^t$
space ship A	$3 \cdot 10^{11}$ m	$(-1, 2)^t \cdot 10^4$ m/s
space ship B	$3 \cdot 10^{11}$ m	$(-2, -4)^t \cdot 10^4$ m/s

at the time ( $\varphi = 0$ ) of the “almost collision”. Use the rounded values  $G = 7 \cdot 10^{-11} \frac{\text{m}^3}{\text{kg s}^2}$  and  $M = 2 \cdot 10^{30}$  kg for the gravitational constant and the mass of the sun.

**Resources:** [Method of Undetermined Coefficients](#)

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<sup>3</sup>a consequence of Newton’s laws of motion



## Solution

Initial values for  $q(\varphi) = 1/r(\varphi)$

representation of position and velocity with polar coordinates:

$$\begin{aligned} p(t) &= r(\varphi(t)) \begin{pmatrix} \cos \varphi(t) \\ \sin \varphi(t) \end{pmatrix} \\ p'(t) &= r'(\varphi(t))\varphi'(t) \begin{pmatrix} \cos \varphi(t) \\ \sin \varphi(t) \end{pmatrix} + r(\varphi(t)) \begin{pmatrix} -\sin \varphi(t) \\ \cos \varphi(t) \end{pmatrix} \varphi'(t) \end{aligned}$$

substituting  $\varphi(0) = 0$ ,  $p(0) = (r(0), 0)^t$  and  $p'(0) = (v_x, v_y)^t \rightsquigarrow$

$$\begin{pmatrix} v_x \\ v_y \end{pmatrix} = r'(0)\varphi'(0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + r(0) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \varphi'(0),$$

i.e.  $\varphi'(0) = v_y/r(0)$ ,  $r'(0) = v_x/\varphi'(0) = v_x r(0)/v_y$

derivative of the reciprocal of the distance to the sun:

$$q'(0) = \left. \frac{d}{d\varphi} \frac{1}{r(\varphi)} \right|_{\varphi=0} = -\frac{r'(0)}{r(0)^2} = -\frac{v_x}{v_y r(0)}$$

Solution of the initial value problem for  $q$

$$q'' + q = C \quad \Longrightarrow$$

$$q(\varphi) = C + c_1 \cos \varphi + c_2 \sin \varphi = C + D \cos(\varphi - \delta), \quad C, D > 0,$$

i.e.  $q$  parametrizes a conic section ( $C > D$ : ellipse,  $C = D$ : parabola,  $C < D$ : hyperbola)

substituting the initial values  $\rightsquigarrow$

$$1/r(0) - C = q(0) - C = D \cos \delta, \quad -\frac{v_x}{v_y r(0)} = q'(0) = D \sin \delta$$

and, using  $\cos^2 \delta + \sin^2 \delta = 1$ ,  $\tan \delta = \sin \delta / \cos \delta$ ,

$$D = \sqrt{\left(\frac{1}{r(0)} - C\right)^2 + \left(\frac{v_x}{v_y r(0)}\right)^2}, \quad \tan \delta = \frac{q'(0)}{q(0) - C} = -\frac{v_x}{v_y(1 - Cr(0))}$$

Solution for the given data

Computation without units,  $r$  and  $v$  measured in m and m/s, respectively

- space ship A:  $r(0) = 3 \cdot 10^{11}$ ,  $v_x = -1 \cdot 10^4$ ,  $v_y = 2 \cdot 10^4$

$$\begin{aligned} C &= GM/|r(0) \cdot v_y|^2 = 7 \cdot 10^{-11} \cdot 2 \cdot 10^{30} / (3 \cdot 10^{11} \cdot 2 \cdot 10^4)^2 \\ &= \frac{7}{18} \cdot 10^{-11} \approx 0.3889 \cdot 10^{-11} \end{aligned}$$

$$\begin{aligned} D &= \sqrt{((1/3) \cdot 10^{-11} - (7/18) \cdot 10^{-11})^2 + (-10^4 / (2 \cdot 10^4 \cdot 3 \cdot 10^{11}))^2} \\ &= \sqrt{(-1/18)^2 + (-1/6)^2} \cdot 10^{-11} = \frac{\sqrt{10}}{18} \cdot 10^{-11} \approx 0.1757 \cdot 10^{-11} \end{aligned}$$

$$\tan \delta = \frac{-10^4}{2 \cdot 10^4(1 - (7/18) \cdot 10^{-11} \cdot 3 \cdot 10^{11})} = -3, \quad \delta \approx 1.8925$$

- space ship B:  $r(0) = 3 \cdot 10^{11}$ ,  $v_x = -2 \cdot 10^4$ ,  $v_y = -4 \cdot 10^4$   
analogous computation  $\rightsquigarrow$

$$C = \frac{7}{72} \cdot 10^{-11} \approx 0.0972 \cdot 10^{-11}, \quad D = \frac{\sqrt{433}}{72} \cdot 10^{-11} \approx 0.2890 \cdot 10^{-11},$$

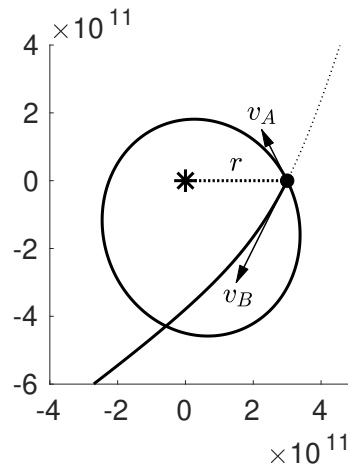
$$\text{and } \tan \delta = -\frac{12}{17}, \quad \delta \approx -0.6147$$

Trajectories of the space ships

The figure shows (coordinates in m, initial velocities strongly magnified):

The trajectory of A is an ellipse <sup>a</sup> ( $D < C$ ), the trajectory of B a hyperbola ( $D > C$ ) with  $r(\varphi) \rightarrow \infty$  (no return to our solar system).

<sup>a</sup>Kepler's first law



## 5.4 Electrical Circuit with MATLAB<sup>®</sup>

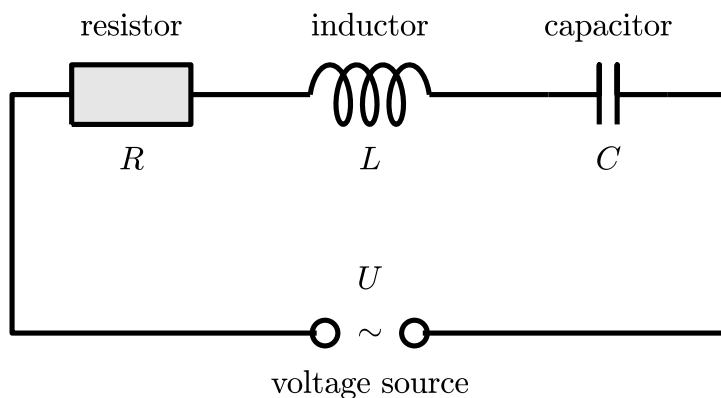
The current  $I$  of the electrical circuit shown below satisfies the differential equation

$$LI'' + RI' + \frac{1}{C}I = U\omega \cos(\omega t).$$

Write a MATLAB<sup>®</sup> program which plots  $I(t)$ ,  $0 \leq \omega t \leq 10\pi$ , for the initial conditions  $I(0) = I'(0) = 0$ , the data  $U = 220 \text{ V}$ ,  $\omega = 50 \text{ s}^{-1}$ , and the values of

$$\omega L, R, \frac{1}{\omega C} \in [10, 200] \Omega,$$

which can be specified interactively via sliders in a graphics window.



**Resources:** [Forced Damped Oscillations](#), [Differential Equations with Matlab](#)

## Solution

Solution of the differential equation

An exact solution is possible; but, in view of its simplicity, a numerical method is preferred for the MATLAB<sup>®</sup> program.

formulating  $LI'' + RI' + \frac{1}{C}I = U\omega \cos(\omega t)$  as a first order system for  $(y_1, y_2) = (I, I')$   $\rightsquigarrow$

$$y_1' = y_2, \quad y_2' = -(R/L)y_2 - (1/(CL))y_1 + (U/L)\omega \cos(\omega t)$$

$\rightsquigarrow$  application of standard numerical solvers such as the MATLAB<sup>®</sup> function `ode45`.

MATLAB<sup>®</sup> function for simulating the electrical circuit

```
function electric_circuit()

% initialization
global ui_data plot_handle U omega R C L

ui_data = []; plot_handle = [];
U = 220; omega = 50;
R = 100; C = 1/(100*omega); L = 100/omega;

% initialize the graphics user interface (GUI)

clf; % create a figure

% text field in the normalized window [0,1]^2
% positioned at (.5,.85) with width .3 and height .05
% prints the current value of R rounded to the next integer
% returns the ui control object
ui_data(1) = uicontrol(...
    'Units','normalized', ...
    'HorizontalAlignment','left', ...
    'Position',[.5,.85,.3,.05],...
    'String',[sprintf('R=%d',round(R)) ' Ohm'],...
    'FontSize',15,...
    'Style','text');
```

```

% slider for R with range [10,200]
% invokes the function slider_update upon user input
ui_data(2) = uicontrol(...
    'Units','normalized', ...
    'Position',[.05,.85,.4,.05], ...
    'Min',10,'Max',200,'Value',R,...
    'Callback',@slider_update,...
    'Style','slider');

% analogous text fields and sliders for 1/(wC) and wL
...

% display the current for the default values
plot_current
end

% callback function
function slider_update(varargin)
global ui_data plot_handle U omega R C L

% read the value of R
% stored in the field .Value of the object ui_data(2)
R = get(ui_data(2),'Value');
% update the text field, i.e. the object ui_data(1)
set(ui_data(1),'String',sprintf('R=%d',round(R)));

% analogous updates for C and L
...

% update the plot of the current
plot_current;

end

% plot of the current
function plot_current
global ui_data plot_handle U omega R C L

% solution of the differential equation for the current I
% y = (I,dI/dt)
f = @(t,y) [y(2); -(R/L)*y(2)-(1/(C*L))*y(1)+(U/L)*omega*cos(omega*t)];

```

```

[T,Y] = ode45(f,[0 2*pi*10/omega],[0; 0]);
% Y(k,1) = I(T(k))

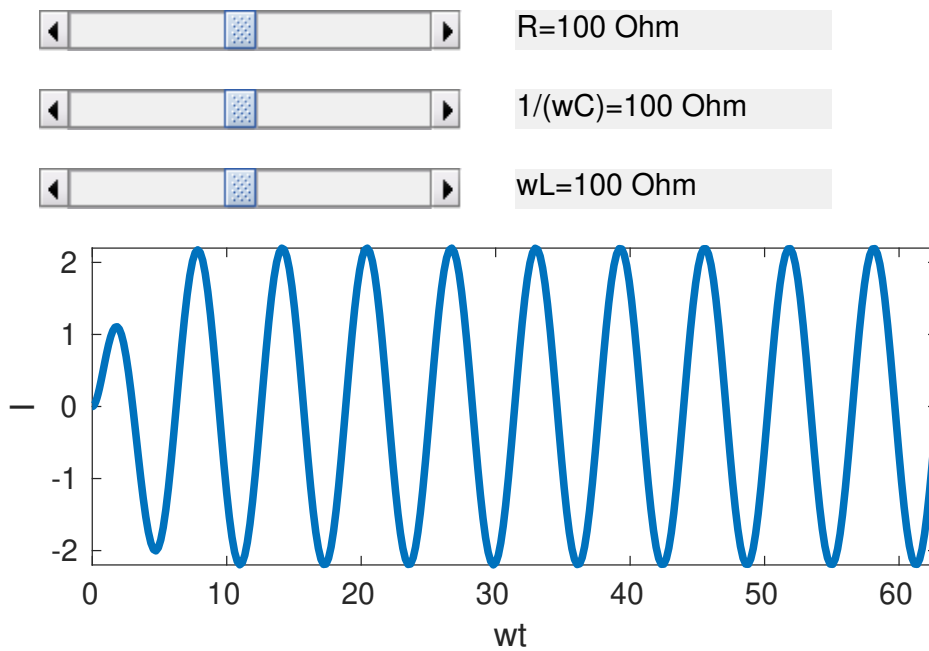
% delete the existing plot
delete(plot_handle)
% specify the plot window inside the gui figure
plot_handle = axes('Position',[0.1 0.2 0.8 0.4]);

% plot I as a function of omega*t
plot(omega*T,Y(:,1),'LineWidth',3)
xlabel('wt'); ylabel('I');
axis tight

end

```

Graphical user interface with plot of the current for the default parameters



It is instructive to consider the extreme cases with slider values equal to  $10\ \Omega$  or  $200\ \Omega$ .

## 5.5 Roller Coaster Ride with MATLAB<sup>®</sup>

Visualize the first 10 seconds of a ride on the roller coaster which is modelled by the height function

$$h(x) = 10 \cos^2(x/20)e^{-x/50}, \quad x \geq 0$$

(units: meters and seconds [m/s]). To this end mark the positions of the wagon on the graph of  $h$  in intervals of 0.5 seconds. Assume that the start velocity  $v_0$  equals 0.1m/s and neglect friction<sup>4</sup>. Moreover, determine the maximal velocity.

**Resources:** [Differential Equations with Matlab](#), [Conservation of Energy](#)

---

<sup>4</sup>More realistic (but also more complicated ...) is a three dimensional trajectory of a roller coaster containing loops - an incentive to study the relevant mathematics in more detail!

## Solution

Differential equation for the position  $(x(t), y(t))$  of the wagon  
total energy (sum of kinetic and potential energy):

$$E(t) = mgh(x(t)) + \frac{m}{2}|v(t)|^2$$

with  $g = 9.81 \text{ m/s}^2$  (gravity),  $v(t) = (x'(t), y'(t))$  [m/s] the velocity, and  $m$  [kg] the mass of the wagon

invariance of  $E \implies E(t) = E(0)$ , i.e.

$$(E) \quad mgh(x(t)) + \frac{m}{2}|v(t)|^2 = mgh(x(0)) + \frac{m}{2}v_0^2$$

$\underbrace{\hspace{10em}}_{=0}$

$v \parallel$  tangent of  $h \implies h'(x(t)) = y'(t)/x'(t)$  and hence

$$|v(t)|^2 = x'(t)^2 + y'(t)^2 = x'(t)^2(1 + h'(x(t))^2)$$

substituting into (E) and dividing by  $m \rightsquigarrow$

$$gh(x(t)) + \frac{1}{2}x'(t)^2(1 + h'(x(t))^2) = gh_0 + \frac{1}{2}v_0^2$$

with  $h_0 = h(0) = 10$

solving for  $x' \rightsquigarrow$

$$x' = \pm \sqrt{(2gh_0 + v_0^2 - 2gh(x))/(1 + h'(x)^2)}$$

Only the positive root is relevant - you do not want to ride backwards on a roller coaster!

MATLAB<sup>®</sup> script

```
% parameter
g = 9.81; tmax = 10; dt = 0.5; x0 = 0; v0 = 0.1;

% height function, describing the roller coaster geometry
h = @(x) 10*cos(x/20).^2.*exp(-x/50);
h0 = h(0);

% derivative (after simplification)
dh = @(x) (-cos(x/20).*sin(x/20)-cos(x/20).^2/5).*exp(-x/50);
```



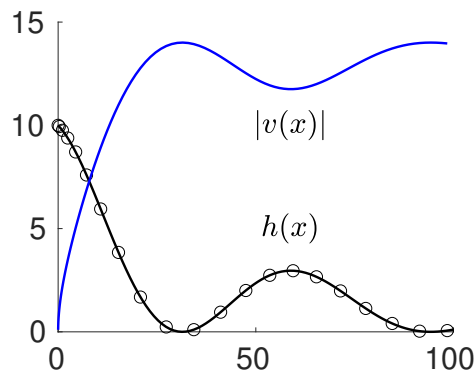
```

% right-hand side of the differential equation
dx = @(t,x) sqrt((2*g*h0+v0^2-2*g*h(x))./(1+dh(x).^2));
% There is no explicit t-dependence;
% but the variable t is required by the Matlab-syntax.

% solution of the initial value problem
[t,x] = ode45(dx,[0:dt:tmax],x0);
% -> t = [0 dt 2*dt ...], x = [x(0) x(dt) x(2*dt) ...]

% graph of the height function with wagon positions
hold on
plot(x,h(x),'ok');
x = linspace(0,xmax); plot(x,h(x),'-k');
hold off

```



The figure also shows the velocity which, however, was not computed with the above program segment.

### Maximal velocity

In view of the invariance of energy (equation (E)),  $|v| = x' \sqrt{1 + h'(x)^2}$  is maximal at the minima  $x_*$  of  $h$ , i.e. for  $h(x_*) = 0$  ( $x_*/20 = \pi/2, 3\pi/2, \dots$ ). Hence, setting  $x(t) = x_*$  in (E), dividing by  $m/2$ , and taking the square root

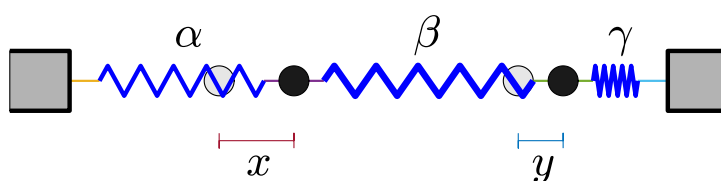
$$v_{\max} = \sqrt{2gh_0 + v_0^2} \approx \sqrt{2 \cdot 9.81 \cdot 10 + 0.1^2} \text{ m/s} \approx 4.0075 \text{ m/s}.$$

## 5.6 Oscillation of Coupled Springs

The differential equations

$$\begin{aligned}x'' &= -(\alpha + \beta)x + \beta y \\y'' &= \beta x - (\beta + \gamma)y\end{aligned}$$

model the oscillation of three coupled springs with spring constants  $\alpha$ ,  $\beta$ , and  $\gamma$ .



Determine the elongations  $x(t)$  and  $y(t)$  of the two discs from the equilibrium positions (gray discs) for  $\alpha = \gamma = 1$ ,  $\beta = 2$  and the initial values  $x(0) = 1$ ,  $y(0) = 2$ ,  $x'(0) = y'(0) = 0$ .

**Resources:** [Diagonalization of a Linear System of Differential Equations, Linear Oscillator](#)

## Solution

Matrix form of the system of differential equations

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \underbrace{\begin{pmatrix} -3 & 2 \\ 2 & -3 \end{pmatrix}}_A \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_Z$$

## Diagonalization

transformation of  $Z'' = AZ$  to diagonal form with the matrix

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

of eigenvectors of  $A$

substituting  $Z = V\tilde{Z} \rightsquigarrow$  decoupled system

$$(V\tilde{Z})'' = A(V\tilde{Z}) \iff \tilde{Z}'' = \underbrace{V^t AV}_\Lambda \tilde{Z}$$

( $V^t = V^{-1}$  since  $V$  is orthogonal) with

$$\Lambda = \begin{pmatrix} -1 & 0 \\ 0 & -5 \end{pmatrix}$$

the diagonal matrix of eigenvalues of  $A$ , i.e.

$$\tilde{z}_1'' = -\tilde{z}_1, \quad \tilde{z}_2'' = -5\tilde{z}_2$$

general solution of the decoupled system

$$\begin{aligned} \tilde{z}_1(t) &= c_{1,1} \cos t + c_{1,2} \sin t \\ \tilde{z}_2(t) &= c_{2,1} \cos(\sqrt{5}t) + c_{2,2} \sin(\sqrt{5}t) \end{aligned}$$

## Initial values

$Z(0) = (x(0), y(0))^t = (1, 2)^t$ ,  $Z'(0) = (x'(0), y'(0))^t = (0, 0)^t \implies$

$$\tilde{Z}(0) = V^t Z(0) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$\tilde{Z}'(0) = V^t Z'(0) = (0, 0)^t$$

substituting these initial values into the general solution  $\rightsquigarrow$

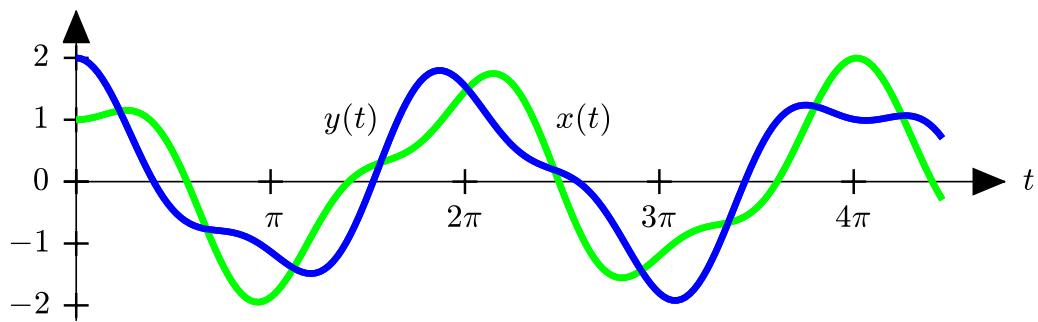
$$c_{1,1} = 3/\sqrt{2}, c_{2,1} = 1/\sqrt{2}, c_{1,2} = c_{2,2} = 0,$$

i.e.

$$\tilde{Z}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} 3 \cos t \\ \cos(\sqrt{5}t) \end{pmatrix}$$

undoing the transformation  $Z = V\tilde{Z}$   $\rightsquigarrow$

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \underbrace{\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}_V \tilde{Z}(t) = \frac{1}{2} \begin{pmatrix} 3 \cos t - \cos(\sqrt{5}t) \\ 3 \cos t + \cos(\sqrt{5}t) \end{pmatrix}$$



aperiodic solution in view of the irrational ratio  $(1/(2\pi)) : (\sqrt{5}/(2\pi))$  of the frequencies of the summands  $\cos t$  and  $\cos(\sqrt{5}t)$

## 5.7 Predator-Prey Model

Determine the critical points of the system of differential equations

$$\begin{aligned}u' &= u(1 - 2u + 2v) \\v' &= v(1 - u)\end{aligned}$$

as well as their type and sketch the vector field of tangent directions.

**Resources:** [Stability of Autonomous Systems of Differential Equations](#),  
[Stability of Linear Systems of Differential Equations](#)

## Solution

### Critical points

solving

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} u' \\ v' \end{pmatrix} = \underbrace{\begin{pmatrix} u(1-2u+2v) \\ v(1-u) \end{pmatrix}}_{F(u,v)}$$

$\rightsquigarrow$  critical points  $p_1 = (0, 0)$ ,  $p_2 = (1/2, 0)$ ,  $p_3 = (1, 1/2)$

### Stability and type

classification with the determinant and trace of the Jacobi matrix

$$F' = \begin{pmatrix} 1-4u+2v & 2u \\ -v & 1-u \end{pmatrix}$$

- $F'(p_1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$   
double eigenvalue  $\lambda = 1 > 0$ , diagonal matrix  $\implies$  unstable knot
- $F'(p_2) = \begin{pmatrix} -1 & 1 \\ 0 & 1/2 \end{pmatrix}$   
 $\det F'(p_2) = -1 < 0 \implies$  saddle (unstable)
- $F'(p_3) = \begin{pmatrix} -2 & 2 \\ -1/2 & 0 \end{pmatrix}$   
 $\det F'(p_3) = 1 > 0$ ,  $\text{trace } F'(p_3) = -2 < 0 \implies$  stable  
knot since  $\det \leq (\text{trace}/2)^2$  (degenerate since 1 is a double eigenvalue  
and  $F'$  is not diagonalizable)

### Vector field of tangent directions

MATLAB<sup>®</sup> -script

```
[u,v] = meshgrid([0.1:0.1:1.4],[0.1:0.1:0.9]);  
F1 = @(u,v) u.*(1+2*v-2*u);  
F2 = @(u,v) v.*(1-u);  
  
% direction field  
quiver(u,v,F1(u,v),F2(u,v));  
  
% critical points  
plot([0;1/2;1],[0;0;1/2],'.k','MarkerSize',30);
```

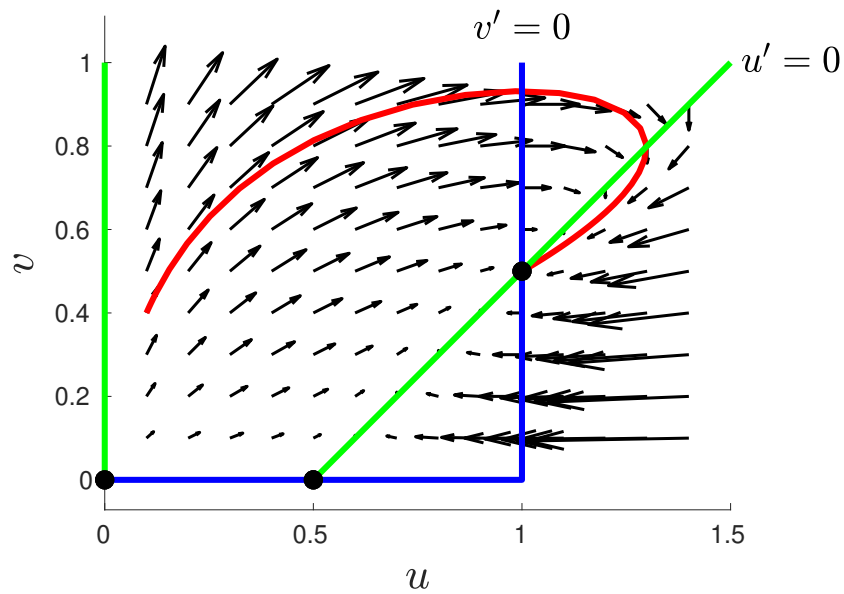
```

% solution curve
% specifying the differential equation in Matlab syntax
F = @(t,uv) [F1(uv(1),uv(2)); F2(uv(1),uv(2))];
% solution starting from (0.1,0.4)
[T,UV] = ode45(F,[0 100],[0.1; 0.4]);
plot(UV(:,1),UV(:,2),'-r','LineWidth',3)

% labels and lines u'=0, v'=0
...

axis equal

```



## 5.8 Animation of Incompressible Flow Around a Circular Island with Maple<sup>TM</sup>

Visualize the motion of a boat around a circular island with radius  $R = 1$  for several starting positions. Use the approximation

$$V(x, y) = \text{grad } U(x, y), \quad U(x, y) = \left(1 + \frac{R^2}{x^2 + y^2}\right) x$$

for the velocity field  $V$ , assuming that the width of the river is large compared to  $R$ .

**Resources:** [Systems of Differential Equations with Maple<sup>TM</sup>](#), [System of Differential Equations](#)

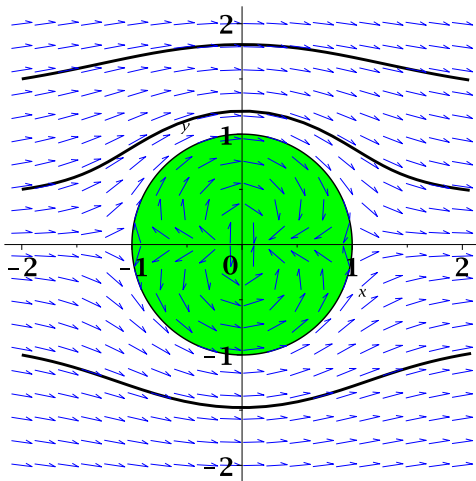


## Solution

### Computation of the boat trajectories

```
with(DEtools): with(plots): with(plottools):
# differential equations for the path x(t),y(t) of the boat
R := 1; U := (1+R^2/(x^2+y^2))*x
Vx := diff(U,x); Vy := diff(U,y)
DEx := diff(x(t),t) = subs(x=x(t),y=y(t),Vx)
DEy := diff(y(t),t) = subs(x=x(t),y=y(t),Vy)
# initial positions
S := {[x(0)=-2,y(0)=-1],[x(0)=-2,y(0)=1/2],[x(0)=-2,y(0)=3/2]}
# time interval and image window
T := t=0..5; X := x=-2..2; Y := y=-2..2
# plotting the island, the velocity field,
# and the boat trajectories
A := disk([0,0],R,color=green)
B := DEplot({DEx,DEy},{x(t),y(t)},T,X,Y,S, ...
... color=blue,linecolor=black,animatecurves=true)
display(A,B)
```

### Animation of the flow



By clicking on the image and choosing the options animation and play in the opening menu, the boat trajectories are successively displayed according to the velocity. The image on the left shows the last frame of this video sequence<sup>a</sup>.

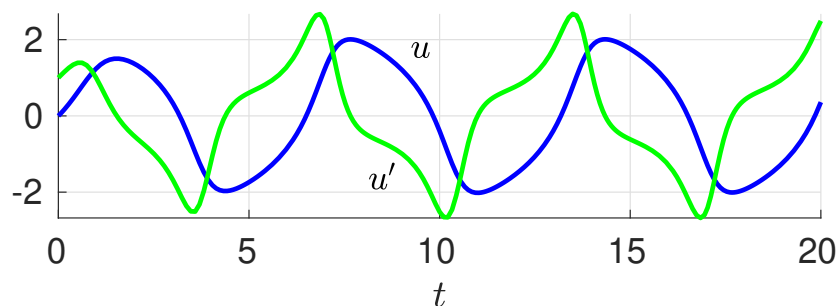
<sup>a</sup>The velocity field inside the island cannot be suppressed in an elegant fashion.

## 5.9 Limit Cycle of the Van der Pol Equation with MATLAB<sup>®</sup>

After a few initial oscillations, the solution  $u$  of the differential equation (model of a nonlinear oscillator)

$$u'' + (u^2 - 1)u' + u = 0$$

becomes periodic.



Determine the period  $T$  and initial values  $u_0, u'_0$  for this limit cycle.

**Resources:** [Differential Equations with MATLAB<sup>®</sup>](#), [System of Differential Equations](#), [Differentiation with Respect to Initial Conditions](#)

## Solution

### Periodicity condition

With the choice  $u_0 = 0$ , one has to solve a nonlinear system for the unknown derivative  $a = u'_0$  and the period  $T$ :

$$P(a, T) - (0, a)^t = (0, 0)^t. \quad (1)$$

In these equations,  $P$  denotes a function which computes  $(u(T), u'(T))^t$  for the initial values  $(u(0), u'(0))^t = (0, a)^t$ .

### Solution of the differential equation

Transforming the second order differential equation  $u'' = (1 - u^2)u' - u$  to standard form (system of differential equations of first order) by introducing the variables  $U = (u, u')^t$ ,

$$U' = \text{vdP}(U) = \begin{pmatrix} U_2 \\ (1 - U_1^2)U_2 - U_1 \end{pmatrix},$$

leads to the following MATLAB<sup>®</sup>-implementation.

```
function UT = P(aT)
a = aT(1); T = aT(2); % initial slope and period
% solution of the system of differential equations
% on the interval [0,T] with the initial values [0;a]
vdP = @(t,U) [U(2); (1-U(1)^2)*U(2)-U(1)];
[t,Ut] = ode45(vdP, [0;T], [0;a]);
% Ut(k,1) = U_1(t(k)), Ut(k,2) = U_2(t(k))
UT = Ut(end,:)'; % value and derivative at time T
end
```

### Solving the equations for periodicity

numerically minimizing the norm of the error

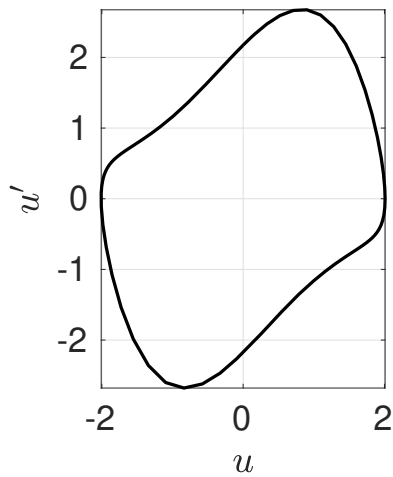
graphical determination of start values

```
a = 2 % slope of u at the zero near t = 13
T = 7 % time difference to the next zero near t = 20
aT = [a;T];
% solving equation (1) by minimizing the norm of the error
err = @(aT) norm(P(aT)-[0;aT(1)])
aT = fminsearch(err,aT)
% -> slope aT(1) and period aT(2)
2.1733 6.6655
```

Plot of the limit cycle in the phase plane

$u' = U_2$  as a function of  $u = U_1$

```
[t,Ut] = ode45(@vdP,[0;aT(2)],[0; aT(1)]);  
plot(Ut(:,1),Ut(:,2))
```

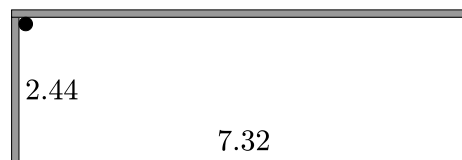


Alternative solution

It would be more elegant to solve the nonlinear system (1) with Newton's method. However, the implementation is substantially more difficult, since one has to solve additional systems of differential equations for the Jacobi matrices.

## 5.10 Goal of the Month with MATLAB<sup>®</sup>

At what angle  $\varphi$  does a soccer ball have to be shot from the penalty point with an initial velocity of 80 km/h in order to hit the goal in the upper left corner, just next to the post and underneath the crossbar?



In a simplified model<sup>5</sup>, the trajectory  $x(t)$  of a soccer ball during a penalty kick can be described by the differential equation

$$x'' = - \begin{pmatrix} 0 \\ g \end{pmatrix} - \gamma \sqrt{(x_1')^2 + (x_2')^2} x', \quad x(0) = \begin{pmatrix} 0 \\ r \end{pmatrix}, \quad x'(0) = v \begin{pmatrix} \cos \varphi \\ \sin \varphi \end{pmatrix},$$

where  $x_1$  is the horizontal distance of the center of the ball from the penalty point (located eleven meters in front of the goal), and  $x_2$  is the vertical coordinate of the ball position. The constant  $g \approx 9.81$  denotes the gravity of the earth, and  $\gamma \approx 0.0118$  is a proportionality factor for air resistance. Finally,  $r \approx 0.11$  is the radius of the soccer ball, and  $v = 80 \cdot 1000/60^2$  its initial velocity (units: meters [m] and seconds [s]).

**Resources:** [Differential Equations with MATLAB<sup>®</sup>](#), [System of Differential Equations](#)

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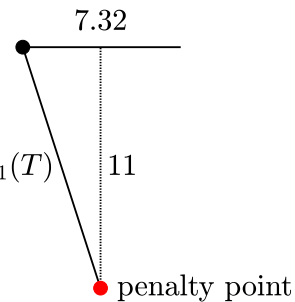
<sup>5</sup>T. Wilhelm, F. Zimmermann: *Die Luft beim Fußballflug*, Praxis der Naturwissenschaften - Physik in der Schule 63, Nr. 1, 2014, S. 28-37

## Solution

### Nonlinear system for the angle and the flight time

Solving the differential equation numerically yields the final ball position  $x(T)$  as a function  $f$  of the parameters  $p = (\varphi, T)^t$  (angle and flight time). The angle  $\varphi$  is then obtained by solving the nonlinear system

$$x(T) = f(p) \stackrel{!}{=} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} \sqrt{11^2 + (7.32/2 - 0.11)^2} \\ 2.44 - 0.11 \end{pmatrix} \approx \begin{pmatrix} 11.56 \\ 2.33 \end{pmatrix}, \quad x_1(T)$$



with  $e$  the position of the center of the ball in the upper left corner of the goal with respect to the penalty point (horizontal distance:  $x_1(T)$ , height:  $2.44 - r$ ).

### Standard form of the differential equation for the numerical solution

$u = (x_1, x_2, x'_1, x'_2)$  with additional variables  $u_3, u_4$  for the velocity  $x'$   $\rightsquigarrow$  first order system of differential equations

$$\begin{pmatrix} u'_1 \\ u'_2 \\ u'_3 \\ u'_4 \end{pmatrix} = \begin{pmatrix} u_3 \\ u_4 \\ -\gamma \sqrt{u_3^2 + u_4^2} u_3 \\ -g - \gamma \sqrt{u_3^2 + u_4^2} u_4 \end{pmatrix}$$

### MATLAB<sup>®</sup> -function $f$ , describing the nonlinear system

```
function e = f(p)
% p: angle phi and flight time T
% e: final ball position

% parameters
gamma = 0.0118; g = 9.81; r = 0.11;
v = 80000/3600; % conversion km/h -> m/s
phi = p(1); T = p(2);

% right-hand side of the differential equation
u_prime = @(t,u) [u(3); u(4); ...
    -gamma*norm(u(3:4))*u(3); -g-gamma*norm(u(3:4))*u(4)];
```

```
% initial values
u_0 = [0; r; v*cos(phi); v*sin(phi)];
```

```
% numerical solution
[t,u] = ode45(u_prime,[0,T],u_0);
e = u(end,1:2)'; % final ball position
end
```

#### Solution of the nonlinear system

initial approximation  $p_s = (\varphi_s, T_s)^t$  based on a straight shot, ignoring gravitation and air resistance:

difference  $d = e - (0, r)^t \approx (11.56, 2.22)^t$  of target position and penalty point  
 $\rightsquigarrow$

$$\varphi_s = \arctan(d_2/d_1) \approx 0.180, \quad T_s = |d|/v = 0.530$$

solution of  $f(p) = e$  by minimizing the least squares error  $\Delta_e = |f(p) - e|^2$   
 $\rightsquigarrow$  MATLAB<sup>®</sup> script

```
% initial approximation
r = 0.11; v = 80000/3600;
e = [sqrt(11^2+(7.32/2-r)^2); 2.44-r]; d = e-[0;r];
phi_s = atan(d(2)/d(1)), T_s = norm(d)/v,
```

```
% minimization of the least squares error
delta_e = @(p) norm(f(p)-e)^2;
p = fminsearch(delta_e,[phi_s;T_s]);
phi = p(1), T = p(2)
```

$$\text{phi} = 0.3209, \quad \text{T} = 0.5885$$

The angle  $\varphi$  is almost twice as large as the angle  $\varphi_s$  of a straight line from the penalty point to the corner of the goal.

## 5.11 Three Body Problem with MATLAB<sup>®</sup>

The trajectories  $t \mapsto P_k(t) \in \mathbb{R}^3$ ,  $k = 1, 2, 3$ , of three planets subject to their gravitational forces satisfy the differential equations

$$P_k'' = \sum_{j \neq k} m_j \frac{P_j - P_k}{|P_j - P_k|^3},$$

if the masses  $m_j$  are normalized appropriately. Solve this system numerically for  $0 \leq t \leq 4$ ,  $m = (10, 10, 1)$ , and the two (almost identical) initial values

$P_1(0)$	$P_2(0)$	$P_3(0)$	$P_1'(0)$	$P_2'(0)$	$P_3'(0)$
$(1, 0, 0)$	$(-1, 0, 0)$	$(\mathbf{0}, 0, 0)$	$(0, 1, 0)$	$(0, -1, 0)$	$(0, 0, 1)$
$(1, 0, 0)$	$(-1, 0, 0)$	$(\mathbf{0.1}, 0, 0)$	$(0, 1, 0)$	$(0, -1, 0)$	$(0, 0, 1)$

Plot the trajectories in both cases.

**Resources:** [Differential Equations with MATLAB<sup>®</sup>](#), [System of Differential Equations](#)



## Solution

### Transformation to standard form

conversion of the second order differential equations to a first order system for the vector

$$u := (P_1, P_2, P_3, P'_1, P'_2, P'_3) \in \mathbb{R}^{18},$$

i.e.  $(u_1, u_2, u_3) = P_1$ ,  $(u_4, u_5, u_6) = P_2$ , ..., and  $(u_{10}, u_{11}, u_{12}) = P'_1$ , ...

- coupling of position  $P_k$  and velocity  $P'_k \rightsquigarrow$

$$(u_1, u_2, u_3)' = (u_{10}, u_{11}, u_{12}), \quad (u_4, u_5, u_6)' = (u_{13}, u_{14}, u_{15}), \quad \dots \quad (1)$$

- equations  $P''_1 = (P'_1)' \rightsquigarrow$

$$(u_{10}, u_{11}, u_{12})' = m_1 \underbrace{\frac{(u_4, u_5, u_6) - (u_1, u_2, u_3)}{|(u_4, u_5, u_6) - (u_1, u_2, u_3)|^3}}_{=: d_{2,1}/n_{2,1}^3} + m_3 d_{3,1}/n_{3,1}^3, \quad (2)$$

with analogous differential equations for  $(u_{13}, u_{14}, u_{15})'$  and  $(u_{16}, u_{17}, u_{18})'$

combining (1) and (2)  $\rightsquigarrow$  first order system

$$(u_1, \dots, u_{18})' = (f_1(u), \dots, f_{18}(u))$$

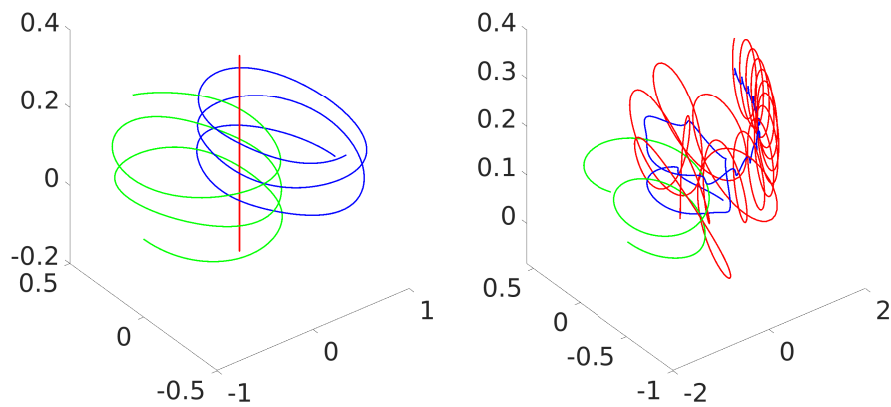
### MATLAB<sup>®</sup> function $f$

```
function Du = f(t,u)
% f(u)->f(t,u) since the Matlab solver requires a t-dependence
m = [10; 10; 1];
% djk = P_j-P_k and njk = |P_j-P_k|
d21 = u(4:6)-u(1:3); n21 = norm(d21);
d31 = u(7:9)-u(1:3); n31 = norm(d31);
d32 = u(7:9)-u(4:6); n32 = norm(d32);

% derivatives
Du = [u(10:18); ...
      m(2)*d21/n21^3+m(3)*d31/n31^3; ...
      -m(1)*d21/n21^3+m(3)*d32/n32^3; ...
      -m(1)*d31/n31^3-m(2)*d32/n32^3];
```

## Numerical solution and plot of the trajectories

```
% first initial values
u0 = [1,0,0, -1,0,0, 0,0,0, 0,1,0, 0,-1,0, 0,0,1];
% numerical solution of the system of differential equations
% output: u(k,1:18) = solution at time t(k)
[t,u] = ode45(@f_3body,[0,4],u0);
% plot of the three trajectories (blue, green, red)
subplot(1,2,1) % left image
plot3(u(:,1),u(:,2),u(:,3),'-b',u(:,4),u(:,5),u(:,6),'-g', ...
      u(:,7),u(:,8),u(:,9),'-r');
% second initial values
u0 = [1,0,0, -1,0,0, 0.1,0,0, 0,1,0, 0,-1,0, 0,0,1];
...
subplot(1,2,2) % right image
...
```



small perturbation with dramatic effect: periodic up and down motion of the small planet → chaotic orbit!

## 5.12 Return to Earth with MATLAB<sup>®</sup>

The position  $p(t)$  and velocity  $v(t)$  ( $|v| > 0$ ) of a space ship can be modeled by the differential equations

$$p' = v, \quad v' = -Gp/|p|^3 - Av/|v|,$$

with  $G = 3.9860 \cdot 10^{14}$ , the product of the gravitational constant and the mass of the earth, and  $-A$  the negative acceleration (units: meters [m] and seconds [s]).

Write a MATLAB<sup>®</sup> function `space_shuttle(R)` which simulates a flight back to the earth, starting from a circular orbit with radius  $R \gg r = 6.378 \cdot 10^6$  (radius of the earth). The interactive program should permit to gradually increase or reduce  $A$  by clicking the left and right arrow keys ( $A = n_A A_{\min}$ ,  $n_A \in \mathbb{N}_0$ ).

**Resources:** [Differential Equations with MATLAB<sup>®</sup>](#), [System of Differential Equations](#)

## Solution

### Program description

The function `space_shuttle(R)` first defines the relevant constants and parameters. Then, a graphic window is initialized, the earth is plotted, and a function `key_action` for reacting on clicks of the arrow keys is activated. The subsequent `while`-loop iterates time-steps,  $t \rightarrow t + \Delta t$ , of Euler's approximation<sup>6</sup>, i.e.

$$\begin{aligned} p(t + \Delta t) &= p(t) + \Delta t v(t), \\ v(t + \Delta t) &= v(t) + \Delta t (-Gp(t)/|p(t)|^3 - Av(t)/|v(t)|), \end{aligned}$$

and draws the segment of the flight trajectory for the interval  $[t, t + \Delta t]$ . Changes of the acceleration  $-A$  by invoking `key_action` ( $A = n_A A_{\min}$ ,  $n_A \rightarrow n_A \pm 1$ ), are accessible to every program segment by storing  $n_A$  as a global variable.

### MATLAB<sup>®</sup> implementation

```
function space_shuttle(R)
% R: radius of the initial circular orbit, e.g. 2 x 10^7
global n_A

% constants (units [m,s]) and parameters
G = 3.9860e14; % gravitational constant * mass of the earth
r = 6.378e6; g = G/(r^2); % radius of the earth, gravity
Amin = g/20; n_A = 0; % acceleration increment and factor
dt = 2*pi*sqrt(R^3/G)/2000; % time-step: orbital period/2000
dt_s = dt/200; % scaling of the image (time-lapse)
p = [R; 0]; % start position
v = [0; sqrt(G/R)]; % initial velocity

% initialization of the image, plotting of the earth
figure; axis equal; hold on
fill(r*cos(linspace(0,2*pi)),r*sin(linspace(0,2*pi)),'b')

% initializing the function for reacting on clicks of arrow keys
set(gcf,'KeyPressFcn',@key_action)

while norm(p) > r % not yet landed
```

---

<sup>6</sup>chosen, to keep the program as simple as possible

```

% time step
p_new = p + dt*v;
v = v+dt*(-G*p/norm(p)^3-n_A*Amin*v/(10*eps+norm(v)));
% Adding 10*eps avoids a division by 0.

% plotting a segment of the flight path
plot([p(1) p_new(1)], [p(2) p_new(2)], 'r');
p = p_new;

% current velocity and acceleration
vt = num2str(round(norm(v))); At = num2str(n_A*Amin/g);
xlabel(['|v|: ' vt ', A: ' At 'g'])
pause(dt_s)
end

function key_action(h,evt)
% h: graphics handle
% evt: structure with the field .Key for a clicked arrow key
global n_A

switch evt.Key
case 'leftarrow'; % increasing acceleration
    n_A = n_A+1;
case 'rightarrow'; % reducing acceleration
    n_A = max(n_A-1,0);
end
end

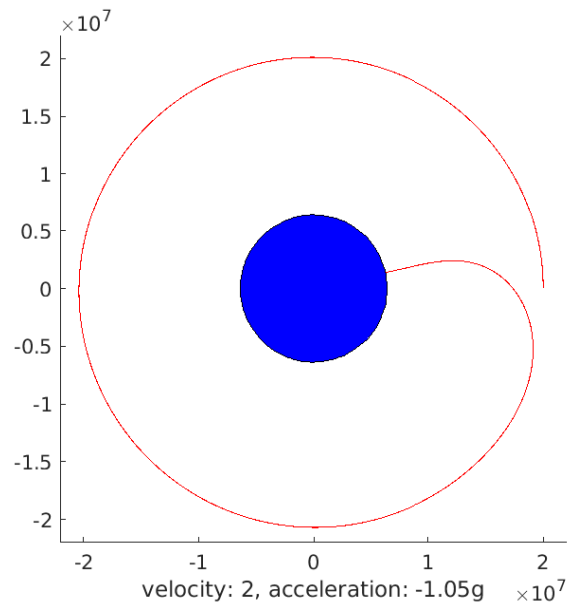
```

### Example of a simulation

In the flight simulation, shown in the figure, the space ship first stays for 25000 seconds in the initial circular orbit ( $n_A = 0$ , no arrow keys are clicked). Then it slows down. For a smooth landing (acceptable final velocity) the negative acceleration has to be adapted to the increasing gravity of the earth. Manually, this is **very** difficult. The depicted landing approach with the relatively low final velocity  $v = 2$  m/s is the result of an autopilot<sup>7</sup>, an additional program segment which updates the acceleration automatically.

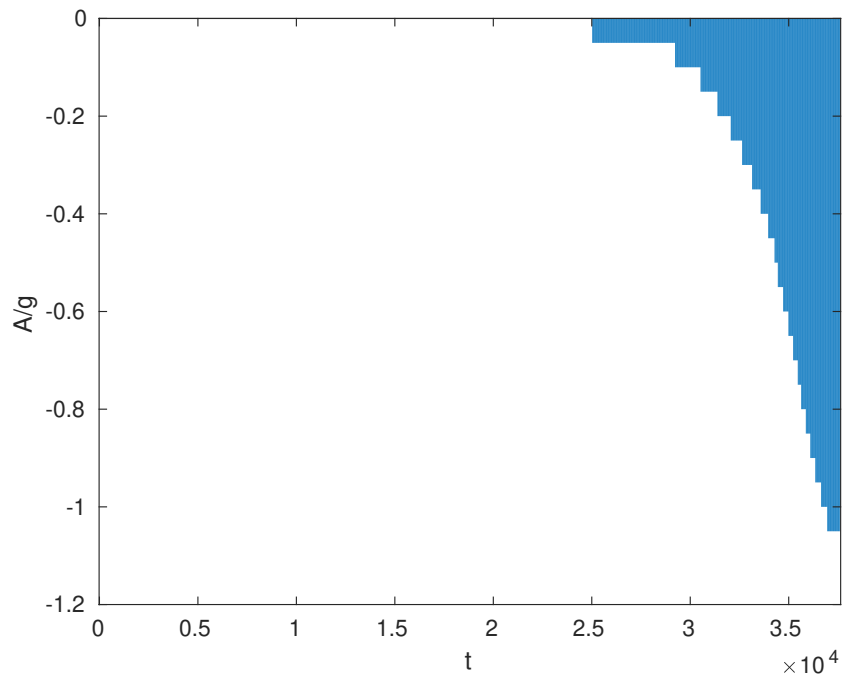
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<sup>7</sup>You should also program an automatic control of the acceleration, if you are not satisfied with your attempts to land manually (as were the authors!).



Acceleration history

The following bar graph shows the acceleration  $A$ , measured in terms of multiples of the gravity  $g$ , as a function of the flight time  $t$ , and documents the exact acceleration history.



# Chapter 6

## Lexicon

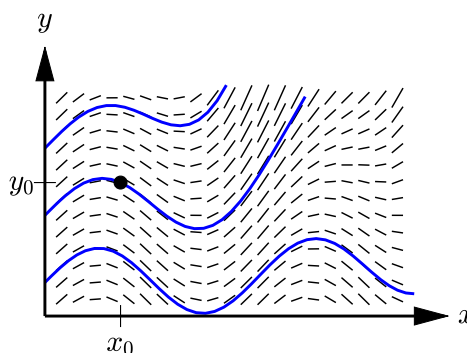
## 6.1 First Order Differential Equations

### First Order Differential Equation

$$y'(x) = f(x, y(x)), \quad \underbrace{y(x_0) = y_0}_{\text{initial condition}}$$

unique solution on an interval  $(x_0 - \delta, x_0 + \delta)$  for continuously differentiable  $f$

**Geometric interpretation:**  
prescribed slopes/tangents of the solutions at each point  $(x, y)$



### Differential Equation for the Exponential Function

$$y' = py + q, \quad p, q \in \mathbb{R}, p \neq 0$$

general solution:  $y(x) = c e^{px} - q/p$

initial value  $y(x_0) = y_0 \rightsquigarrow$  integration constant  $c = (y_0 + q/p) e^{-px_0}$  and

$$y(x) = (y_0 + q/p) e^{p(x-x_0)} - q/p$$

$p < 0 \implies \lim_{x \rightarrow \infty} y(x) = -q/p$  for any initial value

### First Order Linear Differential Equation

$$y'(x) = p(x)y(x) + f(x)$$

general solution:  $y = y_h + y_p$  with

$$y_h(x) = c \exp(P(x)), \quad P(x) = \int p(x) dx$$

the general solution of the homogeneous differential equation  $y' = py$  and

$$y_p(x) = \int_{x_0}^x \exp(P(x) - P(\xi)) f(\xi) d\xi$$



a particular solution, which can be constructed with the ansatz  $y_p(x) = C(x) \exp(P(x))$  (*variation of constants*)

initial value  $y(x_0) = y_0 \implies c = y_0 \exp(-P(x_0))$

**Special case:** constant coefficient  $p$

$$y(x) = y_0 \exp(p(x - x_0)) + \int_{x_0}^x \exp(p(x - \xi)) f(\xi) d\xi$$

### Method of Undetermined Coefficients

particular solutions  $y_p$  of the differential equation

$$y'(x) - py(x) = f(x), \quad p \neq 0,$$

for special right-hand sides  $f$

$f(x)$	$y_p(x)$
$\sum_{k=0}^n a_k x^k$	$\sum_{k=0}^n b_k x^k$
$a e^{\lambda x}, \lambda \neq p$	$\frac{a}{\lambda - p} e^{\lambda x}$
$a e^{px}$ ( <i>resonance</i> )	$ax e^{px}$
$a \cos(\omega x) + b \sin(\omega x)$	$\tilde{a} \cos(\omega x) + \tilde{b} \sin(\omega x)$

general solution:  $y(x) = y_p(x) + c e^{px}$

### Separable Differential Equation

$$g(y(x))y'(x) = f(x)$$

indefinite integrals of  $f$  and  $g \rightsquigarrow$  implicit form of the general solution

$$G(y(x)) = F(x) + c, \quad F' = f, G' = g$$

initial value  $y_0 = y(x_0) \rightsquigarrow$  integration constant  $c = G(y_0) - F(x_0)$

### Bernoulli Differential Equation

$$y' + py = qy^k, \quad k \neq 0, 1$$

substitution  $z = y^{1-k} \rightsquigarrow$  linear differential equation

$$z' = p(k-1)z + (1-k)q$$

**Special case:**  $p, q$  constant

$$y(x) = \left( \frac{q}{p} + c \exp(p(k-1)x) \right)^{\frac{1}{1-k}}$$

### Linear Substitution

$$y' = f(ax + by + cy) \xrightarrow{z=ax+by+cy} \frac{z'}{b+cf(z)} = 1 \quad (\text{separable})$$

### Differential Equation with Homogeneous Right-hand Side

$$y'(x) = f(y(x)/x)$$

substitution  $xz(x) = y(x) \rightsquigarrow$  separable differential equation

$$xz'(x) = f(z(x)) - z(x) \iff \frac{z'}{f(z) - z} = x$$

### Exact Differential Equation

$$p(x, y) + q(x, y)y' = 0 \iff p dx + q dy = 0, \quad p = \partial_x F, \quad q = \partial_y F$$

$\rightsquigarrow$  implicit solution  $F(x, y) = c$

necessary condition for integrability (existence of  $F$  with  $\text{grad } F = (p, q)^t$ ):

$$\partial_y p = \partial_x q$$

sufficient on a simply connected domain

**Construction of  $F$ :** Integrating  $F_x(x, y) = p(x, y)$  with respect to  $x$  yields  $F(x, y) = P(x, y) + c(y)$ , and  $c$  is obtained by integrating  $c'(y) = q(x, y) - P_y(x, y)$  with respect to  $y$ .

### Integrating Factor

$$p dx + q dy = 0 \xrightarrow{\text{multiply with an integrating factor } a} \underbrace{(ap) dx + (aq) dy = 0}_{\text{exact differential equation}}$$

i.e.  $a$  is chosen so that  $\partial_y(ap) = \partial_x(aq)$

### Discretization Error

error  $\Delta$  of a numerical scheme

$$\tilde{y}(x+h) = \tilde{y}(x) + hf(x, \tilde{y}(x), \tilde{y}(x+h))$$

for approximating a differential equation  $y'(x) = f(x, y(x))$ :

$$\Delta(x, h) = y(x+h) - y(x) - hf(x, y(x), y(x+h)),$$

i.e.  $\Delta$  is the discrepancy, when substituting an exact solution  $y$  of the differential equation into the difference approximation

### Differential Equations with Maple™

```
with (DEtools) # load the differential equations package

# analytic solution
dsolve(de) # general solution of the differential equation de
dsolve({de,c}) # impose the initial/boundary conditions c
# example: y''+y'=y+x, y(0)=0, y'(1)=1
de := diff(y(x),x,x)+diff(y(x),x)=y(x)+x
c := y(0)=0,D(y)(1)=1

# numerical solution
sol := dsolve({de,c},numeric)
# -> procedure for evaluating the solution
# example: y' = cos(x*y), y(0)=0
sol := dsolve({diff(y(x),x)=cos(x*y(x)), y(0)=0},numeric)
sol(1) # -> value at x=1
```

## 6.2 Second Order Differential Equations

### Linear Oscillator

$$u''(t) + \omega_0^2 u(t) = c \cos(\omega t), \quad \omega_0 > 0$$

superposition of free and forced oscillations:  $u = u_h + u_p$  with

$$u_h(t) = a \cos(\omega_0 t) + b \sin(\omega_0 t)$$
$$u_p(t) = \begin{cases} \frac{c}{\omega^2 - \omega_0^2} (\cos(\omega_0 t) - \cos(\omega t)), & \omega \neq \omega_0 \\ \frac{c}{2\omega} t \sin(\omega t), & \omega = \omega_0 \quad (\text{resonance}) \end{cases}$$

initial conditions  $\implies a = u(0), b = u'(0)/\omega_0$

### Homogeneous Second Order Differential Equation with Constant Coefficients

$$u''(t) + pu'(t) + qu(t) = 0$$

type of solution determined by the zeros of the *characteristic polynomial*  
 $\lambda^2 + p\lambda + q$

- two real zeros  $\lambda_1 \neq \lambda_2$ :  $u(t) = ae^{\lambda_1 t} + be^{\lambda_2 t}$
- a double zero  $\lambda$ :  $u(t) = (a + bt)e^{\lambda t}$
- two complex conjugate zeros  $-p/2 \pm \varrho i$ :  $u(t) = e^{-pt/2}(a \cos(\varrho t) + b \sin(\varrho t))$

initial conditions  $u(0) = u_0, u'(0) = u'_0 \rightsquigarrow$  linear system for the constants  $a$  and  $b$

### Method of Undetermined Coefficients

particular solutions  $u_p$  for the differential equation

$$u''(t) + pu'(t) + qu(t) = f(t)$$

- polynomials  $f(t) = \sum_{k=0}^n a_k t^k \rightsquigarrow u_p(t) = \sum_{k=0}^n b_k t^k$ , if  $q \neq 0$   
multiplication of  $\sum_{k=0}^n \dots$  with  $t$  ( $t^2$ ), if  $q = 0$  ( $q = p = 0$ )

- exponential functions  $f(t) = e^{\lambda t} \rightsquigarrow u_p(t) = c e^{\lambda t}$ , if  $\lambda^2 + p\lambda + q \neq 0$   
multiplication of  $c e^{\lambda t}$  with  $t$  ( $t^2$ ), if  $\lambda$  is a simple (double) root of the characteristic polynomial ( $u = f$  solves the homogeneous differential equation  $u'' + pu' + qu = 0$  in this case)
- trigonometric functions  $f(t) = e^{\alpha t}(a \sin(\omega t) + b \cos(\omega t)) \rightsquigarrow u_p(t) = e^{\alpha t}(\tilde{a} \sin(\omega t) + \tilde{b} \cos(\omega t))$   
multiplication of  $e^{\alpha t}(\dots)$  with  $t$ , if  $\alpha \pm i\omega$  are the zeros of the characteristic polynomial

superposition of particular solutions  $u_p$  for a sum  $f$  of different terms

### Euler's Differential Equation

$$t^2 u''(t) + pu'(t) + qu(t) = 0, \quad t > 0$$

ansatz  $u(t) = t^\lambda \rightsquigarrow \lambda(\lambda - 1) + p\lambda + q = 0$

general solution

- two real exponents  $\lambda_1, \lambda_2$ :  $u(t) = c_1 t^{\lambda_1} + c_2 t^{\lambda_2}$
- a single real exponent  $\lambda$ :  $u(t) = t^\lambda(c_1 + c_2 \ln t)$
- complex conjugate exponents  $r \pm si$ :  $u(t) = t^r(c_1 \cos(s \ln t) + c_2 \sin(s \ln t))$

**Alternative:** substituting  $t = e^\tau$ ,  $u(t) = v(\tau) \rightsquigarrow$

$$v'' + (p - 1)v' + qv = 0$$

### Forced Damped Oscillations

$$u''(t) + 2ru'(t) + \omega_0^2 u(t) = c \cos(\omega t), \quad r > 0$$

qualitative behavior determined by the solutions  $u_h$  of the homogeneous differential equation ( $c = 0$ )

- strong damping ( $r > \omega_0$ ):  $u_h = a e^{\lambda_1 t} + b e^{\lambda_2 t}$ ,  $\lambda_{1,2} = -r \pm \sqrt{r^2 - \omega_0^2}$
- critical damping ( $r = \omega_0$ ):  $u_h = (a + bt)e^{-rt}$

- weak damping ( $r < \omega_0$ ):  $u_h = e^{-rt}(a \cos(\lambda t) + b \sin(\lambda t))$ ,  $\lambda = \sqrt{\omega_0^2 - r^2}$

particular solution (periodic)

$$u_p(t) = C \cos(\omega t + \delta)$$

with amplitude  $C = c/\sqrt{(\omega_0^2 - \omega^2)^2 + (2r\omega)^2}$  and phase  $\delta = \arg(\omega_0^2 - \omega^2 - i2r\omega)$

general solution:  $u = u_p + u_h$

**Resonance frequency:** maximizes  $C$

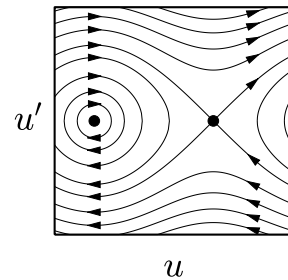
$$\omega_* = \sqrt{\omega_0^2 - 2r^2} \quad \text{for } r < \omega_0/\sqrt{2}$$

### Phase Plane

autonomous differential equation

$$u'' = f(u, u')$$

- unique solution curves  $t \mapsto (u(t), u'(t))$  for any initial point  $(u_0, u'_0)$
- critical points: constant solutions  $u(t) \equiv u_0$ , reached as limits only
- curve intersection only at critical points



differential equation for  $v(u) = u'(t)$ :  $u''(t) = du'(t)/dt = (du'(t)/du)(du/dt)$   
 $\rightsquigarrow$

$$\frac{dv}{du}v = f(u, v)$$

### Conservation of Energy

one-dimensional motion  $t \mapsto u(t)$  with velocity  $v = u'$ , governed by a potential  $P$

$$u'' + P'(u) = 0$$

constant sum of kinetic and potential energy

$$E(u, v) = \frac{1}{2}v^2 + P(u), \quad v = u'$$

$\rightsquigarrow$  implicit representation of solution curves in the phase plane

## 6.3 Systems of Differential Equations

### System of Differential Equations

standard form:

$$u'(t) = f(t, u(t)), \quad u(t_0) = a, \quad u = (u_1, \dots, u_n)^t, \quad f : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n$$

transformation to first order using additional variables for higher derivatives, e.g.  $u_1 = y$ ,  $u_2 = y'$ ,  $u_3 = y''$ , etc.

- existence of a solution in a neighborhood of  $t_0$  for continuous  $f$  (Theorem of Peano)
- uniqueness for Lipschitz continuous  $f$  (Picard-Iteration), i.e. if

$$|f(t, u) - f(t, \tilde{u})| \leq L|u - \tilde{u}|$$

### Differentiation with Respect to Initial Conditions

differentiating the initial value problem

$$u'(t) = f(t, u(t)), \quad u(t_0) = (a_1, \dots, a_n)^t \quad (1)$$

with respect to  $a \rightsquigarrow$

$$u'_a(t) = f_u(t, u(t))u_a(t), \quad u_a(t_0) = E, \quad (2)$$

with the Jacobi matrix  $u_a = (\partial u / \partial a_1, \dots, \partial u / \partial a_n)$ , the Jacobi matrix  $f_u$  of  $f$  with respect to the second argument, and the  $(n \times n)$  unit matrix  $E$   
 (1),(2): coupled system of  $n + n^2$  differential equations for  $u$  and  $u_a$

### Linear System of Differential Equations

$$u'(t) = \underbrace{A(t)}_{n \times n} u(t) + b(t)$$

general solution:  $u(t) = u_p(t) + \Gamma(t)c$  with a particular solution  $u_p$  and a *fundamental matrix*  $\Gamma$ , the rows of which contain  $n$  linearly independent solutions of the homogeneous system  $u' = Au$

initial condition  $u(t_0) = a \rightsquigarrow c = \Gamma(t_0)^{-1}(u(t_0) - u_p(t_0))$

### Wronski Determinant

determinant of a *fundamental matrix*  $\Gamma$  of a linear system of differential equations

$$\Gamma' = A\Gamma \quad \Longrightarrow \quad (\det \Gamma)' = \text{trace } A (\det \Gamma), \text{ i.e.}$$

$$\det \Gamma(t) = \det \Gamma(t_0) \exp \left( \int_{t_0}^t \text{trace } A(s) \, ds \right)$$

### Variation of Constants

solving a linear system of differential equations,

$$u'(t) = A(t)u(t) + b(t),$$

using the ansatz  $u(t) = \Gamma(t)c(t)$  with a *fundamental matrix*  $\Gamma$  ( $\Gamma' = A\Gamma$ )  $\rightsquigarrow$

$$u(t) = u_h(t) + u_p(t) = \Gamma(t) \left( \Gamma(t_0)^{-1}u(t_0) + \int_{t_0}^t \Gamma(s)^{-1}b(s) \, ds \right)$$

### Diagonalization of a Linear System of Differential Equations

eigensolutions:

$$Av = \lambda v, \quad u(t) = e^{\lambda t} v \quad \Longrightarrow \quad u'(t) = Au(t)$$

The existence of a basis of eigenvectors  $v_k$  permits the diagonalization of  $A$ ,

$$Q^{-1}AQ = \text{diag}(\lambda_1, \dots, \lambda_n), \quad Q = (v_1, \dots, v_n),$$

and a decoupling of the inhomogeneous system of differential equations  $u'(t) = Au(t) + b(t)$ :

$$d'_k(t) = \lambda_k d_k(t) + c_k(t), \quad u(t) = Qd(t), \quad c = Q^{-1}b$$

with the solution

$$d_k(t) = e^{\lambda_k t} \left( e^{-\lambda_k t_0} d_k(t_0) + \int_{t_0}^t e^{-\lambda_k s} c_k(s) \, ds \right).$$



**Special case:** general solution of the homogeneous system  $u' = Au$

$$u(t) = \sum_k \gamma_k v_k e^{-\lambda_k t}$$

with the coefficients  $\gamma_k$  determined by an initial condition

### Jordan Form of a System of Linear Differential Equations

applying the transformations

$$J = Q^{-1}AQ, \quad u(t) = Qv(t), \quad c(t) = Q^{-1}b(t)$$

to  $u'(t) = Au(t) + b(t) \rightsquigarrow$

$$\begin{aligned} v'_n(t) &= \lambda_n v_n(t) + c_n(t) \\ v'_{n-1}(t) &= \lambda_{n-1} v_{n-1}(t) + \varrho_n v_n(t) + c_{n-1}(t) \\ &\vdots \\ v'_1(t) &= \lambda_1 v_1(t) + \varrho_2 v_2(t) + c_1(t) \end{aligned}$$

with  $\lambda_k$  the eigenvalues of  $A$  (diagonal elements of  $J$ ) and  $\varrho_k \in \{0, 1\}$   
 bidiagonal form of the system of differential equations  $\rightsquigarrow$  successive de-  
 termination of  $v_k$  starting with the  $n$ -th component

### Elimination for a Linear System of Differential Equations

elimination of  $u(t)$  from the system of differential equations

$$\begin{aligned} E_u : u' + au &= f(t, v) \\ E_v : v' + bu &= g(t, v) \end{aligned}$$

by forming the difference  $D : bE_u - \left(\frac{d}{dt} + a\right) E_v \rightsquigarrow$  second order differ-  
 ential equation for  $v(t)$ :

$$D : -v''(t) - av'(t) = bf(t, v(t)) - \frac{d}{dt}g(t, v(t)) - ag(t, v(t))$$

**Equivalent alternative:** For  $b \neq 0$ , solve  $E_v$  for  $u(t)$  and substitute into  $E_u$ .

For a system of  $n$  first order differential equations  $u' = Au + f$  a repetition of this procedure yields an  $n$ -th order differential equation for one of the components  $u_k$ . Substituting its solution, the dimension of the system can be reduced.

## Stability of Linear Systems of Differential Equations

stability of  $u'(t) = Au(t)$ , characterized in terms of the eigenvalues  $\lambda_k$  of the matrix  $A$

- stable, i.e.  $\lim_{t \rightarrow \infty} |u(t)| = 0$  for all initial values  $u(0)$   
 $\iff \operatorname{Re} \lambda_k < 0$ <sup>1</sup>
- neutrally stable, i.e.  $|u(t)| \leq c$ , and there exist initial values for which  $|u(t)| \not\rightarrow 0$   
 $\iff \operatorname{Re} \lambda_k \leq 0, \exists \lambda_\ell \text{ with } \operatorname{Re} \lambda_\ell = 0 \text{ and } \operatorname{Re} \lambda_\ell = 0 \implies \text{equal geometric and algebraic multiplicity}$
- unstable, i.e.  $\lim_{t \rightarrow \infty} |u(t)| = \infty$  for some initial value  $u(0)$   
 $\iff \exists \lambda_\ell \text{ with } \operatorname{Re} \lambda_\ell > 0 \text{ or with } \operatorname{Re} \lambda_\ell = 0 \text{ and smaller geometric than algebraic multiplicity}$

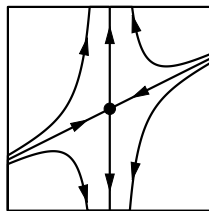
**Special case of two-dimensional systems:  $A : 2 \times 2$**

stability  $\iff \det A > 0 \wedge \operatorname{trace} A < 0$

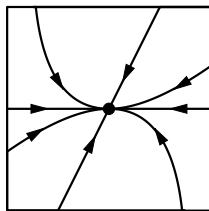
stable knot (stable spiral):  $\det A \leq ((\operatorname{trace} A)/2)^2 (\dots > \dots)$

The figures show typical examples of qualitatively different nondegenerate cases (no eigenvalue equal to zero), which can be classified with the Jordan form of  $A$ ,

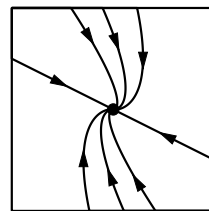
$$J = \begin{pmatrix} \lambda & s \\ 0 & \varrho \end{pmatrix}, \quad s \in \{0, 1\}$$



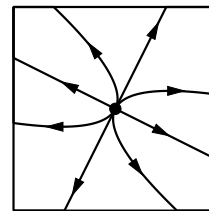
unstable saddle  
 $\lambda \varrho < 0$



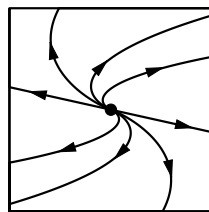
stable knot  
 $\lambda < 0, \varrho < 0, s = 0$



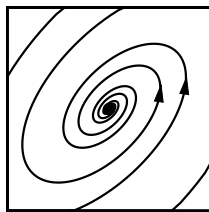
stable knot  
degenerate:  $\lambda < 0, s = 1$



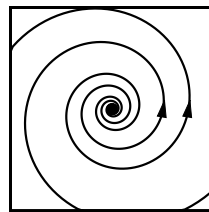
unstable knot  
 $\lambda > 0, \varrho > 0, s = 0$



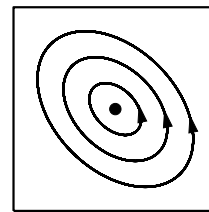
unstable knot  
degenerate:  $\lambda > 0, s = 1$



stable spiral  
 $\lambda = r + i\omega = \bar{\varrho}, r < 0$



unstable spiral  
 $\lambda = r + i\omega = \bar{\varrho}, r > 0$



neutral center  
 $\lambda = i\omega = \bar{\varrho}$

<sup>1</sup>the simplest and most important condition

## Stability of Autonomous Systems of Differential Equations

stability of  $u' = f(u)$  for a critical point  $u_*$  ( $f(u_*) = (0, \dots, 0)^t$ ), characterized in terms of the linearization

$$v' = f'(u_*)v, \quad v(t) = u(t) - u_*$$

stability of  $u_*$ , i.e.  $\lim_{t \rightarrow \infty} u(t) = u_*$  for all initial values in a neighborhood of  $u_*$   $\iff \operatorname{Re} \lambda < 0$  for all eigenvalues  $\lambda$  of the Jacobi matrix  $f'(u_*)$   
classification (stable knot or spiral) analogous to the approximating linear system

## Differential Equations with MATLAB<sup>®</sup>

```
[T,U] = ode45(f,[t0,tend],u0)

% solves u'(t) = f(t,u(t)), u(t0) = u0 on the interval [t0,tend]
% f = @(t,u) ... function handle
%   scalar or column vector, depending on the dimension of u
% T: times where the solution u is evaluated
% U(k,:): solution at T(k)

% example: u1' = u2, u2'=t+u1, u(0) = (1,2)
f = @(t,u) [u(2); t+u(1)], u0 = [1;2]

alternative solvers: ode23, ode113, ...
```

## Systems of Differential Equations with Maple<sup>™</sup>

```
with (DEtools) # load the differential equations package

# analytical solution
dsolve({de_x,de_y}) % solves a two dimensional system
dsolve({de_x,de_y,ic_x,ic_y}) # imposes initial conditions

# numerical solution, returns an evaluation procedure
sol := dsolve({de_x,de_y,ic_x,ic_y},numeric)
sol(t) # -> [t,x(t),y(t)]

# plot the solution curves for several initial conditions
DEplot({de_x,de_y},{x(t),y(t)},trange,{[ic_x,ic_y], ...})
```

```
# sample data
de_x := diff(x(t),t) = t*y(t); de_y := diff(y(t),t) = -t*x(t)
ic_x := x(0) = 1; ic_y := y(0) = 0
trange := t=0..2
```

## 6.4 Laplace Transform

### Laplace Transform

$$\mathcal{L} : u \mapsto U = \mathcal{L}u$$

$$U(s) = \int_0^{\infty} u(t)e^{-st} dt, \quad \operatorname{Re} s \geq a, \quad u(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} U(s)e^{st} ds, \quad b \geq a$$

if  $|u(t)e^{-at}|$  is integrable on  $[0, \infty)$

### Laplace Transform of Exponential Functions

$$u(t) = t^n e^{at} \xrightarrow{\mathcal{L}} U(s) = \frac{n!}{(s-a)^{n+1}}, \quad \operatorname{Re} s > \operatorname{Re} a$$

$a = \lambda \pm i\omega$ , formula of Euler-Moivre  $\rightsquigarrow$

$$e^{\lambda t} \cos(\omega t) \longrightarrow \frac{s-\lambda}{(s-\lambda)^2 + \omega^2}, \quad e^{\lambda t} \sin(\omega t) \longrightarrow \frac{\omega}{(s-\lambda)^2 + \omega^2}$$

### Translation and Scaling of Laplace Transforms

- $u(t-a) \xrightarrow{\mathcal{L}} e^{-as}U(s)$ ,  $a > 0$  and  $u(t) = 0$  for  $t \leq 0$
- $e^{at}u(t) \xrightarrow{\mathcal{L}} U(s-a)$
- $u(at) \xrightarrow{\mathcal{L}} a^{-1}U(s/a)$ ,  $a > 0$

### Laplace Transform of Periodic Functions

$$u(t) = u(t+T) \quad \Longrightarrow \quad U(s) = \frac{\int_0^T e^{-st}u(t) dt}{1 - e^{-Ts}}$$

## Differentiation and Integration of Laplace Transforms

$$\begin{aligned} u^{(n)}(t) &\xrightarrow{\mathcal{L}} s^n U(s) - s^{n-1}u(0) - s^{n-2}u'(0) - \dots - u^{(n-1)}(0) \\ t^n u(t) &\xrightarrow{\mathcal{L}} (-1)^n U^{(n)}(s) \\ v(t) = \int_0^t u(r) dr &\xrightarrow{\mathcal{L}} V(s) = U(s)/s \end{aligned}$$

## Convolution and Laplace Transforms

$$(u \star v)(t) = \int_0^t v(t-r)u(r) dr \xrightarrow{\mathcal{L}} U(s)V(s)$$

## Laplace Transform of Linear First Order Differential Equations

$$\begin{aligned} u'(t) + pu(t) = f(t), u(0) = a &\xrightarrow{\mathcal{L}} U(s) = \frac{1}{s+p}(F(s) + a) \\ \Phi(s) = (s+p)^{-1} \xrightarrow{\mathcal{L}^{-1}} \varphi(t) = e^{-pt} &\implies u = u_h + u_p = a\varphi + \varphi \star f \end{aligned}$$

## Laplace Transform of Linear Second Order Differential Equations

$$\begin{aligned} u''(t) + pu'(t) + qu(t) = f(t), u(0) = a, u'(0) = b \\ \xrightarrow{\mathcal{L}} U(s) = \frac{1}{s^2 + ps + q} (F(s) + as + ap + b) \end{aligned}$$

$\rightsquigarrow$  representation of the solution as convolution:

$$u = \underbrace{a\varphi' + (ap + b)\varphi}_{u_h} + \underbrace{\varphi \star f}_{u_p}, \quad \varphi(t) = \begin{cases} \frac{e^{\lambda t} - e^{\varrho t}}{\lambda - \varrho}, & \lambda \neq \varrho \\ te^{\lambda t}, & \lambda = \varrho \end{cases}$$

with  $\lambda, \varrho$  the zeros of the characteristic polynomial  $\Phi^{-1}(s) = s^2 + ps + q$  and  $\varphi$  the inverse Laplace transform of  $\Phi$