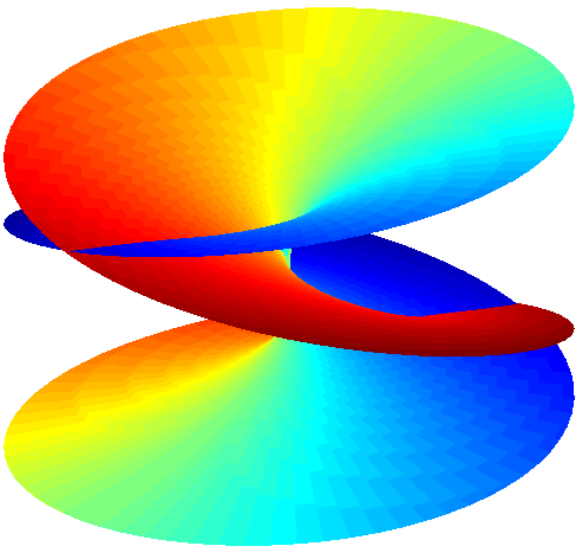
MathTraining

Problems and Solutions for Complex Analysis



Klaus Höllig

Partially translated from
Aufgaben und Lösungen zur Höheren Mathematik 1 und 3, 4. Auflage
by Klaus Höllig and Jörg Hörner

Preface

The problem set of the book covers all basic topics of a first course on complex analysis. It can be used to practice for exams, to facilitate the completion of homework assignments, and to review course material. Interactive variants to model problems with detailed solutions permit the student reader to test his comprehension of the relevant techniques. In addition to the collection of problems, a small mathematics lexicon contains brief descriptions of the relevant theorems, methods, and definitions.

There exists also a sportive aspect of mathematics - challenging problems requiring ideas beyond the standard techniques. The problems in the chapter *Calculus Highlights* are perhaps too difficult for undergraduates. They are included to initiate or strengthen fascination for mathematics. It is definitely not a mistake to practice substantially harder than necessary ...!

The book is partially translated from the books

Aufgaben und Lösungen zur Höheren Mathematik 1

and

Aufgaben und Lösungen zur Höheren Mathematik 3

by Jörg Hörner and the author. It supplements these textbooks by providing detailed solutions to tests for the chapters on *Complex Analysis*. Moreover, the book includes additional problems, in particular problem variants for the topics of the tests.

The author wishes the readers success in their studies **and** hopes that mathematics will become one of their favorite subjects!

Klaus Höllig

Contents

Introduction

The book contains problems with detailed solutions, problem variants with interactive result verification, and a mathematics lexicon for the principal topics which are usually subject of a first course on *Complex Analysis*:

- Complex Numbers,
- Differentiation and Conformal Mapping,
- Integration,
- Taylor and Laurent Series,
- Differential Equations.

The problem set can be used to practice for exams, to facilitate the completion of homework assignments, and to deepen the comprehension of course material. How is this accomplished most effectively? Remembering his own student days, the author makes the following recommendations to a student reader.

Consider, as an example, a problem from the chapter on *Integration*:

3.1 Complex Integration over a Circle

Compute

$$\int_C \frac{\mathrm{e}^z}{z^2+\pi^2}\,\mathrm{d}z$$

for the counterclockwise oriented circle C around i with radius 3.

Resources: Residue Theorem, Residue

Before looking at the solution of the problem, review the relevant theory or methods (resources). Clicking on the links *Residue Theorem* and *Residue* leads to the following brief descriptions of relevant techniques from the *Lexicon* in chapter 7.

Residue Theorem

$$\int_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_k \operatorname*{Res}_{a_k} f$$

for a complex differentiable function f with finitely many singularities a_k on a bounded domain D with counterclockwise oriented boundary C

Residue

$$\operatorname*{Res}_{z=a} f(z) = \operatorname*{Res}_a f = \frac{1}{2\pi\mathrm{i}}\int_C f(z)\,\mathrm{d}z, \quad D \supset C : t \mapsto a + r\mathrm{e}^{\mathrm{i}t},\ r < R$$

for a complex differentiable function f on the annulus $D:\ 0 < |z-a| < R$

- removable singularity: $\operatorname*{Res}_a f = 0$
- simple pole: $\operatorname*{Res}_a f = \lim\limits_{z\to a}(z-a)f(z)$
- pole of order n: $\operatorname*{Res}_a f = \lim\limits_{z\to a}\dfrac{1}{(n-1)!}\,(\mathrm{d}/\mathrm{d}z)^{n-1}\left((z-a)^n f(z)\right)$
- essential singularity:
  coefficient c_{-1} of the Laurent series $f(z) = \sum\limits_{n=-\infty}^{\infty} c_n (z-a)^n$

Maple:

```
residue(f(z),z=a)
```

By the *Residue Theorem*, the integral over the circle C can be computed by determining the residues of the integrand $f(z) = \mathrm{e}^z/(z^2+\pi^2)$ at singularities a in the disc D with boundary C. For the simple pole of f at $a = \pi\mathrm{i}$, the formula $\operatorname*{Res}_a f = \lim_{z\to a}(z-a)f(z)$ in the definition of a *Residue* applies.

Try to solve the problem with these instructions. Then compare your computations with the solution given in the book:

Solution

Application of the residue theorem

complex line integral $\quad\to\quad$ sum of residues:

$$I = \int\limits_C \frac{\mathrm{e}^z}{z^2+\pi^2}\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{a\in D} \operatorname*{Res}_{z=a} \frac{\mathrm{e}^z}{z^2+\pi^2}$$

with C the counterclockwise oriented boundary of the disc $D:\ |z-\mathrm{i}|<3$

Computation of the residues and the integral

binomial formula $\quad\rightsquigarrow$

$$\frac{\mathrm{e}^z}{z^2+\pi^2} = \frac{\mathrm{e}^z}{(z-\pi\mathrm{i})(z+\pi\mathrm{i})}$$

poles $z_\pm = \pm\pi\mathrm{i}$
$|z_- - \mathrm{i}| = |-\pi\mathrm{i}-\mathrm{i}| = \pi+1 > 3 \quad\Longrightarrow\quad z_- \notin D$, i.e. only $z_+ = \pi\mathrm{i}$ is relevant and

$$\operatorname*{Res}_{z=\pi\mathrm{i}} \frac{\mathrm{e}^z}{z^2+\pi^2} = \lim_{z\to\pi\mathrm{i}} (z-\pi\mathrm{i}) \frac{\mathrm{e}^z}{(z-\pi\mathrm{i})(z+\pi\mathrm{i})} = \frac{\mathrm{e}^{\pi\mathrm{i}}}{2\pi\mathrm{i}} = -\frac{1}{2\pi\mathrm{i}}$$

$\rightsquigarrow \quad I = 2\pi\mathrm{i}(-1/(2\pi\mathrm{i})) = -1$

The solutions are written in a keyword-like style, as you would employ when you comment your solutions in an exam or for homework problems. For example, the phrase

"binomial formula $\quad\rightsquigarrow\quad$ …"

stands for

"Applying the binomial formula, we obtain …".

Other examples of typical phrases are "simplification $\quad\rightsquigarrow\quad$ …", "chain rule $\Longrightarrow$ …", or "characterisitic equation: …". There is just as much detail included as is necessary for the mathematical arguments.

To gain more practice with the solution technique, it is highly recommended to solve some (preferably all …) of the problem variants, following the principal model problem for each topic. For *Integration over a Circle* the variants are:

Problem Variants

■ $\displaystyle\int\limits_C \frac{\cos z}{\mathrm{e}^z - 1}\,\mathrm{d}z$

$?\pi\mathrm{i}$:

[ ] check

■ $\displaystyle\int\limits_C \frac{\sin z}{4z^2 - \pi^2}\,\mathrm{d}z$

$?\mathrm{i}$:

[ ] check

■ $\displaystyle\int\limits_C \frac{z^2 - 9}{z^2 + 9}\,\mathrm{d}z$

$-?\pi$:

[ ] check

You can check your solution by typing your answer in the field adjacent to the check - box, replacing every question mark by a character (digit or letter). Convert your result to a decimal, truncated to the number of digits indicated. For example,

$$2/3 \to 0.6666\ldots \xrightarrow{?.??} 0.66, \qquad \text{answer}: \boxed{066}\,.$$

Note that the period is omitted; only the characters corresponding to the question marks are typed.

The solutions for the three problem variants are

(1) $2\pi\mathrm{i}$,

(2) i,

(3) -6π.

Hence, the correct input is

$$(1) \quad ?\pi\mathrm{i} \to \boxed{2} \qquad (2) \quad ?\mathrm{i} \to \boxed{1} \qquad (3) \quad -?\pi \to \boxed{6}.$$

As mentioned in the beginning, the problem set can also assist you in completing homework assignments. Just look for a similar problem and study its solution. Similarly, for methods and examples presented in class, practice with the relevant problems.

The above remarks pertain to the first five chapters which exclusively discuss the solution of standard problems. Usually, such problems constitute the major portion of an exam or homework assignment. Hence, to review the basic techniques involved is of primary importance. Applying these techniques to more advanced problems is a natural next step. The chapter *Calculus Highlights* contains examples of rather challenging applications. You do not have to be disappointed, if you cannot solve any of these problems; they are definitely very difficult. It is legitimate to immediately look at the solutions and learn how the methods from the previous chapters are applied in an advanced setting. Also, as mentioned in the Preface, it is not a mistake to practice substantially harder than necessary ...!

You have solved some of the problems in chapter 6 without resorting to the solutions. Then ...

... **you can take pride in your mastery of the principal techniques for solving problems in Complex Analysis!**

With the previous explanations aimed at student readers, instructors could (obviously) also benefit from the interactive problem collection. The solutions of the model problems can be used as examples in class and some of the variants assigned as homework problems. Students will welcome the possibility of checking results before submitting or presenting their solutions in the exercise sections.

Disclaimer: Although the solutions and answers to the variants have been thoroughly checked, mistakes can always occur[1]. Please, write to the author (`Klaus.Hoellig@gmail.com`) if you find any errors.

[1] A statement by a teaching assistant to encourage students, which the author will always remember: "This year, the final exam is not too difficult - your professor could check the results without committing any errors!".

Chapter 1

Complex Numbers

1.1 Product of Complex Numbers

Compute $(1 - \mathrm{i})^3(2 + \mathrm{i})$.

Resources: Complex Arithmetic Operations

Problem Variants

- $(3 + \mathrm{i})^2(2 - \mathrm{i})^2$

??−??i:

[ ] check

- $(1 - 2\mathrm{i})^3(2 + 3\mathrm{i})$

−??−??i:

[ ] check

- $(2 + \mathrm{i})^3(1 - 2\mathrm{i})^3$

−??−???i:

[ ] check

Solution

binomial formula $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$ and $\mathrm{i}^2 = -1$ $\quad\rightsquigarrow$

$$(1-\mathrm{i})^3 = 1 - 3\mathrm{i} - 3 + \mathrm{i} = -2(1+\mathrm{i})$$

multiplication with $2+\mathrm{i}$ $\quad\rightsquigarrow$

$$\begin{aligned} (1-\mathrm{i})^3(2+\mathrm{i}) &= -2(1+\mathrm{i})(2+\mathrm{i}) \\ &= -2(2+2\mathrm{i}+\mathrm{i}-1) = -2(1+3\mathrm{i}) = -2-6\mathrm{i} \end{aligned}$$

1.2 Quotient of Complex Numbers

Compute $\dfrac{4-3\mathrm{i}}{2+\mathrm{i}}$.

Resources: Complex Arithmetic Operations

Problem Variants

- $\dfrac{2+5\mathrm{i}}{4-3\mathrm{i}}$

−?.??+?.??i:

[ ] check

- $\dfrac{4}{2+3\mathrm{i}}$

?.??−?.??i:

[ ] check

- $\dfrac{2+\mathrm{i}}{2-4\mathrm{i}}$

?.?i:

[ ] check

Solution

expanding by $2-\mathrm{i}$ and applying the binomial formula $(a+b)(a-b)=a^2-b^2$
$\rightsquigarrow$

$$\begin{aligned}\frac{4-3\mathrm{i}}{2+\mathrm{i}} &= \frac{(4-3\mathrm{i})(2-\mathrm{i})}{(2+\mathrm{i})(2-\mathrm{i})}=\frac{8-4\mathrm{i}-6\mathrm{i}-3}{4+1}\\ &= \frac{5-10\mathrm{i}}{5}=1-2\mathrm{i}\end{aligned}$$

1.3 Polar Form of a Complex Number

Determine the polar form $re^{\mathrm{i}\varphi}$ of $\sqrt{3} - \mathrm{i}$.

Resources: Polar Form of Complex Numbers

Problem Variants

- $-2 - 2\mathrm{i}$

?.?? exp($-$i?.??π):

[ ] check

- $-1 + \sqrt{3}\mathrm{i}$

? exp(i?.??π):

[ ] check

- $1/\sqrt{(2)} + \mathrm{i}/\sqrt{2}$

? exp(i?.??π):

[ ] check

Solution

application of the transformation rules

$$r = \sqrt{x^2 + y^2}, \quad \varphi = \arctan(y/x) + \sigma\pi \quad \text{with} \quad \sigma = \begin{cases} 0 & \text{for } x \geq 0 \\ 1 & \text{for } x < 0, y \geq 0 \\ -1 & \text{for } x < 0, y < 0 \end{cases}$$

for the rectangular form $z = x + \mathrm{i}y$ and the polar form $z = r\mathrm{e}^{\mathrm{i}\varphi}$, $\varphi \in (-\pi, \pi]$, of a complex number z to $z = \sqrt{3} - \mathrm{i} \quad \rightsquigarrow$

$$\begin{aligned} r &= \sqrt{3+1} = 2 \\ \varphi &\underset{x \geq 0,\, \sigma = 0}{=} \arctan(-1/\sqrt{3}) = -\pi/6 \end{aligned}$$

1.4 Product in Rectangular and Polar Form

Compute $z = (2 + 2\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\pi/3}$ in rectangular and polar form.

Resources: Polar Form of Complex Numbers, Complex Arithmetic Operations

Problem Variants

■ $z = (1 + \sqrt{3}\,\mathrm{i})\,\mathrm{e}^{-3\pi\mathrm{i}/4}$

$z = ?\ \exp(-?\pi\mathrm{i}/12) = (\sqrt{?} - \sqrt{?} - (\sqrt{2} + \sqrt{?})\,\mathrm{i})/2$:

[ ] check

■ $z = (\sqrt{2} - \sqrt{2}\,\mathrm{i})\,\mathrm{e}^{-5\pi\mathrm{i}/6}$

$z = ?\ \exp(??\pi\mathrm{i}/12) = (-(\sqrt{2} + \sqrt{?}) + (\sqrt{?} - \sqrt{?})\,\mathrm{i})/2$:

[ ] check

■ $z = (\sqrt{3} - \mathrm{i})\,\mathrm{e}^{2\pi\mathrm{i}/3}$

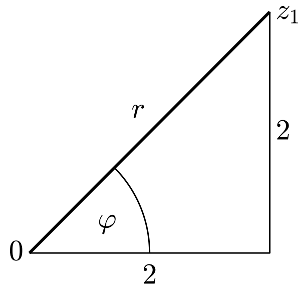
$z = ?\ \exp(\pi\mathrm{i}/?) = ??$:

[ ] check

Solution

<u>Polar form</u>

converting $z_1 = 2 + 2\mathrm{i}$ to polar form $r\,\mathrm{e}^{\mathrm{i}\varphi}$ (cf. the depicted isosceles right triangle) $\rightsquigarrow$



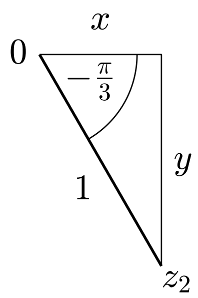
$$\begin{aligned} r &= \sqrt{2^2 + 2^2} = 2\sqrt{2} \\ \varphi &\underset{\mathrm{Re}\, z_1 \geq 0}{=} \arctan(\mathrm{Im}\, z_1 / \mathrm{Re}\, z_1) \\ &= \arctan 2/2 = \pi/4 \end{aligned}$$

forming the product $\rightsquigarrow$

$$z = (2 + 2\mathrm{i})\,\mathrm{e}^{-\mathrm{i}\pi/3} = 2\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}\,\mathrm{e}^{-\mathrm{i}\pi/3} = 2\sqrt{2}\,\mathrm{e}^{\mathrm{i}(\pi/4 - \pi/3)} = 2\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\pi/12}$$

<u>Rectangular form</u>

converting $z_2 = \mathrm{e}^{-\mathrm{i}\pi/3}$ to rectangular form $x + \mathrm{i}y$ (cf. the depicted half of an equilateral triangle with side length 1) $\rightsquigarrow$



$$\begin{aligned} z_2 &= \cos(-\pi/3) + \mathrm{i}\,\sin(\pi/3) \\ &= 1/2 - (\sqrt{3}/2)\,\mathrm{i} \end{aligned}$$

forming the product $\rightsquigarrow$

$$(2 + 2\mathrm{i})\,(1/2 - (\sqrt{3}/2)\,\mathrm{i}) = 1 + \sqrt{3} + (1 - \sqrt{3})\,\mathrm{i}$$

<u>Remark</u>

comparing the polar and the rectangular form, Euler's formula $\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$ $\implies$

$$\begin{aligned} \cos(\pi/12) &= \cos(-\pi/12) = \frac{1 + \sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2} + \sqrt{6}}{4}, \\ \sin(\pi/12) &= -\sin(-\pi/12) = \frac{\sqrt{6} - \sqrt{2}}{4} \end{aligned}$$

1.5 Quotients Involving Rectangular and Polar Form

Determine and simplify the rectangular form of $\dfrac{\mathrm{i}}{1+\mathrm{e}^{\mathrm{i}\pi/4}}$.

Resources: Complex Numbers, Polar Form of Complex Numbers

Problem Variants

■ $z = \dfrac{2}{\mathrm{e}^{\mathrm{i}\pi/3} - \mathrm{i}}$

$? + \sqrt{?} + ?\mathrm{i}$:

[ ] check

■ $z = \dfrac{3}{\mathrm{e}^{-\mathrm{i}\pi/6} - \sqrt{2}}$

$(\sqrt{?} + (? + \sqrt{?})\,\mathrm{i})/2$:

[ ] check

■ $z = \dfrac{1}{\mathrm{e}^{-3\mathrm{i}\pi/4} + \sqrt{2}}$

$(\sqrt{?} + \sqrt{?}\,\mathrm{i})/2$:

[ ] check

Solution

Conversion to rectangular form

$e^{i\varphi} = \cos\varphi + i\,\sin\varphi \quad \rightsquigarrow \quad$ rectangular form of the denominator:

$$1 + e^{i\pi/4} = 1 + \cos(\pi/4) + i\,\sin(\pi/4) = 1 + 1/\sqrt{2} + i/\sqrt{2}$$

simplifying the quotient by expanding with $\sqrt{2} \quad \rightsquigarrow$

$$z = \frac{i}{1 + e^{i\pi/4}} = \frac{i}{(1 + 1/\sqrt{2}) + (1/\sqrt{2})\,i} = \frac{\sqrt{2}\,i}{(\sqrt{2}+1) + i}$$

Making the denominator real

expanding the quotient $c/(a + b\,i)$ with $a - b\,i$, using that $(a + b\,i)(a - b\,i) = a^2 + b^2 \quad \rightsquigarrow$

$$z = \frac{\sqrt{2}\,i\,\big((\sqrt{2}+1) - i\big)}{\big((\sqrt{2}+1) + i\big)\big((\sqrt{2}+1) - i\big)} = \frac{2\,i + \sqrt{2}\,i + \sqrt{2}}{(\sqrt{2}+1)^2 + 1} = \frac{\sqrt{2} + (2 + \sqrt{2})\,i}{4 + 2\sqrt{2}}$$

Making the denominator rational

expanding with $4 - 2\sqrt{2}$, again using the binomial formula to simplify the resulting denominator $(4 + 2\sqrt{2})(4 - 2\sqrt{2}) \quad \rightsquigarrow$

$$\begin{aligned} z &= \frac{\big(\sqrt{2} + 2\,i + \sqrt{2}\,i\big)\big(4 - 2\sqrt{2}\big)}{4^2 - (2\sqrt{2})^2} \\ &= \frac{4\sqrt{2} + 8\,i + 4\sqrt{2}\,i - 4 - 4\sqrt{2}\,i - 4i}{16 - 8} = \frac{(\sqrt{2} - 1) + i}{2} \end{aligned}$$

1.6 Computing with Complex Numbers

Compute $\dfrac{z^2+2z\bar{z}}{\bar{z}^2-2z/\bar{z}}$ for $z=1+\mathrm{i}$.

Resources: Complex Arithmetic Operations

Problem Variants

■ $\dfrac{z+\bar{z}^2}{z^2-\bar{z}}$, $z=2-\mathrm{i}$

$-?.??+?.??\mathrm{i}$:

[ ] check

■ $\dfrac{z-|z|}{\bar{z}/z}$, $z=4+3\mathrm{i}$

$-?.??-?.??\mathrm{i}$:

[ ] check

■ $\dfrac{z+1/z}{\bar{z}}$, $z=1+\mathrm{i}/2$

$?.?-?.?\mathrm{i}$:

[ ] check

Solution

$z\bar{z} = |z|^2$, $z/\bar{z} = z^2/|z|^2$, $\bar{z}^2 = \overline{z^2}$ with $z = 1 + \mathrm{i}$ and $|1 + \mathrm{i}| = \sqrt{1+1} = \sqrt{2}$ $\rightsquigarrow$

$$\begin{aligned} \frac{z^2 + 2z\bar{z}}{\bar{z}^2 - 2z/\bar{z}} &= \frac{(1+\mathrm{i})^2 + 2|1+\mathrm{i}|^2}{\overline{(1+\mathrm{i})^2} - 2(1+\mathrm{i})^2/|1+\mathrm{i}|^2} \\ &= \frac{2\mathrm{i} + 4}{-2\mathrm{i} - 2(2\mathrm{i})/2} = \frac{2\mathrm{i}+4}{-4\mathrm{i}} = -\frac{1}{2} + \mathrm{i} \end{aligned}$$

1.7 Simplification of Complex Expressions

Determine the rectangular and polar form of

$$\frac{(1-\mathrm{i})^5\,\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}}{(\sqrt{3}+3\mathrm{i})\;+\;2\,\mathrm{e}^{-\mathrm{i}\pi/6}}\,.$$

Resources: Complex Arithmetic Operations, Powers of Complex Numbers

Problem Variants

■ $\dfrac{(1-\mathrm{i})^3+\mathrm{e}^{\mathrm{i}\pi/4}}{(1+\mathrm{i})^2}$

$-?.??+?.??\mathrm{i}$:

[ ] check

■ $\dfrac{(1-\mathrm{i})^2}{\mathrm{e}^{4\pi\mathrm{i}/3}+\mathrm{e}^{5\pi\mathrm{i}/6}}$

$?.??+?.??\mathrm{i}$:

[ ] check

■ $\dfrac{\mathrm{e}^{-2\pi\mathrm{i}/3}+2\mathrm{e}^{\pi\mathrm{i}/3}}{(3-\mathrm{i})^2}$

$-0.0???+0.0???\mathrm{i}$:

[ ] check

Solution

Numerator

computing $p = (1-\mathrm{i})^5\,\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}$ in polar form, $1-\mathrm{i} = \sqrt{2}\,\mathrm{e}^{-\mathrm{i}\pi/4}$ $\leadsto$

$$\begin{aligned} p &= \left(\sqrt{2}\,\mathrm{e}^{-\mathrm{i}\pi/4}\right)^5 \sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4} = 2^{(5+1)/2}\,\mathrm{e}^{-\mathrm{i}5\pi/4+\mathrm{i}\pi/4} \\ &= 8\,\mathrm{e}^{-\mathrm{i}\pi} = 8\,(\cos\pi + \mathrm{i}\sin\pi) = 8\,(-1+0\mathrm{i}) = -8 \end{aligned}$$

alternative: using the rectangular form $\mathrm{e}^{\mathrm{i}\pi/4} = (1+\mathrm{i})/\sqrt{2}$ and the binomial formulas $\leadsto$

$$\begin{aligned} p &= (1-\mathrm{i})^5(1+\mathrm{i}) = [(1-\mathrm{i})(1+\mathrm{i})]\,[(1-\mathrm{i})^2]^2 \\ &= [1-\mathrm{i}^2]\,[1-2\mathrm{i}+\mathrm{i}^2]^2 = 2\cdot(-2\mathrm{i})^2 = -8 \quad \checkmark \end{aligned}$$

Denominator

computing $q = (\sqrt{3}+3\mathrm{i}) + 2\,\mathrm{e}^{-\mathrm{i}\pi/6}$ in rectangular form, $2\,\mathrm{e}^{-\mathrm{i}\pi/6} = \sqrt{3}-\mathrm{i}$ $\leadsto$

$$q = (\sqrt{3}+3\mathrm{i}) + (\sqrt{3}-\mathrm{i}) = 2\sqrt{3}+2\mathrm{i}$$

conversion to polar form:
$|q| = \sqrt{4\cdot 3+4} = 4\,,\quad \arg q = \arctan(1/\sqrt{3}) = \pi/6 \quad \leadsto$

$$q = 4\,\mathrm{e}^{\mathrm{i}\pi/6}$$

Rectangular and polar form of the fraction

$$z = \frac{p}{q} = -\frac{8}{2\sqrt{3}+2\mathrm{i}} = -\frac{8(2\sqrt{3}-2\mathrm{i})}{4\cdot 3+4} = -\sqrt{3}+\mathrm{i}$$

conversion to polar form $z = r\mathrm{e}^{\mathrm{i}\varphi}$:

$$\begin{aligned} r &= \sqrt{3+1} = 2 \\ \varphi &= \arctan(-1/\sqrt{3}) + \sigma\pi = -\pi/6 + \pi = 5\pi/6 \end{aligned}$$

($\sigma = 1$, since $\operatorname{Re} z = -\sqrt{3} < 0$ and φ in $(-\pi, \pi]$)
$\leadsto \quad z = 2\,\mathrm{e}^{\mathrm{i}5\pi/6}$

alternative: direct computation of the polar form

$$z = \frac{p}{q} = \frac{8\,\mathrm{e}^{\mathrm{i}\pi}}{4\,\mathrm{e}^{\mathrm{i}\pi/6}} = 2\,\mathrm{e}^{\mathrm{i}5\pi/6}$$

1.8 Trigonometric Identities

Rewrite $2\sin^2(2t)$ as a trigonometric polynomial

$$\frac{a_0}{2} + \sum_{k=1}^{4} \left(a_k \cos(kt) + b_k \sin(kt)\right) .$$

Resources: Formula of Euler Moivre

Problem Variants

■ $\cos^3(3t)$

---

largest coefficient =?.??:

[ ] check

■ $\sin(3t)\cos(2t)$

---

largest coefficient =?.?:

[ ] check

■ $\sin^3 t \cos t$

---

largest coefficient =?.??:

[ ] check

Solution

writing $\sin\varphi$ with the formula of Euler-Moivre as $(\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi})/(2\mathrm{i})$ $\rightsquigarrow$

$$2\sin^2(2t) = 2\left(\frac{\mathrm{e}^{2\mathrm{i}t} - \mathrm{e}^{-2\mathrm{i}t}}{2\mathrm{i}}\right)^2 = -\frac{\mathrm{e}^{4\mathrm{i}t} - 2 + \mathrm{e}^{-4\mathrm{i}t}}{2}$$

writing $\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{e}^{-\mathrm{i}\varphi}$ with the formula of Euler-Moivre as $2\cos\varphi$ $\rightsquigarrow$

$$2\sin^2(2t) = 1 - \cos(4t)$$

1.9 Complex Square Root

Determine the polar form and the rectangular form of the square roots of $z = \sqrt{3} + \mathrm{i}$.

Resources: Complex Root

Problem Variants

■ $z = 3 - 4\mathrm{i}$

---

coordinate form: ?−?i:

[ ] check

■ $z = 1 + \sqrt{3}\mathrm{i}$

---

coordinate form: $(\sqrt{?} - \sqrt{?}\mathrm{i})/2$:

[ ] check

■ $z = 4 + 4\mathrm{i}$

---

coordinate form: $\sqrt{?\sqrt{?}+?} + \sqrt{?\sqrt{?}-?}\mathrm{i}$:

[ ] check

Solution

Polar form of the square roots

absolute value r and argument φ of $z = \sqrt{3} + \mathrm{i}$

$$r = |z| = \sqrt{3+1} = 2, \quad \varphi = \arg z = \arctan(1/\sqrt{3}) = \pi/6$$

$\rightsquigarrow \quad z = r\,\mathrm{e}^{\mathrm{i}\varphi} = 2\,\mathrm{e}^{\mathrm{i}\pi/6}$ and $w = \sqrt{z} = \pm\sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/12}$, i.e.

$$w_1 = \sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/12}, \quad w_2 = \sqrt{2}\,\mathrm{e}^{-\mathrm{i}11\pi/12}$$

$(-1 = \mathrm{e}^{\pm\mathrm{i}\pi})$

Rectangular form of the square roots

ansatz $w = \sqrt{z} = x + \mathrm{i}y \quad \rightsquigarrow$

$$\sqrt{3} + \mathrm{i} = (x + \mathrm{i}y)^2$$

comparing the real and imaginary part $\quad \rightsquigarrow$

$$\sqrt{3} = x^2 - y^2, \quad 1 = 2xy$$

substituting $y = 1/(2x)$ into the first equation and multiplying with $x^2 \quad \rightsquigarrow$

$$x^4 - \sqrt{3}\,x^2 - 1/4 = 0$$

formula for the solutions of a quadratic equation, $x \in \mathbb{R}$ and consequently $x^2 \geq 0 \quad \rightsquigarrow$

$$x^2 = \sqrt{3}/2 + \sqrt{3/4 + 1/4} = 1 + \sqrt{3}/2, \quad x = \pm\sqrt{1 + \sqrt{3}/2} = \pm\frac{1+\sqrt{3}}{2},$$

since $1 + \sqrt{3}/2 = (1+\sqrt{3})^2/4$
expanding with $\sqrt{3} - 1$ and applying the third binomial formula $\quad \rightsquigarrow$

$$y = \frac{1}{2x} = \pm\frac{1}{1+\sqrt{3}} = \pm\frac{\sqrt{3}-1}{2}$$

1.10 Complex Powers

Determine all values of $(1+\mathrm{i})^{4/3}$.

Resources: Complex Powers

Problem Variants

■ $(\sqrt{3}+\mathrm{i})^{3/2}$

---

power with positive real and imaginary part =?+?i:

[ ] check

■ $(-16)^{3/4}$

---

power with positive real and imaginary part =?.??+?.??i:

[ ] check

■ $(2\mathrm{i}-2)^{2/3}$

---

power with positive real and negative imaginary part =?.??−?i:

[ ] check

Solution

conversion of $z = x + \mathrm{i}y = 1 + \mathrm{i}$ to polar form $r\,\mathrm{e}^{\mathrm{i}\varphi}$:

$$r = \sqrt{x^2 + y^2} = \sqrt{1+1} = \sqrt{2},\ \varphi = \arctan(y/x) = \arctan(1/1) = \pi/4$$

$\rightsquigarrow \quad z = \sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}$

principal value of the power $z^{4/3}$:

$$\begin{aligned} p_0 &= \left(2^{1/2}\,\mathrm{e}^{\mathrm{i}\pi/4}\right)^{4/3} = 2^{2/3}\,\mathrm{e}^{\mathrm{i}\pi/3} \\ &= \sqrt[3]{4}\,(\cos(\pi/3) + \mathrm{i}\sin(\pi/3)) = \sqrt[3]{4}\,(1/2 + \mathrm{i}\sqrt{3}/2) \end{aligned}$$

multiplication of z with $1 = \mathrm{e}^{2\pi\mathrm{i}k}$, $k = 1, 2 \quad \rightsquigarrow \quad$ two additional values:

$$p_k = \left(z\,\mathrm{e}^{2\pi\mathrm{i}k}\right)^{4/3} = p_0\,\left(\mathrm{e}^{2\pi\mathrm{i}k}\right)^{4/3} = \sqrt[3]{4}\,\mathrm{e}^{\mathrm{i}\pi(1+8k)/3}$$

choosing the standard interval $(-\pi, \pi]$ for the angle $\pi(1+8k)/3$ (subtraction of multiples of 2π) $\quad \rightsquigarrow$

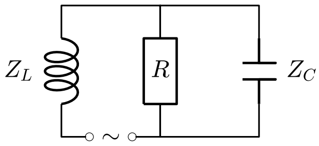
$$\begin{aligned} p_1 &= \sqrt[3]{4}\,\mathrm{e}^{\mathrm{i}\pi(9/3)} = \sqrt[3]{4}\,\mathrm{e}^{\mathrm{i}\pi} = -\sqrt[3]{4} \\ p_2 &= \sqrt[3]{4}\,\mathrm{e}^{\mathrm{i}\pi(17/3)} = \sqrt[3]{4}\,\mathrm{e}^{-\mathrm{i}\pi/3} = \sqrt[3]{4}\,(1/2 - \mathrm{i}\sqrt{3}/2) \end{aligned}$$

1.11 Complex Resistance of an RCL-Circuit

Determine the total complex resistance of the depicted electrical circuit for

$$R = 200, \quad Z_L/\mathrm{i} = 300, \quad \mathrm{i}Z_C = 100$$

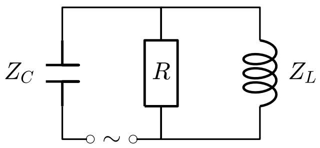
(units: Ohm [Ω]).



Resources: Complex Arithmetic Operations

Problem Variants

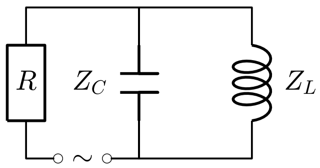
■



???−?i:

check

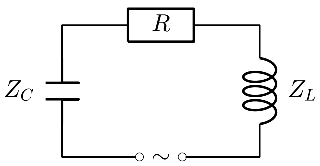
■



???−???i:

check

■



???+???i:

check

Solution

Resistor and conductor, connected in parallel

combined complex resistance Z_{RC} with

$$1/Z_{RC} = 1/R + 1/Z_C$$

$R = 200,\ Z_C = 100/\mathrm{i} \quad \rightsquigarrow$

$$\frac{1}{Z_{RC}} = \frac{1}{200} + \frac{\mathrm{i}}{100} = \frac{1+2\mathrm{i}}{200}, \quad Z_{RC} = \frac{200}{1+2\mathrm{i}} = \frac{200(1-2\mathrm{i})}{1+4} = 40 - 80\mathrm{i}$$

Total complex resistance Z of the RCL-circuit

combining the RC-element with the conductor $\quad \rightsquigarrow$

$$Z = Z_{RC} + Z_L$$

$Z_L/\mathrm{i} = 300 \quad \rightsquigarrow$

$$Z = (40 - 80\mathrm{i}) + 300\mathrm{i} = 40 + 220\mathrm{i}$$

1.12 Solution of a Quadratic Equation

Determine the solutions of the quadratic equation

$$z^2 - 6z + 25 = 0\,.$$

Resources: Complex Root

Problem Variants

■ $z^2/2 - 2z + 2 = \mathrm{i}$

? ± (?+?i:

[ ] check

■ $z^2 - 4z = 2\mathrm{i}z - 8\mathrm{i}$

?+?i ± (?−?i):

[ ] check

■ $z^2 - 4z - 5 = 6\mathrm{i}z - 12\mathrm{i}$

?+?i±?:

[ ] check

Solution

formula for the solution of a quadratic equation $az^2 + bz + c = 0$:

$$z_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$a = 1,\ b = -6,\ c = 25 \quad \rightsquigarrow$

$$z_{\pm} = \frac{6 \pm \sqrt{6^2 - 4 \cdot 25}}{2} = \frac{6 \pm \sqrt{-64}}{2} = 3 \pm 4\mathrm{i}$$

1.13 Zeros of a Cubic Polynomial

The polynomial

$$p(z) = z^3 - 3z^2 + (3 - 2\mathrm{i})z - 1 + 2\mathrm{i}$$

vanishes at $z_1 = 1$. Determine the other zeros.

Resources: Complex Arithmetic Operations

Problem Variants

■ $p(z) = z^3 - z^2 + 2z + 4,\ z_1 = -1$

---

largest absolute value of a solution = ?:

[ ] check

■ $p(z) = z^3 - (1 + \mathrm{i})z^2 + z - 1 - \mathrm{i},\ z_1 = \mathrm{i}$

---

largest absolute value of a solution = ?.??:

[ ] check

■ $p(z) = z^3 + \mathrm{i})z^2 + (\mathrm{i} - 7)z + 6 - 6\mathrm{i},\ z_1 = 2$

---

largest absolute value of a solution = ?:

[ ] check

Solution

Reducing the Order

The cubic polynomial

$$p(z) = z^3 - 3z^2 + (3-2\mathrm{i})z - 1 + 2\mathrm{i}$$

has three zeros z_k, counting multiplicities, and hence admits the factorization

$$p(z) = (z - z_1)(z - z_2)(z - z_3)\,.$$

Dividing by the linear factor $(z - 1)$, corresponding to the known zero $z_1 = 1$, yields the quadratic polynomial $q(z) = (z - z_2)(z - z_3)$, and allows to determine the two remaining zeros z_2 and z_3:

$$\begin{array}{lllll}
(\,z^3 & -3z^2 & +(3-2\mathrm{i})z & -1+2\mathrm{i}\,) : (z-1) & = \underbrace{z^2 - 2z + 1 - 2\mathrm{i}}_{q(z)} \\
\underline{z^3} & \underline{-z^2} & & & \\
 & -2z^2 & +(3-2\mathrm{i})z & & \\
 & \underline{-2z^2} & \underline{+2z} & & \\
 & & (1-2\mathrm{i})z & -1+2\mathrm{i} & \\
 & & \underline{(1-2\mathrm{i})z} & \underline{-1+2\mathrm{i}} & \\
 & & & 0 &
\end{array}$$

Zeros of the quadratic polynomial

solving

$$q(z) = z^2 - 2z + 1 - 2\mathrm{i} = 0$$

with the formula for the solutions of quadratic equations $\quad\rightsquigarrow$

$$z_{2,3} = 1 \pm \sqrt{1 - (1 - 2\mathrm{i})} = 1 \pm \sqrt{2\mathrm{i}}$$

computing the square root using the polar form $\mathrm{i} = \mathrm{e}^{\mathrm{i}\pi/2}$ $\quad\rightsquigarrow$

$$\sqrt{2\mathrm{i}} = \sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4} = 1 + \mathrm{i}$$

and hence

$$z_2 = 2 + \mathrm{i}, \quad z_3 = -\mathrm{i}$$

1.14 Circle in the Complex Plane

Determine the midpoint and radius of the circle

$$C:\ |z+\mathrm{i}| = 2|z-1|\,.$$

Resources: Circle in the Complex Plane

Problem Variants

- $C:\ |z-3| = 2|z-4\mathrm{i}|$

$C:\ |z+?-?.??\mathrm{i}| <?.??:$

[ ] check

- $C:\ |z-1+\mathrm{i}| = |z|/2$

$C:\ |z-?.??+?.??\mathrm{i}| <?.??:$

[ ] check

- $C:\ |z-1| = 3|z-2|$

$C:\ |z-?.???| <?.???:$

[ ] check

Solution

squaring and setting $z = x + \mathrm{i}y \quad \rightsquigarrow$

$$\begin{aligned} |z+\mathrm{i}| = 2|z-1| \quad &\iff \quad |x+(y+1)\mathrm{i}|^2 = 4|(x-1)+y\mathrm{i}|^2 \\ &\iff \quad x^2+(y+1)^2 = 4(x-1)^2+4y^2 \end{aligned}$$

rearranging terms and dividing by 3 $\quad \rightsquigarrow$

$$-1 = x^2 - \frac{8}{3}x + y^2 - \frac{2}{3}y$$

completing the squares $\quad \rightsquigarrow$

$$\underbrace{-1 + \left(\frac{4}{3}\right)^2 + \left(\frac{1}{3}\right)^2}_{8/9} = \left(x - \frac{4}{3}\right)^2 + \left(y - \frac{1}{3}\right)^2 ,$$

i.e. an equation of a circle with midpoint $(4/3, 1/3) \,\widehat{=}\, 4/3 + (1/3)\mathrm{i}$ and radius $\sqrt{8/9} = 2\sqrt{2}/3$

Chapter 2

Differentiation and Conformal Mapping

2.1 Real and Imaginary Part of a Complex Function

Write $f(z) = \sin z$ in the form $u(x, y) + \mathrm{i}v(x, y)$ with $z = x + \mathrm{i}y$.

Resources: Complex Function, Formula of Euler Moivre

Problem Variants

■ $f(z) = \cos z$

$u(\pi, \ln 2) = -?.??$:

[ ] check

■ $f(z) = z^3$

$u(2, 1) = ?$:

[ ] check

■ $f(z) = \dfrac{2 + z}{1 + z}$

$u(1, 2) = ?.??$:

[ ] check

Solution

formulas of Euler Moivre,

$$\sin z = \frac{\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}}, \quad \mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t$$

with $z = x + \mathrm{i}y$ and $t = x \quad \Longrightarrow$

$$\sin z = \frac{\mathrm{e}^{\mathrm{i}x-y} - \mathrm{e}^{-\mathrm{i}x+y}}{2\mathrm{i}}$$

and, using $\mathrm{e}^{a+b} = \mathrm{e}^a\mathrm{e}^b$,

$$\begin{aligned} \sin z &= \frac{\mathrm{e}^{-y}(\cos x + \mathrm{i}\sin x) - \mathrm{e}^{y}(\cos x - \mathrm{i}\sin x)}{2\mathrm{i}} \\ &= \frac{\sin x(\mathrm{e}^{-y} + \mathrm{e}^{y})}{2} + \mathrm{i}\frac{\cos x(-\mathrm{e}^{-y} + \mathrm{e}^{y})}{2} \end{aligned}$$

identifying the real and imaginary part $\quad \rightsquigarrow$

$$\begin{aligned} u(x,y) &= \operatorname{Re}\sin z = \sin x \cosh y \\ v(x,y) &= \operatorname{Im}\sin z = \cos x \sinh y \end{aligned}$$

2.2 Complex Differentiability

For which z is the function

$$f(z) = |z-1|^2 + \bar{z}^2$$

complex differentiable?

Resources: Complex Derivative and Cauchy-Riemann Differential Equations

Problem Variants

■ $f(z) = 2z\bar{z} + \bar{z}^3$

---

z with largest imaginary part =?.?+?.??i:

[ ] check

■ $f(z) = \dfrac{1}{z\bar{z}} + \bar{z}$

---

z =?:

[ ] check

■ $f(z) = \dfrac{1+z^2}{1-\bar{z}^2}$

$z = 0$ and $z = \pm$?i:

[ ] check

Solution

Identifying and differentiating the real and imaginary part of f

$f(z) = |z-1|^2 + \bar{z}^2$, $z = x + \mathrm{i}y \quad \rightsquigarrow$

$$f(x+\mathrm{i}y) = |(x-1)+\mathrm{i}y|^2 + (x-\mathrm{i}y)^2 = (x-1)^2 + y^2 + x^2 - y^2 - 2xy\mathrm{i}$$

$\Longrightarrow$

$$u(x,y) = \operatorname{Re} f(x+\mathrm{i}y) = 2x^2 - 2x + 1, \quad v(x,y) = \operatorname{Im} f(x+\mathrm{i}y) = -2xy$$

partial derivatives

$$u_x = 4x - 2, \quad u_y = 0, \quad v_x = -2y, \quad v_y = -2x$$

Determining the points with a complex derivative

necessary and sufficient: Cauchy-Riemann differential equations

$$u_x = v_y, \quad u_y = -v_x$$

substituting the derivatives (1) $\quad \rightsquigarrow$

$$4x - 2 = -2x, \quad 0 = 2y \quad \Longleftrightarrow \quad (x,y) = (1/3, 0)\,,$$

i.e. f is complex differentiable only at the point $z = 1/3$

2.3 Cauchy-Riemann Differential Equations

For which values of the parameters a and b are

$$u(x,y) = 2x^3 - 6xy^2 - 3x^2 + ay^2, \quad v(x,y) = 6x^2y - 2y^b - 6xy$$

the real and imaginary part of a complex differentiable function f? Derive a formula for f as a function of $z = x + \mathrm{i}y$.

Resources: Complex Derivative and Cauchy-Riemann Differential Equations

Problem Variants

■ $u(x,y) = 2x^2 - 2y^a + 2x$, $v(x,y) = 4xy + by$

$a + b =?$:

[ ] check

■ $u(x,y) = x^3 + 6x^2 - 3xy^2 + 12x - ay^2$, $v(x,y) = 12xy + 12y - y^3 + 3x^by$

$a + b =?$:

[ ] check

■ $u(x,y) = 9x^2 - 6x - 9y^a + 1$, $v(x,y) = bxy - 6y$

$a + b =? =$:

[ ] check

Solution

Determination of the parameters a and b

Cauchy-Riemann differential equations (condition for complex differentiability):

$$u_x = v_y, \quad u_y = -v_x$$

computing the partial derivatives of $u(x,y) = 2x^3 - 6xy^2 - 3x^2 + ay^2$ and $v(x,y) = 6x^2y - 2y^b - 6xy \quad \rightsquigarrow$

$$\begin{aligned} u_x &= 6x^2 - 6y^2 - 6x, & v_y &= 6x^2 - 2by^{b-1} - 6x \\ u_y &= -12xy + 2ay, & -v_x &= -12xy + 6y \end{aligned}$$

comparing coefficents and exponents $\quad \Longrightarrow \quad b = 3$ and $a = 3$, i.e.

$$f(x + \mathrm{i}y) = (2x^3 - 6xy^2 - 3x^2 + 3y^2) + \mathrm{i}\,(6x^2y - 2y^3 - 6xy) \qquad (1)$$

Expression for f in terms of z

substituting

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2\mathrm{i}}$$

in (1) $\quad \rightsquigarrow \quad$ expression for f in terms of z

All terms involving the complex conjugate $\bar{z}$ of z cancel, since for a complex differentiable function $\partial f / \partial \bar{z} = 0$. Carrying out the substitution is straightforward, but very tedious; the assistance of Maple™ is helpful.

```
u := 2*x^3-6*x*y^2-3*x^2+3*y^2
v := 6*x^2*y-2*y^3-6*x*y
f := simplify(subs(x=(z+z_bar)/2,y=(z-z_bar)/(2*I),u+I*v))
```

$$f := 2z^3 - 3z^2$$

2.4 Complex Derivative and Jacobi Matrix

Determine the complex derivative $\mathrm{d}w/\mathrm{d}z$ and the Jacobi matrix $\partial(u,v)/\partial(x,y)$ of the function

$$x + \mathrm{i}y = z \mapsto \mathrm{e}^{1/z} = w = u + \mathrm{i}v\,.$$

Resources: Complex Derivative and Cauchy-Riemann Differential Equations

Problem Variants

■ $w = \exp(z^2)$

$u_x(1,0) =$?.??:

[ ] check

■ $w = \dfrac{z}{1+z}$

$u_x(1,0) =$?.??:

[ ] check

■ $w = \cos^2 z$

$u_x(1,0) = -$?.??:

[ ] check

Solution

Complex Derivative

chain rule $\quad \rightsquigarrow$

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^{1/z} = -\frac{1}{z^2}\,\mathrm{e}^{1/z}$$

real representation: $\dfrac{1}{z} = \dfrac{1}{x+\mathrm{i}y} = \dfrac{x-\mathrm{i}y}{x^2+y^2},\ \dfrac{1}{z^2} = \dfrac{x^2-y^2-2\mathrm{i}xy}{(x^2+y^2)^2} \quad \rightsquigarrow$

$$\begin{aligned} \frac{\mathrm{d}w}{\mathrm{d}z} &= -\frac{x^2-y^2-2\mathrm{i}xy}{R^2}\,\exp((x-\mathrm{i}y)/R) \\ \exp((x-\mathrm{i}y)/R) &= \exp(x/R)\,(\cos(y/R)-\mathrm{i}\sin(y/R)) \end{aligned} \tag{1}$$

with $R = x^2+y^2$

Jacobi matrix

$w = u + \mathrm{i}v$, independence of the direction for complex derivatives $\quad \Longrightarrow$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\partial w}{\partial x} = u_x + \mathrm{i}v_x$$

comparison with (1) $\quad \rightsquigarrow$

$$\begin{aligned} u_x &= \operatorname{Re}\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\exp(x/R)}{R^2}\,\big((y^2-x^2)\,\cos(y/R) + 2xy\,\sin(y/R)\big) \\ v_x &= \operatorname{Im}\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\exp(x/R)}{R^2}\,\big((x^2-y^2)\,\sin(y/R) + 2xy\,\cos(y/R)\big) \end{aligned}$$

Cauchy-Riemann differential equations

$$\partial w/\partial x = -\mathrm{i}\partial w/\partial y \quad \Longleftrightarrow \quad u_x = v_y,\ v_x = -u_y$$

$\rightsquigarrow$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} u_x & -v_x \\ v_x & u_x \end{pmatrix}$$

alternative: computing the partial derivatives of

$$\exp(1/z) = \exp((x-\mathrm{i}y)/R) = \underbrace{\exp(x/R)\cos(y/R)}_{u(x,y)} - \mathrm{i}\,\underbrace{\exp(x/R)\sin(y/R)}_{v(x,y)}$$

2.5 Complex Potential

Show that the function

$$u(x,y) = (\mathrm{e}^x + \mathrm{e}^{-x})\cos y$$

possesses a complex potential $f = u + \mathrm{i}v$ and determine the conjugate harmonic function v.

Resources: Complex Potential, Complex Derivative and Cauchy-Riemann Differential Equations

Problem Variants

■ $u(x,y) = 3x^2 + 2x - 3y^2$

$v(1,2) =$??:

[ ] check

■ $u(x,y) = \dfrac{x^3 + x + xy^2}{x^2 + y^2}$

$v(1,2) =$?.?:

[ ] check

■ $u(x,y) = \dfrac{x^2 - y^2}{x^4 + 2x^2y^2 + y^4}$

$v(1,2) = -$?.??:

[ ] check

Solution

Existence of a complex potential

Since

$$u(x, y) = 2\cosh x \, \cos y$$

is globally defined, it is necessary and sufficient that u is harmonic, i.e. $\Delta u = u_{xx} + u_{yy} = 0$.

computing the partial derivatives $\rightsquigarrow$

$$\begin{aligned} u_{xx} &= (2\sinh x \, \cos y)_x = 2\cosh x \, \cos y \\ u_{yy} &= (-2\cosh x \, \sin y)_y = -2\cosh x \, \cos y = -u_{xx} \quad \checkmark \end{aligned}$$

Construction of a conjugate harmonic function v

For a complex potential

$$f(x, y) = u(x, y) + \mathrm{i}v(x, y)\,,$$

v is chosen, so that f is complex differentiable, i.e. u and v satisfy the Cauchy-Riemann differential equations

$$u_x = v_y, \quad u_y = -v_x\,.$$

integrating the first differential equation, $2\sinh x \, \cos y = v_y$ $\rightsquigarrow$

$$v(x, y) = 2\sinh x \, \sin y + c(x)$$

substituting into the second differential equation $\rightsquigarrow$

$$-2\cosh x \, \sin y = -(2\cosh x \, \sin y + c'(x))\,,$$

i.e. $c(x) = C$ (constant) and hence

$$v(x, y) = 2\sinh x \, \sin y + C$$

2.6 Möbius Transformation

Determine the Möbius-Transformation

$$z \mapsto w = \frac{az+b}{cz+d},$$

which maps $0, \mathrm{i}, \infty$ to $\mathrm{i}, \infty, 0$.

Resources: Möbius Transformation

Problem Variants

■ $0, -1, -\mathrm{i} \;\mapsto\; -\mathrm{i}, 0, \infty$

$a, b, c, d =?, ?, ?, \mathrm{i}$:

[ ] check

■ $0, 1, -1 \;\mapsto\; 0, \mathrm{i}, \mathrm{i}/3$

$a, b, c, d = \mathrm{i}, ?, ?, -?$:

[ ] check

■ $0, 1, \infty \;\mapsto\; -1, -1+2\mathrm{i}, 1$

$a, b, c, d = 1, ?+?\mathrm{i}, ?, -?-?\mathrm{i}$:

[ ] check

Solution

Möbius transformation:

$$z \mapsto w = \frac{az+b}{cz+d}$$

Since $w(0) \neq \infty \quad \Longrightarrow \quad d \neq 0$, $d = 1$ is a possible (convenient) choice. The remaining three coefficients can be determined with the interpolation conditions.

- $z = 0 \mapsto w = \mathrm{i} \quad \Longrightarrow \quad \mathrm{i} = \dfrac{0+b}{0+1}$, i.e. $b = \mathrm{i}$
- $z = \mathrm{i} \mapsto w = \infty \quad \Longrightarrow \quad \infty = \dfrac{a\mathrm{i}+\mathrm{i}}{c\mathrm{i}+1} \quad \Longleftrightarrow \quad$ denomiator $= 0$, i.e. $c = \mathrm{i}$
- $z = \infty \mapsto w = 0 \quad \Longrightarrow \quad 0 = \dfrac{a\infty+\mathrm{i}}{\mathrm{i}\infty+1} \quad \Longleftrightarrow \quad 0 = \lim\limits_{z\to\infty} \dfrac{az+\mathrm{i}}{\mathrm{i}z+1}$, i.e. $a = 0$

$\rightsquigarrow \quad w = \mathrm{i}/(\mathrm{i}z+1)$

<u>Alternative solution A</u>

invariance of the cross-ratio $\quad \Longrightarrow$

$$\frac{w-w_2}{w-w_3} : \frac{w_1-w_2}{w_1-w_3} = \frac{z-z_2}{z-z_3} : \frac{z_1-z_2}{z_1-z_3}$$

substituting $(z_k, w_k) :\ (0,\mathrm{i}),\ (\mathrm{i},\infty),\ (\infty,0) \quad \rightsquigarrow$

$$\frac{w-\infty}{w-0} : \frac{\mathrm{i}-\infty}{\mathrm{i}-0} = \frac{z-\mathrm{i}}{z-\infty} : \frac{0-\mathrm{i}}{0-\infty} \quad \Longleftrightarrow \quad \underbrace{\frac{w-\infty}{\mathrm{i}-\infty}}_{=\infty/\infty=1} \frac{\mathrm{i}}{w} = \frac{z-\mathrm{i}}{-\mathrm{i}} \underbrace{\frac{-\infty}{z-\infty}}_{=\infty/\infty=1}$$

Solving for w leads to an equivalent expression for the Möbius transformation.

<u>Alternative solution B</u>

solution of the underdetermined linear system

$$w_k(cz_k+d) = az_k+b, \quad k = 1,2,3$$

2.7 Fixed Point of a Möbius Transformation

Determine the repelling fixed point of the Möbius transformation

$$z \mapsto w = \frac{2z - \mathrm{i}}{z + 1 - \mathrm{i}}\,.$$

Resources: Möbius Transformation

Problem Variants

■ $z \mapsto w = \dfrac{2\mathrm{i}z - 1}{z + 2\mathrm{i}}$

−?:

[ ] check

■ $z \mapsto w = \dfrac{4z + 1}{3z + 2}$

−?.??:

[ ] check

■ $z \mapsto w = \dfrac{1}{2z + 3\mathrm{i}}$

−?:

[ ] check

Solution

Determination of the fixed points

fixed point equation

$$z = w = \frac{2z - \mathrm{i}}{z + 1 - \mathrm{i}} \quad \Longleftrightarrow \quad z^2 - (1 + \mathrm{i})z + \mathrm{i} = 0$$

applying the formula for the solution of a quadratic equation $\rightsquigarrow$

$$z = \frac{1 + \mathrm{i} \pm \sqrt{(1 + \mathrm{i})^2 - 4\mathrm{i}}}{2} = \frac{1 + \mathrm{i} \pm \sqrt{-2\mathrm{i}}}{2} = \frac{1 + \mathrm{i} \pm (1 - \mathrm{i})}{2},$$

i.e. $z_+ = 1$, $z_- = \mathrm{i}$

Type of the fixed points

A fixed point is repelling (attracting) if $|\mathrm{d}w/\mathrm{d}z| > 1$ (< 1).
evaluating the derivative

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{2(z + 1 - \mathrm{i}) - (2z - \mathrm{i})}{(z + 1 - \mathrm{i})^2} = \frac{2 - \mathrm{i}}{(z + 1 - \mathrm{i})^2}$$

at $z_\pm$ $\rightsquigarrow$

$$\left.\frac{\mathrm{d}w}{\mathrm{d}z}\right|_{z=1} = \frac{2 - \mathrm{i}}{(2 - \mathrm{i})^2} = \frac{1}{2 - \mathrm{i}}, \quad \left.\frac{\mathrm{d}w}{\mathrm{d}z}\right|_{z=\mathrm{i}} = 2 - \mathrm{i}$$

$\Longrightarrow$ $z = \mathrm{i}$ with $|\mathrm{d}w/\mathrm{d}z|_{z=\mathrm{i}}| = |2 - \mathrm{i}| = \sqrt{5} > 1$ is the repelling fixed point.

2.8 Image of a Circle Under a Möbius Transformation

Determine the image of the circle $C: |z| = 1$ under the Möbius transformation

$$z \mapsto w = \frac{1}{2z + \mathrm{i}}.$$

Resources: Circle in the Complex Plane

Problem Variants

■ $z \mapsto w = \dfrac{2z - \mathrm{i}}{2z - 1}$

---

radius and midpoint in the w-plane: ?.??, ?.??−?.??i:

check

■ $z \mapsto w = \dfrac{3z - 2}{3z - 1}$

---

radius and midpoint in the w-plane: ?.???, ?.???:

check

■ $z \mapsto w = \dfrac{3\mathrm{i}}{1 - 4z}$

---

radius and midpoint in the w-plane: ?.?, −?.?i:

check

Solution

Equation of the circle in the w-plane

expressing z in terms of w (inverting the transformation) $\rightsquigarrow$

$$w = \frac{1}{2z + \mathrm{i}} \quad \Longleftrightarrow \quad 2zw + \mathrm{i}w = 1 \quad \Longleftrightarrow \quad z = \frac{1 - \mathrm{i}w}{2w}$$

substituting into the equation $|z| = 1$ $\rightsquigarrow$

$$\left| \frac{1 - \mathrm{i}w}{2w} \right| = 1 \quad \Longleftrightarrow \quad |1 - \mathrm{i}w| = 2|w| \tag{1}$$

Midpoint and radius

squaring equation (1) with $w = u + \mathrm{i}v$ $\rightsquigarrow$

$$|1 - \mathrm{i}u + v|^2 = 4|u + \mathrm{i}v|^2 \quad \Longleftrightarrow \quad (1 + v)^2 + u^2 = 4u^2 + 4v^2$$

rearranging terms, dividing by 3, and completing the square $\rightsquigarrow$

$$1/3 = u^2 + v^2 - (2/3)v \quad \Longleftrightarrow \quad \underbrace{1/3 + 1/9}_{4/9} = u^2 + (v - 1/3)^2$$

equation of a circle with radius $\sqrt{4/9} = 2/3$ and midpoint $(0, 1/3) \mathrel{\widehat{=}} w = \mathrm{i}/3$

2.9 Conformal Mapping

Construct a complex differentiable function $f : z \mapsto w$ with $f(0) = 1$, which maps the strip $S : 0 < \operatorname{Im} z < 1$ onto the disc $D : |w| < 1$. Illustrate the preservation of angles by plotting the image of a coordinate grid of the z-plane.

Resources: Elementary Conformal Mappings

Problem Variants

■ strip $S : 0 < \operatorname{Re} z < \pi \;\rightarrow$ half plane $H : 1 < \operatorname{Im} w,\ f(0) = 1 + \mathrm{i}$

$f(\pi/2) =$?i:

[ ] check

■ disc $D : |z| < 1 \;\rightarrow$ quadrant $Q : \operatorname{Re} w, \operatorname{Im} w > 0,\ f(1) = 1$

$f(-\mathrm{i}) =$?:

[ ] check

■ sector $A : z = r\,\mathrm{e}^{\mathrm{i}\varphi},\ 0 < \varphi < \pi/4 \;\rightarrow$ strip $S : 1 < \operatorname{Re} z < 2,\ f(1) = 1$

$f(\exp(\pi)) =$?−?i:

[ ] check

Solution

The function f is constructed by composing elementary conformal maps.

Strip $S:\ 0 < \operatorname{Im} z < 1\ \rightarrow$ half plane $H:\ \operatorname{Im}\xi > 0$

$$S \ni z = x + \mathrm{i}y \overset{g}{\mapsto} \xi = \mathrm{e}^{\pi z} = \mathrm{e}^{\pi x}\,\mathrm{e}^{\mathrm{i}\pi y}$$

$\mathrm{e}^{\pi x} \in (-\infty, \infty) \quad \Longrightarrow \quad$ For $0 < y < 1$, H is a sector with opening angle π, i.e. the half plane of points with positive imaginary part.

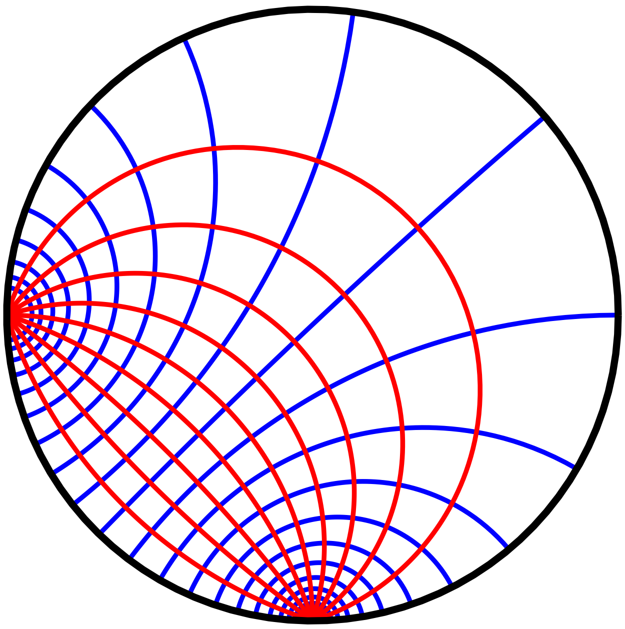
Half plane $H:\ \operatorname{Im}\xi > 0\ \rightarrow$ disc $D:\ |w| < 1$

$$H \ni \xi \overset{h}{\mapsto} w = \frac{1 + \mathrm{i} - \xi}{\xi + \mathrm{i} - 1} \in D$$

The function h is a Möbius transformation which maps $\xi = 0, 1, \infty$ to $w = -\mathrm{i}, 1, -1$.
In view of the consistent orientation of the three points, determining the Möbius transformation, the half plane H and the disc D are lying to the left of ∂H and ∂D, and hence are mapped onto each other.

Mapping of the coordinate grid

image curves of the vertical (blue) and horizontal grid lines (red)



MATLAB® -script for generating the graphic.

```
hold on

f = @(z) (1+i-exp(pi*z))./(exp(pi*z)+i-1);

% images of vertical grid lines (x=-1,-0.9,...,1)
% represented by the points x(:,k), y(:,k)
[x,y] = meshgrid([-1:0.1:1],[0:0.01:1])
w = f(x+i*y);
plot(real(w),imag(w),'-b')

% images of horizontal grid lines (y=0,0.1,...,1)
% represented by the points x(k,:), y(k,:)
[x,y] = meshgrid([-10:0.01:10],[0:0.1:1]);
w = f(x+i*y);
plot(real(w)',imag(w)','-r')

% unit circle
t = linspace(0,2*pi);
plot(cos(t),sin(t),'-k')

hold off
```

Chapter 3

Integration

3.1 Complex Integral over a Line Segment

Compute $\int_C |z|^2 \, \mathrm{d}z$ for the straight path C from i to 1.

Resources: Complex Line Integral

Problem Variants

■ $\int_C (\operatorname{Re} z + \operatorname{Im} z) \, \mathrm{d}z$, $C: \, 2 \to 1 + 3\mathrm{i}$

−?+?i:

[ ] check

■ $\int_C \bar{z}^3 \, \mathrm{d}z$, $C: \, 0 \to 1 + 2\mathrm{i}$

−?.??−?i:

[ ] check

■ $\int_C (\operatorname{Re} z \operatorname{Im} z) \, \mathrm{d}z$, $C: \, 3 \to 2 - \mathrm{i}$

?.??+?.??i:

[ ] check

Solution

Parametrization

straight path C from a to b

$$t \mapsto z(t) = a + t(b-a), \quad 0 \le t \le 1$$

$a = \mathrm{i}$, $b = 1$ $\quad\rightsquigarrow$

$$z(t) = \mathrm{i} + t(1-\mathrm{i}) = t + (1-t)\mathrm{i}$$

Integral

definition of the complex line integral

$$\int_C f\,\mathrm{d}z = \int_a^b f(z(t))\,\underbrace{z'(t)}_{\mathrm{d}z/\mathrm{d}t}\,\mathrm{d}t$$

for a path from a to b
$z(t) = t + (1-t)\mathrm{i}$, $\mathrm{d}z = (1-\mathrm{i})\,\mathrm{d}t$ and $|x+\mathrm{i}y|^2 = x^2 + y^2$ $\quad\rightsquigarrow$

$$\begin{aligned}\int_C |z|^2\,\mathrm{d}z &= \int_0^1 |t + (1-t)\mathrm{i}|^2\,(1-\mathrm{i})\,\mathrm{d}t = (1-\mathrm{i})\int_0^1 (t^2 + (1-t)^2)\,\mathrm{d}t \\ &= (1-\mathrm{i})\,\left[t^3/3 - (1-t)^3/3\right]_{t=0}^{t=1} = \frac{2}{3}(1-\mathrm{i})\end{aligned}$$

3.2 Complex Integral over a Circular Segment

Compute

$$\int_C \operatorname{Re} z \, \operatorname{Im} z \, \mathrm{d}z$$

for the quarter circle C in the first quadrant with midpoint 0 and radius 1 (oriented counterclockwise).

Resources: Complex Line Integral

Problem Variants

■ $\int_C |z|^2 \, \mathrm{d}z$, C: circle with midpoint 1 and radius 2

?πi:

[ ] [check]

■ $\displaystyle\int_C \frac{\mathrm{d}z}{\bar{z}}$, C: semicircle in the halfplane $H : \operatorname{Re} z \geq 0$ with midpoint i and radius 3

?.??:

[ ] [check]

■ $\int_C (\operatorname{Re} z)^2 \, \mathrm{d}z$, C: circle with midpoint $1 + \mathrm{i}$ and radius 1

?πi:

[ ] [check]

Solution

Parametrization

segment of a circle with center p and radius r:

$$C:\ \varphi \mapsto z(\varphi) = p + r\mathrm{e}^{\mathrm{i}\varphi}, \quad \varphi_{\min} \le \varphi \le \varphi_{\max}$$

quarter circle with $p = 0$, $r = 1$:

$$z(\varphi) = \mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi, \quad 0 \le \varphi \le \pi/2$$

Integration

definition of the complex line integral

$$\int_C f(z)\,\mathrm{d}z = \int_{\varphi_{\min}}^{\varphi_{\max}} f(z(\varphi))\,\underbrace{z'(\varphi)}_{\mathrm{d}z/\mathrm{d}\varphi}\,\mathrm{d}\varphi$$

with $z'(\varphi) = \mathrm{i}\mathrm{e}^{\mathrm{i}\varphi} = \mathrm{i}\cos\varphi - \sin\varphi \quad \rightsquigarrow$

$$\begin{aligned} \int_0^{\pi/2} \operatorname{Re} z\ \operatorname{Im} z\,\mathrm{d}z &= \int_0^{\pi/2} \cos\varphi\sin\varphi(\mathrm{i}\cos - \sin\varphi)\,\mathrm{d}\varphi \\ &= \left[-\mathrm{i}\frac{\cos^3\varphi}{3} - \frac{\sin^3\varphi}{3}\right]_{\varphi=0}^{\varphi=\pi/2} = \frac{\mathrm{i}}{3} - \frac{1}{3} \end{aligned}$$

3.3 Line Integral of a Complex Differentiable Function

Compute

$$\int_C z^2 \, \mathrm{d}z$$

for a path C from $1-\mathrm{i}$ to $1+\mathrm{i}$.

Resources: Complex Line Integral

Problem Variants

- $\displaystyle\int_{C:0\to\mathrm{i}} z\mathrm{e}^z \, \mathrm{d}z$

$-?.??$:

[ ] check

- $\displaystyle\int_{C:2\to 1-\mathrm{i}} \frac{4}{z^3} \, \mathrm{d}z$

$?.?-?\mathrm{i}$:

[ ] check

- $\displaystyle\int_{C:0\to\pi+\mathrm{i}\ln 2} \cos z \, \mathrm{d}z$

$?.??\mathrm{i}$:

[ ] check

Solution

existence of an antiderivative F for f, i.e. $f = F'$ $\implies$

$$\int_C f(z)\,\mathrm{d}z = [F]_a^b = F(b) - F(a)$$

for any path C from a to b
application to

$$f(z) = z^2, \quad F(z) = z^3/3, \quad a = 1 - \mathrm{i}, \quad b = 1 + \mathrm{i}$$

$\rightsquigarrow$

$$\int_C z^2\,\mathrm{d}z = \left[\frac{z^3}{3}\right]_{z=1-\mathrm{i}}^{z=1+\mathrm{i}} = \frac{1}{3}(1+\mathrm{i})^3 - \frac{1}{3}(1-\mathrm{i})^3$$

simplifying, using the binomial formula $(u+v)^3 = u^3 + 3u^2v + 3uv^2 + v^3$ $\rightsquigarrow$

$$\left(\frac{1}{3} + \mathrm{i} - 1 - \frac{1}{3}\mathrm{i}\right) - \left(\frac{1}{3} - \mathrm{i} - 1 + \frac{1}{3}\mathrm{i}\right) = \frac{4}{3}\mathrm{i}$$

3.4 Antiderivative and Complex Line Integral

Compute

$$\int_C \frac{3 + 4\sin(z)}{\exp(2z)}\,\mathrm{d}z$$

for a path C from 0 to πi.

Resources: Complex Line Integral

Problem Variants

■ $\int_C \cos z\ \exp z\,\mathrm{d}z,\ C:\ 0 \to \pi\mathrm{i}$

−?.??−?.??i:

check

■ $\int_C \sin^2 z\,\mathrm{d}z,\ C:\ 0 \to \pi\mathrm{i}$

−??.?i:

check

■ $\int_C z\exp z\,\mathrm{d}z,\ C:\ 0 \to \pi\mathrm{i}$

?−?.??i:

check

Solution

Antiderivative

Formel von Euler-Moivre, $\sin z = \dfrac{1}{2\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}z} - \mathrm{e}^{-\mathrm{i}z}\right) \quad \rightsquigarrow$

$$f(z) = \frac{3 + 4\sin(z)}{\exp(2z)} = 3\,\mathrm{e}^{-2z} + \frac{2}{\mathrm{i}}\left(\mathrm{e}^{(-2+\mathrm{i})z} - \mathrm{e}^{(-2-\mathrm{i})z}\right)$$

antiderivative $\mathrm{e}^{\lambda z}/\lambda$ of $\mathrm{e}^{\lambda z} \quad \rightsquigarrow \quad$ expression for the antiderivative of f

$$F(z) = -\frac{3}{2}\mathrm{e}^{-2z} + \frac{2\,\mathrm{e}^{(-2+\mathrm{i})z}}{-2\mathrm{i}-1} - \frac{2\,\mathrm{e}^{(-2-\mathrm{i})z}}{-2\mathrm{i}+1}$$

Line integral

applying the formula $\displaystyle\int\limits_{C:a\to b} f(z)\,\mathrm{d}z = [F]_a^b$, valid for a complex differentiable function in a domain containing the path C, and using that $\mathrm{e}^{2\pi\mathrm{i}} = 1 \quad \rightsquigarrow$

$$\begin{aligned} \int_C f(z)\,\mathrm{d}z &= \left[-\frac{3}{2}\,\mathrm{e}^{-2z} + \frac{2\,\mathrm{e}^{(-2+\mathrm{i})z}}{-2\mathrm{i}-1} - \frac{2\,\mathrm{e}^{(-2-\mathrm{i})z}}{-2\mathrm{i}+1}\right]_0^{\pi\mathrm{i}} \\ &= \left(-\frac{3}{2} + \frac{2\,\mathrm{e}^{-\pi}}{-2\mathrm{i}-1} - \frac{2\,\mathrm{e}^{\pi}}{-2\mathrm{i}+1}\right) - \left(-\frac{3}{2} + \frac{2}{-2\mathrm{i}-1} - \frac{2}{-2\mathrm{i}+1}\right) \end{aligned}$$

expanding with $2\mathrm{i}-1$ and $2\mathrm{i}+1$ to make the denominators real $\quad \rightsquigarrow$

$$\frac{(4\mathrm{i}-2)\mathrm{e}^{-\pi}}{5} - \frac{(4\mathrm{i}+2)\mathrm{e}^{\pi}}{5} - \frac{4\mathrm{i}-2}{5} + \frac{4\mathrm{i}+2}{5} = \frac{4}{5}\left(1 - \cosh(\pi) - 2\mathrm{i}\sinh(\pi)\right)$$

3.5 Residues of a Function with Simple Poles

Determine the residues of the function $f(z) = \dfrac{z^3+1}{z^2-4}$.

Resources: Residue

Problem Variants

- $f(z) = \dfrac{1}{z^2+z-2}$

---

largest residue ?.??:

[ ] check

- $f(z) = \dfrac{3z+4}{z(z^2-1)}$

---

largest residue ?.?:

[ ] check

- $f(z) = \dfrac{3z+4}{z(z^2-1)}$

---

sum of the absolute values of the residues ?:

[ ] check

Solution

At a simple pole at a of a function f,

$$\operatorname{Res}_a f = (z-a)f(z)|_{z=a}\,.$$

For a rational function, this expression can be computed by factoring the denominator.
application to $f(z) = \dfrac{z^3+1}{z^2-4} = \dfrac{z^3+1}{(z-2)(z+2)}$:

- $a = 2$: multiplication with $(z-2)$ $\rightsquigarrow$

$$\operatorname{Res}_2 f(z) = \left.\frac{z^3+1}{z+2}\right|_{z=2} = \frac{9}{4}$$

- $a = -2$: multiplication with $(z+2)$ $\rightsquigarrow$

$$\operatorname{Res}_{-2} f(z) = \left.\frac{z^3+1}{z-2}\right|_{z=-2} = \frac{7}{4}$$

3.6 Residue at a Pole of Higher Order or at an Essential Singularity

Determine the residue of $f(z) = \dfrac{\sin(3z)}{z^4}$ at $z = 0$.

Resources: Residue

Problem Variants

- $f(z) = \dfrac{3}{\cos(2z) - 1}$, $z = 0$

?:

[ ] check

- $f(z) = \dfrac{1}{z^3 + z^2 - z - 1}$, $z = -1$

$-$?.??:

[ ] check

- $f(z) = z^2 e^{1/(2z)}$, $z = 0$

0.0???:

[ ] check

Solution

Taylor series of the sine function $\rightsquigarrow$ Laurent series at $z = 0$

$$f(z) = \frac{\sin(3z)}{z^4} = \frac{3z - (3z)^3/6 + O(z^5)}{z^4} = \frac{3}{z^3} - \frac{9/2}{z} + O(z)$$

and $\operatorname*{Res}_{z=0} f(z) = -(9/2)$ (coeffizient of $1/z$)
Alternative: application of the formula for a residue at a pole of oder 3:

$$\operatorname*{Res}_{0} f = \lim_{z \to 0} \frac{1}{2} \, (\mathrm{d}/\mathrm{d}z)^2 \, (z^3 f(z))$$

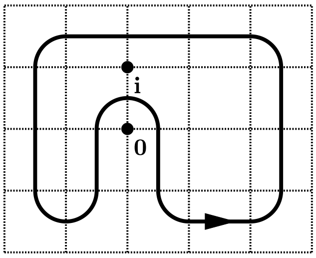
(more complicated for this problem)

3.7 Line Integral of a Rational Function

Integrate the function

$$f(z) = \frac{z^4}{z^3 - z^2 + z - 1}$$

over the depicted path.



Resources: Residue Theorem, Residue

Problem Variants

■ $f(z) = \dfrac{1}{z^3 - z^2 - 2z}$

?.??i:

[ ] check

■ $f(z) = \dfrac{1}{z^3 + z^2 - 4z + 6}$, pole at $z = -3$

−?.??i:

[ ] check

■ $f(z) = \dfrac{1}{z^3 + 2z^2 + 2z}$,

−?.??i:

[ ] check

Solution

Poles

Guessing the zero $z_1 = 1$ of the denominator $q(z) = z^3 - z^2 + z - 1$ of f (pole of f), the other poles z_k can be determined by dividing q by the linear factor $z - 1$:

$$\begin{array}{rrrrl} (\,z^3 & -z^2 & +z & -1\,) : (z-1) & = \quad z^2 + 1 =: \tilde{q}(z) \\ z^3 & -z^2 & & & \\ \hline & 0 & +z & -1 & \\ & & z & -1 & \\ \hline & & & 0 & \end{array}$$

zeros of $\tilde{q}(z) = q(z)/(z-1)$ $\quad\rightsquigarrow\quad$ poles $z_2 = \mathrm{i}$, $z_3 = -\mathrm{i}$ and the factorization

$$f(z) = \frac{z^4}{z^3 - z^2 + z - 1} = \frac{z^4}{(z-1)(z-\mathrm{i})(z+\mathrm{i})} \tag{1}$$

Application of the residue theorem

The line integral can be computed by adding the residues of f at the singularities z_k in the domain enclosed by the path C, traversed in counterclockwise direction:

$$\int\limits_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_{z_k \in D} \operatorname*{Res}_{z_k} f\,. \tag{2}$$

In the given problem, only two of the poles, $z_1 = 1$ and $z_2 = \mathrm{i}$, ly inside D. applying the formula

$$\operatorname*{Res}_{a} f = \lim_{z \to a} (z-a) f(z)$$

for the residue at a simple pole to the factorization (1) of f $\quad\rightsquigarrow$

- $\operatorname*{Res}_{1} f = \left.\dfrac{z^4}{(z-\mathrm{i})(z+\mathrm{i})}\right|_{z=1} = \dfrac{1}{(1-\mathrm{i})(1+\mathrm{i})} = \dfrac{1}{2}$
- $\operatorname*{Res}_{\mathrm{i}} f = \left.\dfrac{z^4}{(z-1)(z+\mathrm{i})}\right|_{z=\mathrm{i}} = \dfrac{1}{(\mathrm{i}-1)2\mathrm{i}} = \dfrac{\mathrm{i}-1}{4}$

adding the residues according to (2) $\quad\rightsquigarrow$

$$\int\limits_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\left(\frac{1}{2} + \frac{\mathrm{i}-1}{4}\right) = 2\pi\mathrm{i}\,\frac{1+\mathrm{i}}{4} = \frac{\pi(\mathrm{i}-1)}{2}$$

3.8 Complex Integration over a Circle

Compute

$$\int_C \frac{\mathrm{e}^z}{z^2+\pi^2}\,\mathrm{d}z$$

for the counterclockwise oriented circle C around i with radius 3.

Resources: Residue Theorem, Residue

Problem Variants

- $\int_C \frac{\cos z}{\mathrm{e}^z - 1}\,\mathrm{d}z$

?πi:

[ ] check

- $\int_C \frac{\sin z}{4z^2-\pi^2}\,\mathrm{d}z$

?i:

[ ] check

- $\int_C \frac{z^2-9}{z^2+9}\,\mathrm{d}z$

$-?\pi$:

[ ] check

Solution

Application of the residue theorem

complex line integral $\quad\rightarrow\quad$ sum of residues:

$$I = \int_C \frac{\mathrm{e}^z}{z^2+\pi^2}\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{a\in D}\operatorname*{Res}_{z=a}\frac{\mathrm{e}^z}{z^2+\pi^2}$$

with C the counterclockwise oriented boundary of the disc $D:\ |z-\mathrm{i}|<3$

Computation of the residues and the integral

binomial formula $\quad\rightsquigarrow$

$$\frac{\mathrm{e}^z}{z^2+\pi^2} = \frac{\mathrm{e}^z}{(z-\pi\mathrm{i})(z+\pi\mathrm{i})}$$

poles $z_\pm = \pm\pi\mathrm{i}$
$|z_- - \mathrm{i}| = |-\pi\mathrm{i}-\mathrm{i}| = \pi+1 > 3 \quad\Longrightarrow\quad z_- \notin D$, i.e. only $z_+ = \pi\mathrm{i}$ is relevant and

$$\operatorname*{Res}_{z=\pi\mathrm{i}}\frac{\mathrm{e}^z}{z^2+\pi^2} = \lim_{z\to\pi\mathrm{i}}(z-\pi\mathrm{i})\frac{\mathrm{e}^z}{(z-\pi\mathrm{i})(z+\pi\mathrm{i})} = \frac{\mathrm{e}^{\pi\mathrm{i}}}{2\pi\mathrm{i}} = -\frac{1}{2\pi\mathrm{i}}$$

$\rightsquigarrow \quad I = 2\pi\mathrm{i}(-1/(2\pi\mathrm{i})) = -1$

3.9 Integral of a Rational Function over $\mathbb{R}$

Compute $\int_{\mathbb{R}} \frac{3}{z^2 - 4z + 5}\, \mathrm{d}z$.

Resources: Residue Theorem, Residue

Problem Variants

- $\int_{\mathbb{R}} \frac{3}{z^2 + 1}\, \mathrm{d}z$

?π:
[ ] check

- $\int_{\mathbb{R}} \frac{2}{z^2 + 2z + 2}\, \mathrm{d}z$

?π:
[ ] check

- $\int_{\mathbb{R}} \frac{4}{z^4 + 4}\, \mathrm{d}z$

?π:
[ ] check

Solution

Poles of f

zeros of the denominator of $f(z) = \dfrac{3}{z^2 - 4z + 5}$:

$$a_\pm = 2 \pm \sqrt{4-5} = 2 \pm \mathrm{i}$$

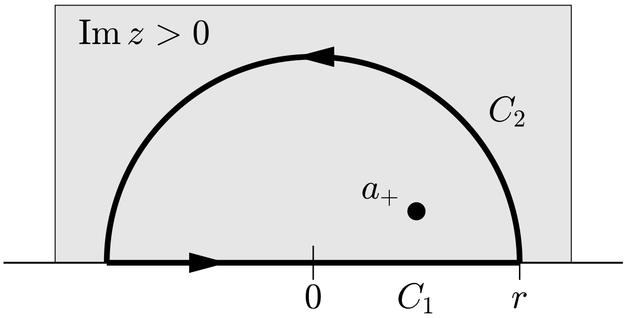
$\rightsquigarrow$ factorization

$$f(z) = \frac{3}{(z-2-\mathrm{i})(z-2+\mathrm{i})}$$

Application of the residue theorem

For the semicircle $C_1 + C_2$ (center 0 and radius r) in the upper half plane (positive imaginary part),

$$\int_{C_1+C_2} f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\operatorname*{Res}_{a_+} f, \quad a_+ = 2 + \mathrm{i}\,.$$



For $r \to \infty$

- $\int_{C_2} f(z)\,\mathrm{d}z \to 0$ since
$$\left|\int_{C_2} \frac{3}{z^2-4z+5}\right| \le \underbrace{(\pi r)}_{\text{length of } C_2} \frac{3}{r(r - |4+5/r|)} \le \frac{3\pi}{r/2} \quad \text{if } r/2 \ge |4+5|\,.$$

- $\displaystyle\int_{C_1} f(z)\,\mathrm{d}z \to \int_{\mathbb{R}} f(z)\,\mathrm{d}z$

$\Longrightarrow$

$$s := \int_{\mathbb{R}} \frac{3}{z^2-4z+5} = 2\pi\mathrm{i}\operatorname*{Res}_{2+\mathrm{i}} f$$

Computation of the residue

applying the formula for the residue at a simple pole and using the factorization of the denominator of f $\quad\rightsquigarrow$

$$\operatorname*{Res}_{2+\mathrm{i}} f = (z-2-\mathrm{i})f(z)|_{z=2+\mathrm{i}} = \left.\frac{3}{z-2+\mathrm{i}}\right|_{z=2+\mathrm{i}} = \frac{3}{2\mathrm{i}}$$

$\Longrightarrow \qquad s = 2\pi\mathrm{i}\cdot(3/(2\mathrm{i})) = 3\pi$

3.10 Trigonometric Integral

Compute $\int_0^{2\pi} \frac{\mathrm{d}t}{2+\cos t}$.

Resources: Residue Theorem, Residue

Problem Variants

- $\int_0^{2\pi} \frac{2}{3-\sin t}\,\mathrm{d}t$

?.??:

check

- $\int_0^{2\pi} \frac{1}{2+\cos t+\sin t}\,\mathrm{d}t$

?.??:

check

- $\int_0^{2\pi} \frac{1}{3+\cos^2 t}\,\mathrm{d}t$

?.??:

check

Solution

Conversion to a complex integral

formula of Euler-Moivre, $\cos t = (\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{-\mathrm{i}t})/2 \quad \rightsquigarrow$

$$f(t) = \frac{1}{2+\cos t} = \frac{2}{4+\mathrm{e}^{\mathrm{i}t}+\mathrm{e}^{-\mathrm{i}t}}$$

substituting the parametrization

$$C:\ t \mapsto z(t) = \mathrm{e}^{\mathrm{i}t}, \quad 0 \le t \le 2\pi\,,$$

of the unit circle C with $\mathrm{d}z = \mathrm{i}z\mathrm{d}t \quad \rightsquigarrow$

$$s := \int_0^{2\pi} \frac{\mathrm{d}t}{2+\cos t} = \int_C \underbrace{\frac{2}{\mathrm{i}(4z+z^2+1)}}_{g(z)}\,\mathrm{d}z$$

Computation of the complex integral

residue theorem $\quad \Longrightarrow$

$$s = \int_C g(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\operatorname*{Res}_a g \tag{1}$$

with a the pole of g inside the circle C with center 0 an radius 1

$$g(z) = \frac{2}{\mathrm{i}(z+2+\sqrt{3})(z+2-\sqrt{3})}\,\mathrm{d}z \qquad \Longrightarrow \qquad a = -2+\sqrt{3} \approx -0.2679$$

$(-2-\sqrt{3}$ lies outside of the circle) and applying the formula for the residue at a simple pole

$$\operatorname*{Res}_a g = (z-a)g(z)|_{z=a} = \frac{2}{\mathrm{i}(-2+\sqrt{3}+2+\sqrt{3})} = \frac{2}{\mathrm{i}(2\sqrt{3})}$$

substituting into (1) $\quad \rightsquigarrow$

$$s = (2\pi\mathrm{i})\,\frac{2}{\mathrm{i}(2\sqrt{3})} = \frac{2\pi}{\sqrt{3}}$$

3.11 Improper Integrals with Trigonometric/Rational Functions

Compute $\displaystyle\int_{\mathbb{R}} \frac{\sin(2x)}{x^2+2x+2}\,\mathrm{d}x$.

Resources: Residue Theorem, Residue

Problem Variants

- $\displaystyle\int_{\mathbb{R}} \frac{\cos x}{x^2+1}\,\mathrm{d}x$

?.??:

[ ] check

- $\displaystyle\int_{\mathbb{R}} \frac{\sin^2(2x)}{x^2+2}\,\mathrm{d}x$

?.??:

[ ] check

- $\displaystyle\int_{\mathbb{R}} \frac{\mathrm{e}^{ix}}{x^2-4x+5}\,\mathrm{d}x$

-?.??+?.??i:

[ ] check

Solution

Poles of f

zeros of the denominator of $f(x) = \dfrac{\sin(2x)}{x^2+2x+2}$ $\quad\rightsquigarrow\quad$ poles

$$a_\pm = -1 \pm \sqrt{1-2} = -1 \pm \mathrm{i}$$

and the factorization

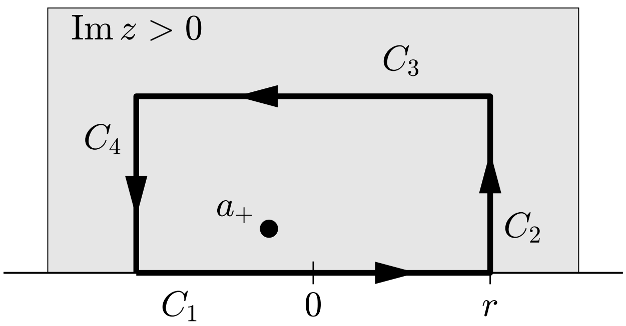
$$p(z) = z^2 + 2z + 2 = (z+1-\mathrm{i})(z+1+\mathrm{i})$$

Application of the residue theorem

formula of Euler-Moivre, $\mathrm{e}^{\mathrm{i}t} = \cos t + \mathrm{i}\sin t$ $\quad\rightsquigarrow\quad$ complex integral:

$$s := \int_{\mathbb{R}} f(x)\,\mathrm{d}x = \operatorname{Im} \int_{\mathbb{R}} \underbrace{\frac{\mathrm{e}^{2\mathrm{i}z}}{p(z)}}_{g(z)}\,\mathrm{d}z = \operatorname{Im}\left(\lim_{r\to\infty}\int_{C_1} g(z)\,\mathrm{d}z\right)$$

with C_1 the path from $-r$ to r



For the rectangle (upper right corner $r+r\mathrm{i}$) with boundary $C_1+C_2+C_3+C_4$ in the upper half plane (positive imaginary part), the residue theorem implies

$$\int_{C_1+C_2+C_3+C_4} g(z)\,\mathrm{d}z = 2\pi\mathrm{i}\operatorname*{Res}_{a_+} g$$

with $a_+ = -1+\mathrm{i}$ the pole of g in the upper half plane.
$\rightsquigarrow$ computation of $\int_{\mathbb{R}} f$ by subtracting from $2\pi\mathrm{i}\operatorname*{Res}_{a_+} g$ the integrals $s_k = \int_{C_k} g(z)\,\mathrm{d}z$, $k=2,3,4$, and letting r tend to ∞

Estimation of the complex line integrals

for $z \in C_k$, $k = 2,3,4$, $|z| \geq r$ and

$$|\underbrace{(z+1-\mathrm{i})(z+1+\mathrm{i})}_{p(z)}| \geq (|z| - |1-\mathrm{i}|)(|z| - |1+\mathrm{i}|) \geq (r/2)^2$$

if $(r/2) \geq |1-\mathrm{i}| = |1+\mathrm{i}| = \sqrt{2}$

estimate of p $\quad\rightsquigarrow\quad$ bounds for $s_k := \int_{C_k} g(z)\,\mathrm{d}z$

- $C_2:\ t \mapsto z(t) = r + \mathrm{i}t,\ 0 \leq t \leq r$

  $$|s_1| = \left| \int_0^r \frac{\mathrm{e}^{2\mathrm{i}r-2t}}{p(r+\mathrm{i}t)}\,\mathrm{d}t \right| \leq \int_0^r \frac{|\mathrm{e}^{2\mathrm{i}r}|\,|\mathrm{e}^{-2t}|}{|p(r+\mathrm{i}t)|}\,\mathrm{d}t \leq \frac{1}{(r/2)^2} \int_0^\infty \mathrm{e}^{-2t}\,\mathrm{d}t = \frac{2}{r}$$

  identical estimate for s_4

- $C_3:\ t \mapsto z(t) = -t + \mathrm{i}r,\ -r \leq t \leq r$
  using $|\int_{-r}^r g(-t+\mathrm{i}r)\,\mathrm{d}t| \leq (2r)\,\max|g|$ $\quad\rightsquigarrow$

  $$|s_3| = \left| \int_{-r}^r \frac{\mathrm{e}^{-2\mathrm{i}t-2r}}{p(t+\mathrm{i}r)}\,\mathrm{d}t \right| \leq (2r)\,\mathrm{e}^{-2r}\,\frac{1}{(r/2)^2} = \frac{8}{r}\,\mathrm{e}^{-2r}$$

-

estimates $\quad\Longrightarrow\quad$ $s_k \to 0$ for $r \to \infty$ and hence

$$\int_{\mathbb{R}} f = \operatorname{Im} \lim_{r\to\infty} \underbrace{\int_{C_1} g(z)\,\mathrm{d}z}_{s_1} = \operatorname{Im} \lim_{r\to\infty} (s_1 + s_2 + s_3 + s_4) = \operatorname{Im}\left(2\pi\mathrm{i} \operatorname*{Res}_{a_+} g\right)$$

Computation of the residue

formula for the residue at a simple pole $\quad\rightsquigarrow$

$$\begin{aligned} \operatorname*{Res}_{a_+} g &= (z-a_+)g(z)|_{z=a_+} = \left.\frac{\mathrm{e}^{2\mathrm{i}z}}{z+1+\mathrm{i}}\right|_{z=-1+\mathrm{i}} \\ &= \frac{\mathrm{e}^{-2\mathrm{i}-2}}{2\mathrm{i}} = \frac{\mathrm{e}^{-2}(\cos(-2) + \mathrm{i}\sin(-2))}{2\mathrm{i}} = \frac{\mathrm{e}^{-2}(-\mathrm{i}\cos 2 - \sin 2)}{2} \end{aligned}$$

and hence

$$\int_{\mathbb{R}} f = \operatorname{Im}\left(2\pi\mathrm{i} \operatorname*{Res}_{a_+} g\right) = \operatorname{Im}\left(\mathrm{e}^{-2}(\pi\cos 2 - \pi\mathrm{i}\sin 2)\right) = -\pi\mathrm{e}^{-2}\sin 2$$

3.12 Fourier Transform of a Rational Function

Compute $\hat{f}(y) = \int_{-\infty}^{\infty} \underbrace{\frac{1}{x^4 + 2x^2 + 1}}_{f(x)} \mathrm{e}^{-\mathrm{i}yx}\, \mathrm{d}x, \quad y \in \mathbb{R}.$

Resources: Residue, Integrals Involving Exponential Functions

Problem Variants

■ $f(x) = \dfrac{3x}{x^2 + 1}$

$\hat{f}(1) = -?.??\mathrm{i}$:

check

■ $f(x) = \dfrac{2}{x^2 + 2x + 2}$

$\hat{f}(1) = ?.?? - ?.??\mathrm{i}$:

check

■ $f(x) = \dfrac{x}{x^4 + 1}$

$\hat{f}(1) = -?.??\mathrm{i}$:

check

Solution

Application of the residue theorem

Integrating over the path

$$C:\ -R\ \to\ R\ \to\ R+R\mathrm{i}\ \to\ -R+R\mathrm{i}\ \to\ -R$$

(enclosing a rectangle in the upper halfplane $H_+ : \operatorname{Im} z > 0$) and letting R tend to ∞, the residue theorem implies

$$\int_{\mathbb{R}} f(x)\, \underbrace{\mathrm{e}^{-\mathrm{i}xy}}_{e_y(x)}\, \mathrm{d}x = 2\pi\mathrm{i} \sum_{\operatorname{Re} z>0} \operatorname*{Res}_z\,(f\, e_y), \quad y \le 0\,, \tag{1}$$

since the integral over the part of the path in the upper half plane H_+ tends to 0. For $y \ge 0$, the sum is replaced by the sum of negative residues in the lower half plane $H_- : \operatorname{Im} z \le 0$.

Computation of the residues

factoring the denominator of f using the binomial formula $\quad\rightsquigarrow$

$$f(z) = \frac{1}{z^4+2z^2+1} = \frac{1}{(z^2+1)^2} = \frac{1}{(z-\mathrm{i})^2(z+\mathrm{i})^2}\,,$$

i.e. f has double poles at $z = \pm\mathrm{i}$
applying the formula for the residue at double pole for $z = \mathrm{i}$ in the upper half plane $H_+ \quad\rightsquigarrow$

$$\begin{aligned}\operatorname*{Res}_{\mathrm{i}}\,(f\, e_y) &= \frac{\mathrm{d}}{\mathrm{d}z}\left(f(z)e_y(z)(z-\mathrm{i})^2\right)\Big|_{z=\mathrm{i}} = \frac{\mathrm{d}}{\mathrm{d}z}\,\frac{\mathrm{e}^{-\mathrm{i}yz}}{(z+\mathrm{i})^2}\Big|_{z=\mathrm{i}} \\ &= \frac{-\mathrm{i}y\,\mathrm{e}^{-\mathrm{i}yz}(z+\mathrm{i})^2 - \mathrm{e}^{-\mathrm{i}yz}\,2(z+\mathrm{i})}{(z+\mathrm{i})^4}\Big|_{z=\mathrm{i}} = \cdots = \mathrm{i}\frac{y-1}{4}\mathrm{e}^y\end{aligned}$$

substituting into (1) $\quad\rightsquigarrow\quad$ Fourier transform of f for $y \le 0$:

$$\begin{aligned}\hat{f}(y) &= \int_{\mathbb{R}} f(x)e_y(x)\,\mathrm{d}x = (2\pi\mathrm{i})\operatorname*{Res}_{\mathrm{i}}\,(f\, e_y) \\ &= (2\pi\mathrm{i})\,\mathrm{i}\frac{y-1}{4}\,\mathrm{e}^y \underset{y\le 0}{=} \frac{\pi}{2}\,(1+|y|)\,\mathrm{e}^{-|y|}\end{aligned}$$

Since Fourier transforms preserve symmetry, $(f(-x) = f(x) \implies \hat{f}(-y) = \hat{f}(y))$, this expression is valid also for $y \ge 0$.

Chapter 4

Taylor and Laurent Series

4.1 Product of a Taylor Series with a Polynomial

Develop $f(z) = \dfrac{z^2 - z/2}{e^{-2z}}$ in a Taylor series at $z = 0$.

Resources: Methods of Taylor Expansion

Problem Variants

■ $f(z) = \dfrac{(z^2 - 1)\sin z}{z}$

---

coefficient of $z^4 = -?.???$:

check

■ $f(z) = \dfrac{\cos(z^2) - 1}{z^2 - 2z}$

---

coefficient of z^6 =?.???:

check

■ $f(z) = (z + 1 + 1/z)\ln(1 + z)$

---

coefficient of $z^5 = -?.??$:

check

Solution

substituting $t = 2z$ into the Taylor series

$$\mathrm{e}^t = \sum_{n=0}^{\infty} \frac{1}{n!} t^n$$

and multiplying with $z^2 - z/2 \quad \rightsquigarrow$

$$f(z) = \frac{z^2 - z/2}{\mathrm{e}^{-2z}} = (z^2 - z/2)\mathrm{e}^{2z} = z^2 \sum_{n=0}^{\infty} \frac{1}{n!}(2z)^n - (z/2) \sum_{n=0}^{\infty} \frac{1}{n!}(2z)^n$$

simplifying and combining the two series $\quad \rightsquigarrow$

$$\begin{aligned} f(z) &= \sum_{n=2}^{\infty} \frac{2^{n-2}}{(n-2)!} z^n - \sum_{n=1}^{\infty} \frac{2^{n-1}/2}{(n-1)!} z^n \\ &= \sum_{n=2}^{\infty} \frac{2^{n-2} n(n-1)}{n!} z^n - \sum_{n=1}^{\infty} \frac{2^{n-2} n}{n!} z^n = \sum_{n=1}^{\infty} \frac{2^{n-2} n(n-2)}{n!} z^n \end{aligned}$$

4.2 Taylor and Geometric Series

Develop $f(z) = \dfrac{2}{z(2-z)}$ in a Taylor series at $z = 1$.

Resources: Methods of Taylor Expansion

Problem Variants

■ $f(z) = \dfrac{3}{4-z^2}$, $z = 0$

---

coefficient of z^6 =?.???:

[ ] check

■ $f(z) = \dfrac{1}{z^2+3z}$, $z = 2$

---

coefficient of $(z-2)^4$ =?.???:

[ ] check

■ $f(z) = \dfrac{1}{(z-2)(z-3)}$, $z = 4$

---

coefficient of $(z-4)^5$ $= -?.??$:

[ ] check

Solution

Partial fraction decomposition

poles of $f(z) = \dfrac{2}{z(2-z)}$: $z = 0$, $z = 2$

$\rightsquigarrow$ ansatz

$$\frac{2}{z(2-z)} = \frac{a}{z} + \frac{b}{2-z}$$

- multiplying with z and setting $z = 0$ $\quad\Longrightarrow\quad$ $2/2 = a + 0$, i.e. $a = 1$
- similarly: $\cdot(2-z)$ and $z = 2$ $\quad\Longrightarrow\quad$ $b = 1$

$\rightsquigarrow \quad f(z) = \dfrac{1}{z} + \dfrac{1}{2-z}$

Expanding f with the aid of the geometric series

$\dfrac{1}{1-q} = \displaystyle\sum_{n=0}^{\infty} q^n \quad \rightsquigarrow$

- $\dfrac{1}{z} = \dfrac{1}{1-(1-z)} \underset{q=(1-z)}{=} \displaystyle\sum_{n=0}^{\infty} (1-z)^n$
- $\dfrac{1}{2-z} = \dfrac{1}{1-(z-1)} \underset{q=(z-1)}{=} \displaystyle\sum_{n=0}^{\infty} (z-1)^n$

combining the expressions $\rightsquigarrow$

$$f(z) = \sum_{n=0}^{\infty} ((-1)^n + 1)(z-1)^n = 2\sum_{m=0}^{\infty} (z-1)^{2m}$$

4.3 Taylor Series of a Square Root

Develop $f(z) = \sqrt{1+2z}$ at $z = -1$ in a Taylor series with $f(-1) = \mathrm{i}$ (principal branch of the square root) and determine the radius of convergence.

Resources: Taylor Series

Problem Variants

■ $f(z) = (1+z)^{2/3}$, $z = 0$

---

coefficient of $z^4 = -?.???$:

[ ] check

■ $f(z) = \ln(3-2z)$, $z = 1$

---

coefficient of $(z-1)^6 = -??.??$:

[ ] check

■ $f(z) = \ln(z^2-1)$, $z = 2$

---

coefficient of $(z-2)^5$ =?.??:

[ ] check

Solution

Taylor expansion

derivatives of $f(z) = (1+2z)^{1/2}$:

$$f'(z) = (1+2z)^{-1/2}, \quad f''(z) = -(1+2z)^{-3/2}, \quad f'''(z) = 3(1+2z)^{-5/2}$$

...

$$f^{(n)}(z) = (-1)^{n-1} \cdot 3 \cdot 5 \cdots (2n-3)(1+2z)^{-(2n-1)/2}$$

substituting

$$\begin{aligned} 3 \cdot 5 \cdots (2n-3) &= \frac{(2n-2)!}{2^{n-1}(n-1)!} \\ (1+2z)^{(2n-1)/2}\big|_{z=-1} &= (-1)^{-n+1/2} = \mathrm{i}(-1)^n \end{aligned}$$

$$\rightsquigarrow \quad f^{(n)}(-1) = -\mathrm{i}\frac{(2n-2)!}{2^{n-1}(n-1)!}$$

definition of Taylor coefficients, $c_n = f^{(n)}(-1)/n!$ $\rightsquigarrow$

$$f(z) = \mathrm{i} - \mathrm{i}\sum_{n=1}^{\infty} \frac{(2n-2)!}{2^{n-1}(n-1)!n!}(z+1)^n$$

Radius of convergence

r = distance of the expansion point $z = -1$ to the singularity of the square root for $1 + 2z = 0$, i.e. for $z = -1/2$

$\Longrightarrow \quad r = |(-1) - (-1/2)| = 1/2$

4.4 Taylor Approximation of an Inverse Function

Determine the cubic Taylor polynomial $q(w)$ of the inverse function the polynomial

$$p(z) = 2 - 5z + z^3$$

at $w = p(1)$.

Resources: Methods of Taylor Expansion

Problem Variants

■ $p(z) = z + z^3$, $z = 0$

---

sum of the Taylor coefficients of $q = ?$:

[ ] check

■ $p(z) = z^2 - z^3$, $z = 0$

---

sum of the Taylor coefficients of $q = -??$:

[ ] check

■ $p(z) = 1 + z^3$, $z = -1$

---

sum of the Taylor coefficients of $q = -0.0???$:

[ ] check

Solution

Derivatives of p

differentiating $p(z) = 2 - 5z + z^3$ and setting $z = 1$ $\rightsquigarrow$

$$p(1) = -2 \quad p'(1) = -2\,^{1}, \quad p''(1) = 6, \quad p'''(1) = 6$$

$\rightsquigarrow$ Taylor representation of p at $z = 1$

$$\begin{aligned} p(z) &= \sum_{n=0}^{3} \frac{p^{(n)}(1)}{n!}\,(z-1)^n \\ &= -2 - 2(z-1) + 3(z-1)^2 + (z-1)^3 \end{aligned}$$

Taylor expansion of the (local) inverse function

computing the derivatives of q by differentiating the identity $q(p(z)) = z$ $\rightsquigarrow$

$$\begin{aligned} q'p' &= 1 \\ q''(p')^2 + q'p'' &= 0 \\ q'''(p')^3 + 3q''p'p'' + q'p''' &= 0 \end{aligned}$$

successively substituting the argument $z = 1 = q(-2)$ of $p^{(k)}$ and the argument $w = -2 = p(1)$ of $q^{(k)}$ using the values for the derivatives of p $\Longrightarrow$

$$\begin{aligned} q'(-2) &= 1/p'(1) = -1/2 \\ q''(-2) &= -(q'(-2)p''(1))/(p'(1))^2 = -((-1/2)\cdot 6)/(-2)^2 = 3/4 \\ q'''(-2) &= -(3\cdot(3/4)(-2)\cdot 6 + (-1/2)\cdot 6)/(-2)^3 = -15/4 \end{aligned}$$

$\rightsquigarrow$ Taylor polynomial

$$\begin{aligned} q(w) &= \sum_{k=0}^{3} \frac{q^{(k)}(-2)}{k!}(w+2)^2 \\ &= 1 - \frac{1}{2}(w+2) + \frac{3}{8}(w+2)^2 - \frac{5}{8}(w+2)^3 + O\left((w+2)^4\right) \end{aligned}$$

[1] $p'(1) \neq 0$ guarantees the existence of a local inverse function, a condition which for complex variables is necessary **and** sufficient.

4.5 Leading Terms of a Laurent Series

Determine the first 4 terms of the Laurent series of $f(z) = \dfrac{1+3z}{\sin^2 z}$ in the annulus $D:\ 0 < |z| < \pi$.

Resources: Methods of Laurent Expansion

Problem Variants

- $f(z) = \dfrac{1}{\cos(2z) - \cos(3z)}$, $D:\ 0 < |z| < 2\pi/5$

---

sum of the first 4 coefficients =?.??:

[ ] check

- $f(z) = \dfrac{1}{\ln^3(2-z)}$, $D:\ 0 < |z-1| < 1$

---

sum of the first 4 coefficients $= -$?.????:

[ ] check

- $f(z) = \tan(z/2)^2$, $D:\ 0 < |z-\pi| < 2\pi$

---

sum of the first 4 coefficients =?.??:

[ ] check

Solution

Taylor series of the sine function $\quad\rightsquigarrow$

$$f(z)=\frac{1+3z}{\sin^2 z}=\frac{1+3z}{\left(z-z^3/6+O(z^5)\right)^2}=\frac{1+3z}{z^2-z^4/3+O(z^6)}$$

Expanding with $1+z^2/3$ and using the binomial formula $(a+b)(a-b)=a^2-b^2$ $\rightsquigarrow$

$$\frac{1}{z^2}\frac{(1+3z)(1+z^2/3)}{\left(1-z^2/3+O(z^4)\right)(1+z^2/3)}=\frac{1}{z^2}\frac{1+3z+z^2/3+z^3}{1+O(z^4)}$$

simplifying with the identity $c/(1+\varepsilon)=c+O(\varepsilon)\quad\rightsquigarrow$

$$f(z)=z^{-2}+3z^{-1}+1/3+z+O(z^2)$$

4.6 Laurent Expansion via Comparison of Coefficients

Determine the first three nonzero terms of the Laurent series for the function $f(z) = \dfrac{1}{e^z - 1 - z}$ at $z = 0$.

Resources: Methods of Laurent Expansion

Problem Variants

- $f(z) = \dfrac{z+1}{\cos z - 1}$, $z = 0$

---

sum of the coefficients $= -?.??$:

check

- $f(z) = \dfrac{z^2}{\ln z}$, $z = 1$

---

sum of the coefficients $=?.??$:

check

- $f(z) = \dfrac{\exp(2z)}{\sin(3z)}$, $z = 0$

---

sum of the coefficients $=?.??$:

check

Solution

Expansion of the denominator

Taylor series of the exponential function $\mathrm{e}^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k \quad \Longrightarrow$

$$d(z) = \mathrm{e}^z - 1 - z = \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + O(z^5)\,,$$

i.e. $f = 1/d$ has a pole of order 2 at $z = 0$:

$$f(z) = c_{-2}z^{-2} + c_{-1}z^{-1} + c_0 + O(z)$$

Computation of the coefficients

comparing the coefficients of 1, z und z^2 in the identity $f(z)d(z) = 1 \quad \Longleftrightarrow$

$$\left(c_{-2}z^{-2} + c_{-1}z^{-1} + c_0 + O(z)\right)\left(\frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 + O(z^5)\right) = 1$$

$\rightsquigarrow$ equations for c_k

- z^0: $c_{-2} \cdot \frac{1}{2} = 1 \quad \Longrightarrow \quad c_{-2} = 2$
- z: $2 \cdot \frac{1}{6} + c_{-1} \cdot \frac{1}{2} = 0 \quad \Longrightarrow \quad c_{-1} = -\frac{2}{3}$
- z^2: $2 \cdot \frac{1}{24} - \frac{2}{3} \cdot \frac{1}{6} + c_0 \cdot \frac{1}{2} = 0 \quad \Longrightarrow \quad c_0 = \frac{1}{18}$

$\rightsquigarrow$ Laurent series

$$f(z) = 2z^{-2} - \frac{2}{3}z^{-1} + \frac{1}{18} + O(z)$$

4.7 Laurent Series in an Annulus with Center ∞

Determine the Laurent series of $f(z) = \dfrac{1}{1+z^2}$ in the annulus $D: |z| > 1$.

Resources: Methods of Laurent Expansion

Problem Variants

- $\dfrac{1}{2-3z}$

---

coefficient of $z^{-4} = -?.???$:

[ ] check

- $\dfrac{1}{z^2-3}$

---

coefficient of z^{-6} =?:

[ ] check

- $\dfrac{z+1}{z+2}$

---

coefficient of $z^{-5} = -??$:

[ ] check

Solution

rewriting the function $f \quad \rightsquigarrow$

$$f(z) = \frac{1}{1+z^2} = \frac{1/z^2}{1-(-1/z^2)}$$

applying the formula $\dfrac{1}{1-q} = \sum_{n=0}^{\infty} q^n$ for a geometric series with $q = -1/z^2$ $\rightsquigarrow$

$$f(z) = \frac{1}{z^2}\sum_{n=0}^{\infty}(-z^{-2})^n = \sum_{n=0}^{\infty}(-1)^n z^{-2n-2} = \sum_{m=-1}^{-\infty}(-1)^{m+1} z^{2m}$$

4.8 Principal Part of a Complex Function

Determine the principal part of $f(z) = 1/(\ln z)^2$ at $z = 1$.

Resources: Laurent Series

Problem Variants

- $\dfrac{z-1}{\cos z - 1}$, $z = 0$

---

sum of the coefficients of z^{-k} =?:

[ ] check

- $\dfrac{1}{(\sqrt{z}-2)^3}$, $z = 4$

---

sum of the coefficients of z^{-k} =??:

[ ] check

- $\dfrac{\cos z}{\sin z - z}$, $z = 0$

---

sum of the coefficients of $z^{-k} = -?.?$:

[ ] check

Solution

Principle part of f

Taylor approximation of the Logarithm,

$$\ln z = (z-1) - \frac{1}{2}(z-1)^2 + O((z-1)^3)$$

$\rightsquigarrow$ pole of order 2 for $f(z) = 1/(\ln z)^2$ at $z = 1$ with the Laurent series

$$f(z) = \frac{1}{(\ln z)^2} = \underbrace{c_{-2}(z-1)^{-2} + c_{-1}(z-1)}_{\text{principle part}} + O(1)$$

Coefficients of $(z-1)^{-k}$

The coefficients c_{-2}, c_{-1} are determined by computing the coefficients of the linear Taylor polynomial of

$$g(z) = (z-1)^2 f(z) = c_{-2} + c_{-1}(z-1) + O((z-1)^2)\,,$$

i.e. $c_{-2+k} = g^{(k)}(a)/k!$

- rule of L'Hôpital $\rightsquigarrow$

$$c_{-2} = g(1) = \lim_{z\to 1} \frac{(z-1)^2}{(\ln z)^2} = \lim_{z\to 1} \frac{2(z-1)}{\underbrace{2(\ln z)/z}_{(z-1)z/\ln z}} = \lim_{z\to 1} \frac{2z-1}{1/z} = 1$$

- Taylor approximations $\ln z = \xi - \frac{1}{2}\xi^2 + O(\xi^3)$ and $1/z = 1 - \xi + O(\xi^2)$ with $\xi = z - 1$ $\rightsquigarrow$

$$\begin{aligned} c_{-1} &= g'(1) = \lim_{z\to 1} \frac{2(z-1)\ln z - 2(z-1)^2/z}{(\ln z)^3} \\ &= \lim_{\xi\to 0} \frac{2\xi^2 - \xi^3 - 2\xi^2 + 2\xi^3 + O(\xi^4)}{\xi^3 + O(\xi^4)} = 1 \end{aligned}$$

Alternative solution

comparing coefficients in the identity $f(z)(\ln z)^2 = 1$ $\iff$

$$(c_{-2}\zeta^{-2} + c_{-1}\zeta^{-1} + O(1))\,\underbrace{(\zeta - \zeta^2/2 + O(\zeta^3))^2}_{\zeta^2-\zeta^3+O(\zeta^4)} = 1, \quad \zeta = z - 1$$

multiplying the expansions on the left side and neglecting terms of order $O(\zeta^2)$ $\rightsquigarrow$

$$c_{-2} - c_{-2}\zeta + c_{-1}\zeta + \cdots = 1\,,$$

i.e. $c_{-2} = 1$, $c_{-1} = 1$

4.9 Laurent Series of Rational Functions

Develop $f(z) = \dfrac{4z}{z^2 + 2z - 3}$ in a Laurent series, which converges in the annulus $1 < |z| < 3$.

Resources: Laurent Series

Problem Variants

- $f(z) = \dfrac{2}{z^2 - 2z}$, $2 < |z|$

---

coefficient of z^{-6} =??:

[ ] check

- $f(z) = \dfrac{1}{z^2 - 1}$, $1 < |z - 2| > 3$

---

coefficient of $(z - 2)^{-5}$ =?.?:

[ ] check

- $f(z) = \dfrac{3}{(z - 2)(z - 1)}$, $1 < |z| < 2$

---

coefficient of z^{-4} $= -$?:

[ ] check

Solution

Partial fraction decomposition

factoring the denominator of f $\quad\rightsquigarrow\quad$ ansatz

$$f(z) = \frac{4z}{z^2+2z-3} = \frac{4z}{(z+3)(z-1)} = \frac{a}{z+3} + \frac{b}{z-1}$$

computation of a and b with the multiplication technique:

- multiplying by $z+3$ and setting $z=-3$ $\quad\rightsquigarrow\quad$ $4z/(z-1)|_{z=-3} = a + b(z+3)/(z-1)|_{z=-3} = a+0$, i.e. $a=3$
- $\cdot(z-1)$ and $z=1$ $\quad\rightsquigarrow\quad$ $b=1$

Expanding the elementary terms

bringing the terms into the form $c/(1-q)$ with $|q|<1$, taking the inequalities $1<|z|<3$ into account, and applying the formula $c/(1-q) = c\sum_{n=0}^{\infty} q^n$ $\rightsquigarrow$

- $\displaystyle \frac{3}{z+3} = \frac{1}{1-(-z/3)} = \sum_{n=0}^{\infty}(-3)^{-n}z^n$
- $\displaystyle \frac{1}{z-1} = \frac{1/z}{1-1/z} = \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = \sum_{n=-1}^{-\infty} z^n$

combining the two expansions $\quad\rightsquigarrow$

$$f(z) = \frac{3}{z+3} + \frac{1}{z-1} = \cdots + z^{-2} + z^{-1} + 1 - \frac{1}{3}z + \frac{1}{9}z + \cdots$$

Chapter 5

Differential Equations

5.1 Exponential Solutions of a Linear Differential Equation

Determine and plot all solutions of the form $u(t) = \mathrm{e}^{\lambda t}$ for the differential equation $u''' = 8u$.

Resources: Polar Form of Complex Numbers, Powers of Complex Numbers

Problem Variants

■ $u''' = -u$

---

sum of the solutions for $t = 1$ = ?.??:

check

■ $u'' = 2\mathrm{i}\,u$

---

sum of the solutions for $t = 1$ = ?.??+?.??i:

check

■ $u'' + 2u' = -2u$

---

sum of the solutions for $t = 1$ = ?.??:

check

Solution

Characteristic equation

substituting $u(t) = \mathrm{e}^{\lambda t}$ into the differential equation $u''' = 8u \quad \Longrightarrow$

$$\lambda^3\,\mathrm{e}^{\lambda t} = 8\,\mathrm{e}^{\lambda t} \quad \Longleftrightarrow \quad \lambda^3 = 8$$

three roots:

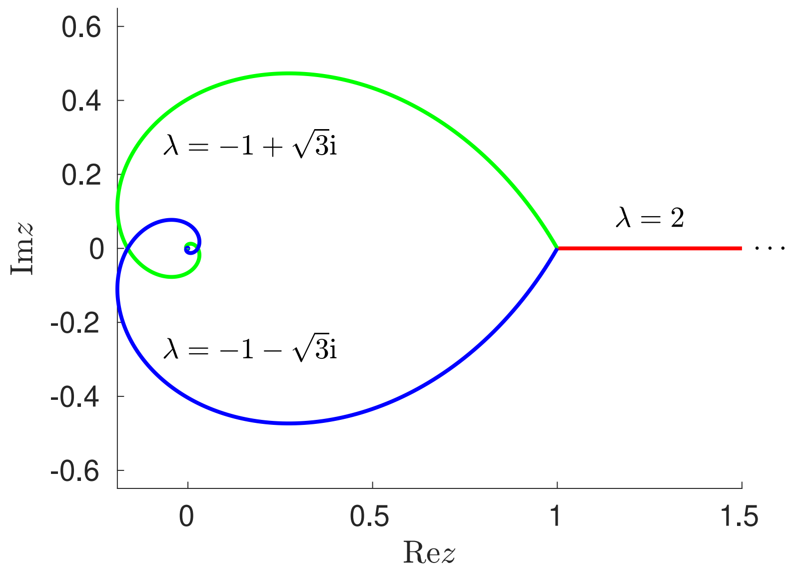
$\lambda_1 = 2,\ \lambda_2 = 2\,\mathrm{e}^{2\pi\mathrm{i}/3} = 2\cos(2\pi/3)+2\mathrm{i}\sin(2\pi/3) = -1+\sqrt{3}\,\mathrm{i},\ \lambda_3 = -1-\sqrt{3}\,\mathrm{i}$

$\rightsquigarrow$ three solutions $u(t) = \mathrm{e}^{\lambda t}$:

$$\begin{aligned} u(t) &= \mathrm{e}^{2t} \\ u(t) &= \mathrm{e}^{(-1+\sqrt{3}\,\mathrm{i})\,t} = \mathrm{e}^{-t}\,(\cos(\sqrt{3}\,t) + \mathrm{i}\sin(\sqrt{3}\,t)) \\ u(t) &= \mathrm{e}^{-t}\,(\cos(\sqrt{3}\,t) + \mathrm{i}\sin(\sqrt{3}\,t)) \end{aligned}$$

For λ_2 and λ_3, $\operatorname{Re}\lambda = -1 < 0$, causing the solution curves spiraling down to 0.

Plotting the exponential solutions



MATLAB® commands for plotting one of the solutions

```
lambda = -1+sqrt(3)*i;
t = linspace(0,10); z = exp(lambda*t);
plot(real(z),imag(z),'-g')   % green curve
```

5.2 Taylor Expansion for a First Order Initial Value Problem

Determine a recurion for the Taylor coefficients u_k of the solution $u(z) = \sum_{k=0}^{\infty} u_k z^k$ of the solution for the initial value problem

$$u' = zu + 1, \quad u(0) = 0\,.$$

Resources: Regular Point of a Differential Equation

Problem Variants

■ $u' = (1+z)u + 1 - z$

$u_7 = 0.0???$:

[ ] check

■ $(1+z^2)u' = u - z$

$u_7 = -0.???$:

[ ] check

■ $u' = (1-z^2)u - z$

$u_7 = 0.0???$:

[ ] check

Solution

substituting the Taylor series

$$u(z) = \sum_{n=0}^{\infty} u_n z^n$$

into the differential equation $u' = zu + 1 \quad \rightsquigarrow$

$$u_1 + 2u_2 z + 3u_3 z^2 + 4u_4 z^3 + \cdots = (u_0 z + u_1 z^2 + u_2 z^3 + \cdots) + 1$$

comparing the coefficients of $z^n \quad \rightsquigarrow \quad$ equations for the Taylor coefficients u_k:

- intial condition $u(0) = 0 \quad \Longrightarrow \quad u_0 = 0$
- z^0: $u_1 = 1$
- z^1: $2u_2 = u_0 \quad \Longrightarrow \quad u_2 = 0$
- z^2: $3u_3 = u_1 \quad \Longrightarrow \quad u_3 = \frac{1}{3}$
- z^3: $4u_4 = u_2 \quad \Longrightarrow \quad u_4 = 0$
- z^4: $5u_5 = u_3 \quad \Longrightarrow \quad u_5 = \frac{1}{5}u_3 = \frac{1}{5}\frac{1}{3} = \frac{1}{1\cdot 3\cdot 5}$

$\ldots \quad \Longrightarrow \quad u_{2m} = 0$ and

$$\begin{aligned} u_{2m-1} &= \frac{1}{2m-1}u_{2m-3} = \frac{1}{(2m-1)(2m-3)}u_{2m-5} = \cdots \\ &= \frac{1}{(2m-1)(2m-3)\cdots 1} = \frac{2^m m!}{(2m)!}, \end{aligned}$$

i.e.

$$u(z) = \sum_{m=1}^{\infty} \frac{2^m m!}{(2m)!} z^{2m-1}$$

It is instructive/fascinating to see how Maple™ handles this initial value problem:

```
dsolve({diff(u(z),z)=z*u(z)+1,u(0)=0})
```

$$u(z) = \frac{\mathrm{e}^{z^2/2}\sqrt{\pi}\sqrt{2}\,\mathrm{erf}(\sqrt{2}z/2)}{2}$$

5.3 Taylor Series Solution of a First Order Differential Equation

Determine the expansion $u(z) = z^\lambda \sum_{k=0}^{\infty} u_k z^k$ of the general solution for the differential equation

$$zu' + (1 + z)u = 0$$

at the singular point $z = 0$.

Resources: Singular Point of a Differential Equation

Problem Variants

■ $2zu' = (3z - 1)u$

---

u_3/u_2 =?.?:

[ ] [check]

■ $zu' = (3 - 2z)u$

$u_3/u_2 = -?.??$:

[ ] [check]

■ $3zu' = (z^2 - 1)u$

$u_4/u_2 = 0.0???$:

[ ] [check]

Solution

Characteristic equation

substituting the ansatz $u(z) = \sum_{n=0}^{\infty} u_n z^{\lambda+n}$ into the differential equation $zu' + (1+z)u = 0 \quad \rightsquigarrow$

$$z \sum_{n=0}^{\infty} (\lambda+n) u_n z^{\lambda+n-1} + \sum_{n=0}^{\infty} u_n z^{\lambda+n} + \sum_{n=0}^{\infty} u_n z^{\lambda+n+1} = 0 \tag{1}$$

comparing the coefficients of z^λ $(n=0) \quad \Longrightarrow$

$$(\lambda+1)u_0 = 0$$

with the non-trivial solution $\lambda = -1$ and $u_0 = c$

Coefficients of the expansion

comparing the coefficients of $z^{\lambda+n}$ with $\lambda = -1$ and $n > 0$ in equation (1) $\Longrightarrow$

$$(n-1+1)u_n + u_{n-1} = 0 \quad \text{bzw.} \quad u_n = -\frac{u_{n-1}}{n}$$

iterating this recursion $\quad \rightsquigarrow$

$$u_n = \frac{u_{n-2}}{n(n-1)} = -\frac{u_{n-3}}{n(n-1)(n-2)} = \cdots = \frac{(-1)^n c}{n!}$$

comparison with the Taylor series of the exponential function $\quad \rightsquigarrow$

$$u(z) = c \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} z^{-1+n} = c\,\frac{\mathrm{e}^{-z}}{z}$$

5.4 Polynomial Solutions of a Differential Equation

For which λ does the differential equation

$$(1-z^2)u'' - 2zu' = \lambda u$$

possess a polynomial solution (Legendre Polynomial)?

Resources: Regular Point of a Differential Equation

Problem Variants

■ $(1-z^2)u'' = \lambda u$

---

smallest value $-\lambda > 70 =??$:

[ ] check

■ $u'' + zu' = \lambda u$

---

smallest value $\lambda > 70 =??$:

[ ] check

■ $zu'' + (1-z)u' = \lambda u$

---

smallest value $-\lambda > 70 =??$:

[ ] check

Solution

Recursion for the Taylor Coefficients

substituting the ansatz $u(z) = \sum_{n=0}^{\infty} u_n z^n$ into the differential equation $(1 - z^2)u'' - 2zu' = \lambda u \quad \rightsquigarrow$

$$\left(\sum_{n=0}^{\infty} n(n-1)u_n z^{n-2} - \sum_{n=0}^{\infty} n(n-1)u_n z^n\right) - 2\sum_{n=0}^{\infty} n u_n z^n = \lambda \sum_{n=0}^{\infty} u_n z^n$$

$$\iff \quad \sum_{n=0}^{\infty}(n+2)(n+1)u_{n+2}z^n = \sum_{n=0}^{\infty}(\lambda + \underbrace{n(n-1)+2n}_{n(n+1)})\, u_n z^n$$

comparison of the coefficients of $z^n \quad \Longrightarrow$

$$u_{n+2} = \frac{\lambda + n(n+1)}{(n+2)(n+1)}\, u_n \tag{1}$$

Polynomial solutions

The recursion (1) terminates, hence yields a polynomial solution, if $\lambda + n(n+1) = 0$ for some $n \in \mathbb{N}$, i.e. if

$$\lambda = -0, -2, -6, -12, -20, -30, -42, -56, -72, -90, -110, \ldots$$

and if the initial conditions are chosen appropriately.
For example, $\lambda = -12$ for $n = 3$, and the initial conditions $u_0 = 0$, $u_1 = 1$ $\Longrightarrow$

$$u_3 = \frac{-12 + 1\cdot(1+1)}{(1+2)(1+1)}u_0 = -\frac{5}{3}, \qquad u_4 = -\frac{6}{12}u_1 = 0,$$
$$u_5 = \frac{-12 + 3\cdot(3+1)}{(3+2)(3+1)}u_3 = 0, \qquad u_6 = u_7 = \cdots = 0$$

$\rightsquigarrow \quad u(z) = z - \frac{5}{3}z^3$

5.5 Taylor Expansion for a Second Order Initial Value Problem

Solve the initial value problem

$$(1 + z^2)u'' = 6u, \quad u(0) = 1,\, u'(0) = 0\,,$$

by expanding u in a Taylor series.

Resources: Regular Point of a Differential Equation

Problem Variants

■ $u'' + zu' = -u, \quad u(0) = 1,\, u'(0) = 0$

---

coefficient of $z^6 = -0.0???$:

check

■ $(3 - z^2)u'' = -2u, \quad u(0) = 0,\, u'(0) = 1$

---

coefficient of $z^7 = -0.00???$:

check

■ $(1 - z^2)u'' - zu' + u = 0, \quad u(0) = 1,\, u'(0) = 0$

---

coefficient of $z^8 = -0.0???$:

check

Solution

Recursion for the Taylor coefficients

substituting the Taylor series $u(z) = \sum_{n=0}^{\infty} u_n z^n$ into the differential equation $(1+z^2)u'' = 6u$ $\rightsquigarrow$

$$\underbrace{\sum_{n=0}^{\infty} n(n-1)u_n z^{n-2}}_{\sum_{n=0}^{\infty}(n+2)(n+1)u_{n+2}z^n} + \sum_{n=0}^{\infty} n(n-1)u_n z^n = 6\sum_{n=0}^{\infty} u_n z^n$$

comparison of the coefficients of z^n $\implies$

$$u_{n+2} = \frac{6-n(n-1)}{(n+2)(n+1)}\, u_n = \frac{3-n}{n+1}\, u_n$$

Initial conditions

$u'(0) = 0 \quad \implies \quad u_1 = u_3 = \cdots = 0$ and $u(0) = 1 \quad \implies$

$$\begin{array}{lllllllll} u_0 &=& 1, & u_2 &=& 3, & u_4 &=& 1, \\ u_6 &=& -\dfrac{1}{5}, & u_8 &=& \dfrac{3}{5\cdot 7}, & u_{10} &=& -\dfrac{3\cdot 5}{5\cdot 7\cdot 9} \end{array}$$

$\rightsquigarrow \quad u_{2m} = (-1)^m \dfrac{3}{(2m-1)(2m-3)}$ for $m \geq 2$ and

$$u(z) = 1 + 3z^2 + \sum_{m=2}^{\infty} (-1)^m \frac{3}{(2m-1)(2m-3)}\, z^{2m}$$

5.6 Differential Equation for a Given Recursion of the Taylor Coeffients

Determine the quadratic polynomial p and the linear polynomial q for which the Taylor coefficients $u_n = u^{(n)}(0)/n!$ of each solution of the differential equation

$$pu'' + qu' + u = 0$$

satisfy the recursion

$$u_{n+2} = u_{n+1} + 2u_n \,.$$

Resources: Regular Point of a Differential Equation

Problem Variants

■ $u_{n+2} = u_{n+1} + u_n$

$p(1) + q(1) =$?.?:

[ ] check

■ $u_{n+2} = -\dfrac{1}{n+1}\, u_n$

$p(1) + q(1) =$?:

[ ] check

■ $(n+2)\, u_{n+2} = u_{n+1} + u_n$

$p(1) + q(1) =$?:

[ ] check

Solution

General form of the recursion

differentiating the Taylor series $u(z) = \sum_{n=0}^{\infty} u_n\, z^n \quad \rightsquigarrow$

$$u'(z) = \sum_{n=0}^{\infty} (n+1) u_{n+1}\, z^n, \quad u''(z) = \sum_{n=0}^{\infty} (n+2)(n+1) u_{n+2}\, z^n$$

comparing the coefficients of z^n in the differential equation

$$pu'' + qu' + u = 0$$

with $p(z) = p_0 + p_1 z + p_2 z^2$ and $q(z) = q_0 + q_1 z \quad \rightsquigarrow$

$$\begin{array}{rcrcrcl} p_0(n+2)(n+1)\, u_{n+2} & + & p_1(n+1)n\, u_{n+1} & + & p_2 n(n-1)\, u_n & & \\ & + & q_0(n+1)\, u_{n+1} & + & q_1 n\, u_n & & \\ & & & + & u_n & = & 0 \end{array}$$

rearranging terms $\quad \rightsquigarrow$

$$u_{n+2} = -\frac{p_1 n + q_0}{p_0(n+2)}\, u_{n+1} - \frac{p_2 n(n-1) + q_1 n + 1}{p_0(n+2)(n+1)}\, u_n$$

Coefficients of p and q

comparison with the given recursion

$$u_{n+2} = u_{n+1} + 2u_n$$

$\rightsquigarrow \quad$ equations for p_k and q_k:

- coefficient of u_n:

$$2 = -\frac{p_2 n(n-1) + q_1 n + 1}{p_0(n+2)(n+1)} \quad \Longleftrightarrow \quad 2p_0 n^2 + 6p_0 n + 4p_0 = -p_2 n^2 + (p_2 - q_1)n - 1$$

$\Longrightarrow \quad p_0 = -1/4,\ p_2 = -2p_0 = 1/2,\ q_1 = -6p_0 + p_2 = 2$

- coefficient of u_{n+1}:

$$1 = -\frac{p_1 n + q_0}{p_0(n+2)} \quad \Longleftrightarrow \quad -\frac{1}{4}n - \frac{1}{2} = -p_1 n - q_0$$

$\Longrightarrow \quad p_1 = 1/4,\ q_0 = 1/2$

$\rightsquigarrow \quad p(z) = -\dfrac{1}{4} + \dfrac{1}{4}z + \dfrac{1}{2}z^2, \quad q(z) = \dfrac{1}{2} + 2z$

5.7 Euler's Differential Equation

Determine the real solution $u(t)$ of the initial value problem

$$z^2u'' + zu + u = 0, \quad u(1) = 0,\, u'(1) = 1\,.$$

Resources: Euler's Differential Equation

Problem Variants

■ $2z^2u'' + 3zu' = u, \quad u(1) = 2,\, u'(1) = 1$

$u(4)$ =?:

[ ] check

■ $z^2u'' - 3zu' + 5u = 0, \quad u(1) = 1,\, u'(1) = 2$

$u(2)$ =?.??:

[ ] check

■ $z^2u'' - zu' + u = 0, \quad u(1) = 0,\, u'(1) = 1$

$u(2)$ =?.??:

[ ] check

Solution

General solution

substituting the ansatz $u(z) = z^\lambda$ into the Euler differential equation $z^2u'' + zu + u = 0 \quad \rightsquigarrow \quad$ characteristic equation

$$\lambda(\lambda - 1) + \lambda + 1 = 0 \quad \Longleftrightarrow \quad \lambda^2 + 1 = 0$$

with the solutions $\lambda_\pm = \pm\mathrm{i}$ and the general solution (linear combination of the particular solutions z^{i} and $z^{-\mathrm{i}}$, determined with the ansatz)

$$u(z) = c_1 z^{\mathrm{i}} + c_2 z^{-\mathrm{i}}$$

Initial conditions

initial conditions $u(1) = 1$, $u'(1) = 1$ and $1^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}\ln 1} = \mathrm{e}^0 = 1 \quad \Longrightarrow$

$$\begin{aligned} u(1) &= c_1 z^{\mathrm{i}} + c_2 z^{-\mathrm{i}}\big|_{z=1} = c_1 + c_2 = 0, \\ u'(1) &= c_1 \mathrm{i} z^{\mathrm{i}} - c_2 \mathrm{i} z^{-\mathrm{i}}\big|_{z=1} \mathrm{i}c_1 - \mathrm{i}c_2 = 1\,, \end{aligned}$$

i.e. $c_1 = \frac{1}{2\mathrm{i}}$, $c_2 = -\frac{1}{2\mathrm{i}}$
formula of Euler-Moivre $\quad \rightsquigarrow$

$$u(z) = \frac{z^{\mathrm{i}} - z^{-\mathrm{i}}}{2\mathrm{i}} = \frac{\mathrm{e}^{\mathrm{i}\ln z} - \mathrm{e}^{-\mathrm{i}\ln z}}{2\mathrm{i}} = \sin(\ln z)$$

5.8 Expansion at a Singular Point

Determine two linearly independent solutions $u(z) = z^{\lambda}\left(1 + \sum_{n=1}^{\infty} u_n z^n\right)$ of the differential equation

$$z^2 u'' + z u' = (1 + z^4) u \,.$$

Resources: Singular Point of a Differential Equation

Problem Variants

■ $z^2 u'' + 2zu' = z^2 u$

$1/u_8$ =?????? for the solution with $\lambda \geq 0$):

check

■ $2z^2 u'' + 3zu' = (1 + z)u$

$1/u_4$ =????? for the solution with $\lambda \geq 0$):

check

■ $z^2 u'' = (6 + z^2)u$

$1/u_4$ =??? for the solution with $\lambda \geq 0$):

check

Solution

Characteristic equation

substituting the ansatz $u(z) = z^\lambda \left(1 + \sum_{n=1}^{\infty} u_n z^n\right)$ into the differential equation $z^2u'' + zu' = (1 + z^4)u \quad \rightsquigarrow$

$$\sum_{n=0}^{\infty}(\lambda+n)(\lambda+n-1)u_n z^n + \sum_{n=0}^{\infty}(\lambda+n)u_n z^n = \sum_{n=0}^{\infty} u_n z^n + \sum_{n=0}^{\infty} u_{n-4} z^n \quad (1)$$

with $u_0 = 1$, $u_{-4} = u_{-3} = u_{-2} = u_{-1} = 0$ comparing the coefficients of z^λ $(n = 0) \quad \rightsquigarrow \quad$ characteristic equation

$$\lambda^2 = 1$$

with the solutions $\lambda_\pm = \pm 1$

Recursion for the coefficients of the expansion

comparing the coefficients of $z^{\lambda+n}$ in equation (1) with $n > 0 \quad \rightsquigarrow \quad$ recursion

$$u_n = \frac{u_{n-4}}{(\lambda_\pm + n)^2 - 1} = \frac{u_{n-4}}{(\lambda_\pm + n + 1)(\lambda_\pm + n - 1)}$$

$\rightsquigarrow \quad$ formulas for the coefficients for both exponents

- $\lambda_+ = 1$: $u_0 = 1$, $u_n = u_{n-4}/((n+2)n) \quad \Longrightarrow$

  $$u_4 = \frac{1}{4 \cdot 6}, \quad u_8 = \frac{1}{4 \cdot 6 \cdot 8 \cdot 10},$$
  $$u_{4m} = \frac{1}{4 \cdot 6 \cdots (4m+2)} = \frac{2}{2^{2m+1} \cdot 1 \cdot 2 \cdots (2m+1)} = \frac{1}{2^{2m}(2m+1)!}$$

  and $u_n = 0$ for $n/4 \notin \mathbb{N}$

- $\lambda_- = -1$: $u_0 = 1$, $u_n = u_{n-4}/(n(n-2)) \quad \Longrightarrow$

  $$u_4 = \frac{1}{2 \cdot 4}, \quad u_8 = \frac{1}{2 \cdot 4 \cdot 6 \cdot 8}, \quad \ldots$$
  $$u_{4m} = \frac{1}{2 \cdot 4 \cdots 4m} = \frac{1}{2^{2m} \cdot 1 \cdot 2 \cdots 2m} = \frac{1}{2^{2m}(2m)!} u_n = 0 \,\text{für}\, n/4 \notin \mathbb{N}$$

  and $u_n = 0$ for $n/4 \notin \mathbb{N}$

$\rightsquigarrow \quad$ linearly independent solutions

$$u_+(z) = z \sum_{m=0}^{\infty} \frac{1}{2^{2m}(2m+1)!} z^{4m}, \quad u_-(z) = \frac{1}{z} \sum_{m=0}^{\infty} \frac{1}{2^{2m}(2m)!} z^{4m}$$

Chapter 6

Calculus Highlights

6.1 Trigonometric Sum

Compute $S_n = \sum_{k=1}^{n-1} 2^k \sin(\pi k/n)$.

Resources: Formula of Euler-Moivre

Solution

Transformation of the sum with the formula of Euler-Moivre

$\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi \quad \Longrightarrow \quad \sin\varphi = \operatorname{Im}\mathrm{e}^{\mathrm{i}\varphi}$ (imaginary part of the exponential function)
consequently, with $\varphi = \pi k/n$,

$$S_n = \sum_{k=1}^{n-1} 2^k \sin(\pi k/n) = \operatorname{Im} \sum_{k=0}^{n-1} 2^k \mathrm{e}^{\mathrm{i}\pi k/n}\,,$$

since the additional summand for $k = 0$ is zero

Application of the formula for a geometric sum and simplification

$\sum_{k=0}^{n-1} q^k = \frac{q^n-1}{q-1}$ with $q = 2\mathrm{e}^{\mathrm{i}\pi/n} \quad \Longrightarrow$

$$S_n = \operatorname{Im} \frac{2^n \mathrm{e}^{\mathrm{i}\pi} - 1}{2\mathrm{e}^{\mathrm{i}\pi/n} - 1} \underset{\mathrm{e}^{\mathrm{i}\pi}=-1}{=} -\operatorname{Im} \frac{2^n + 1}{2\mathrm{e}^{\mathrm{i}\pi/n} - 1}$$

expanding the fraction by $2\mathrm{e}^{-\mathrm{i}\pi/n} - 1 = 2(\cos(-\pi/n) + \mathrm{i}\sin(-\pi/n)) - 1 \quad \rightsquigarrow$

$$S_n = -\operatorname{Im} \frac{(2^n + 1)[2\cos(\pi/n) - 2\mathrm{i}\sin(\pi/n) - 1]}{(2\mathrm{e}^{\mathrm{i}\pi/n} - 1)[2\mathrm{e}^{-\mathrm{i}\pi/n} - 1]}$$

denominator $= 4 - 2\mathrm{e}^{\mathrm{i}\pi/n} - 2\mathrm{e}^{-\mathrm{i}\pi/n} + 1 = 5 - 4\cos(\pi/n) \quad \Longrightarrow$

$$S_n = \text{imaginary part of the fraction} = \frac{2(2^n + 1)\sin(\pi/n)}{5 - 4\cos(\pi/n)}$$

Checking the result with Maple™

```
simplify(sum(sin(2*Pi*k/n),k=1..n-1));
```

6.2 Algebraic Expressions for Cosine and Sine

Find, analogously to $\cos(\pi/4) = \sin(\pi/4) = \sqrt{1/2}$, explicit expressions for $\cos(\pi/8)$, $\sin(\pi/8)$, and $\tan(\pi/8)$.

Resources: Formula of Euler Moivre, Polar Form of Complex Numbers

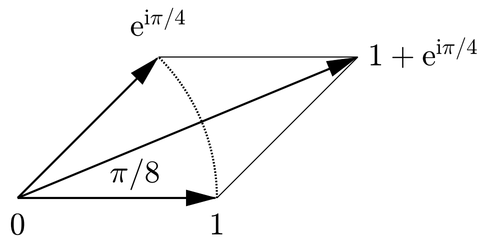
Solution

Cosine and Sine

associate with the angle φ the complex number $\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi$ on the unit circle of the Gaussian plane (formula of Euler-Moivre)

$\pi/8 = (0+\pi/4)/2 \quad \Longrightarrow$

$$\mathrm{e}^{\mathrm{i}\pi/8} \parallel \mathrm{e}^{\mathrm{i}\cdot 0} + \mathrm{e}^{\mathrm{i}\pi/4} = 1 \,+\, (1+\mathrm{i})/\sqrt{2}$$



normalizing ($|\mathrm{e}^{\mathrm{i}\varphi}| = 1$) $\quad \rightsquigarrow$

$$\mathrm{e}^{\mathrm{i}\pi/8} = \frac{1+1/\sqrt{2}+\mathrm{i}/\sqrt{2}}{\sqrt{(1+1/\sqrt{2})^2+(1/\sqrt{2})^2}}$$

simplifying the root in the denominator $\quad \rightsquigarrow$

$$\text{denominator} = \sqrt{1+2/\sqrt{2}+1/2+1/2} = \sqrt{2}\sqrt{1+1/\sqrt{2}}$$

formula of Euler-Moivre $\quad \Longrightarrow$

$$\cos(\pi/8) = \operatorname{Re}\mathrm{e}^{\mathrm{i}\pi/8} = \frac{1+1/\sqrt{2}}{\text{denominator}} = \frac{\sqrt{1+1/\sqrt{2}}}{\sqrt{2}} = \frac{\sqrt{2+\sqrt{2}}}{2}$$

analogously $\sin(\pi/8) = \operatorname{Im}\mathrm{e}^{\mathrm{i}\pi/8}$
alternatively,

$$\sin(\pi/8) = \sqrt{1-\cos^2(\pi/8)} = \sqrt{1-(2+\sqrt{2})/4} = \frac{\sqrt{2-\sqrt{2}}}{2}$$

Tangens

$$\tan(\pi/8) = \frac{\sin(\pi/8)}{\cos(\pi/8)} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}}$$

expanding with $\sqrt{2-\sqrt{2}}$ and using the third binomial formula $\quad \rightsquigarrow$

$$\frac{2-\sqrt{2}}{\sqrt{4-2}} = \frac{2-\sqrt{2}}{\sqrt{2}} = \sqrt{2}-1$$

Remark

Similarly, it is possible to obtain algebraic expressions for the trigonometric functions of the angles $\pi/16$, $\pi/32$, ..., $\pi/12$, $\pi/24$, ... and multiples and sums of these angles.

6.3 Visualisization of Complex Functions with MATLAB®

Visualisize the function

$$x + \mathrm{i}y = z \;\mapsto\; z^3 = w = u + \mathrm{i}v$$

by plotting the graph of the function $(x, y) \mapsto u$, using a colormap based on the imaginary part v of w. Apply this method also to visualize the multiple valued inverse function $w \;\mapsto\; z = w^{1/3}$ (3 z-values for every $\mathbb{C} \ni w \neq 0$) [1].

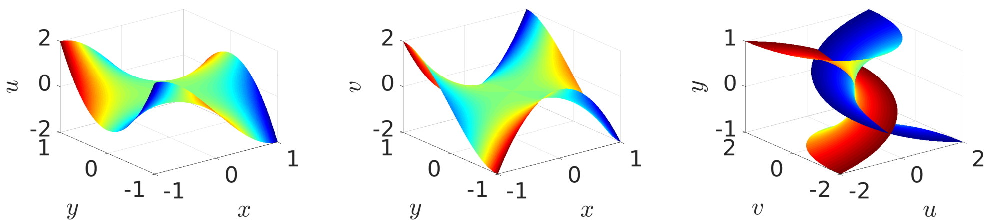
Resources: Complex Function

[1]Experiment with other functions, for example with $w = \sin z$.

Solution

MATLAB® -script

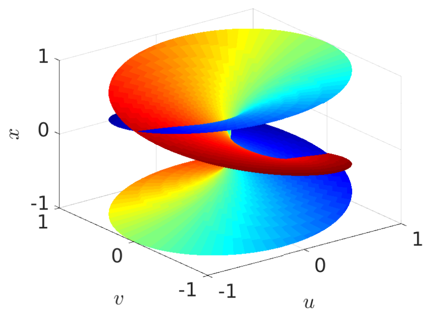
```
view([0.7,0.7,0.3])   % view direction
colormap(jet)   % color map
[x,y] = meshgrid([-1:0.01:1]);   % evaluating grid
z = x+i*y; w = z.^3; u = real(w); v = imag(w);
% graph of (x,y)->u
% choice of colors according to v
% no drawing of the edges
surf(x,y,u,v,'EdgeColor','none');   % left image
% alternative: graph of (x,y)->v
% coice of colors according to u
surf(x,y,v,u,'EdgeColor','none');   % middle image
% inverse function: graph of (u,v)->y
% coice of colors according to x
surf(u,v,y,x,'EdgeColor','none')   % right image
```



Alternative solution

Polarkoordinaten $z = re^{i\varphi}$

```
[r,phi] = meshgrid([0:0.01:1], ...
   [-1:0.01:1]*3*pi);
u = r.*cos(phi); v = r.*sin(phi);
w = r.^(1/3).*exp(i*phi/3);
x = real(w); y = imag(w);
% graph: (u,v)->x, color: y
surf(u,v,x,y,'EdgeColor','none')
```



6.4 Julia Sets with MATLAB®

The Julia set, corresponding to a parameter $c \in \mathbb{C}$, is the boundary of the set A_c of complex numbers z_0 for which the sequence, defined by the recursion

$$z_{n+1} = p(z_n) = z_n^2 + c\,,$$

stays bounded.

Plot an approximation of A_c for several values of c and visualize the distance to the boundary with a suitable choice of colors[2].

Resources: Complex Function

[2]Experiment also with polynomials p of higher degree.

Solution

Algorithmic considerations

$|z_n| \geq R = |c| + C$ with $C \geq 2 \quad \Longrightarrow$

$$\begin{aligned} |z_{n+1}| &= |z_n^2 + c| \geq |z_n|^2 - |c| \geq |c|^2 + 2C|c| + C^2 - |c| \\ &\geq (2C-1)|c| + C^2 \underset{C\geq 2}{\geq} |c| + C + 1 = R + 1\,, \end{aligned}$$

i.e. $|z_{n+k}| \geq R + k$, and, consequently, $|z_{n+k}| \to \infty$

numerical criterium: minimal N with $|z_N| \geq R \quad \rightsquigarrow \quad$ color index
The larger N is for a start value z_0, the smaller is the distance to the Julia set.

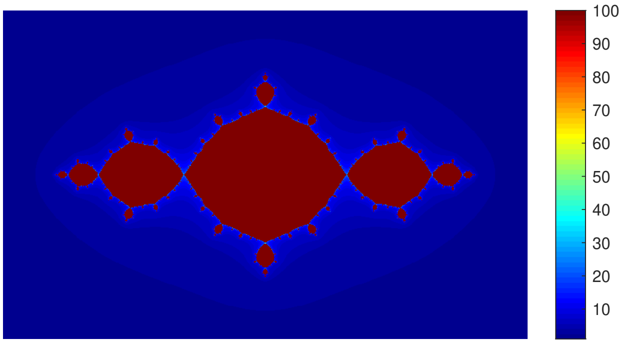
Points z_0 with $|z_n| < R$ for sufficiently large n are very likely to lie in the set A_c (maximal color index).

MATLAB® Script

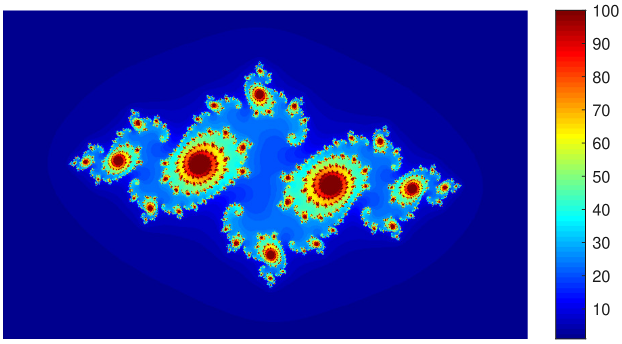
```
% parameter
c = 0.4+0.2*i;   % -> last of the following images
nmax = 100;    % maximal number of iterations
R = 10;   % radius for divergence
d = 0.005;    % distance of the pixels
% image frame
xmin=-2; xmax=2; ymin=-1.25; ymax=1.25;
% start values and color indices
[x,y] = meshgrid([xmin:d:xmax],[ymin:d:ymax]);
z = x+i*y; N = zeros(size(z));
% simultaneous iteration for all start values
for n=1:nmax
   z = z.^2 + c;   % iteration step
   % no convergence -> augmenting the color indices
   N = N+(abs(z)<R);
end
% conversion to a pixel image
imagesc(N), colormap(jet), colorbar
```

Sets A_c for several parameters

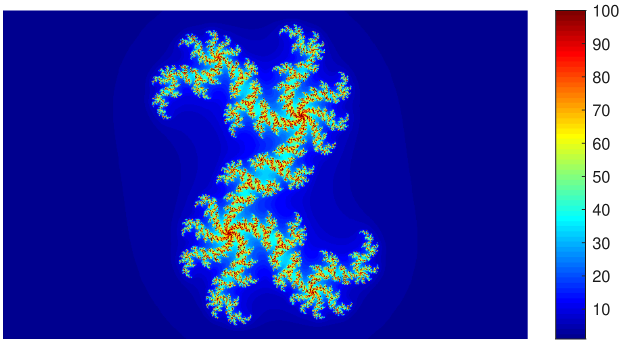
$c = -1$



$c = -0.75 + 0.16\mathrm{i}$



$c = 0.4 + 0.2\mathrm{i}$

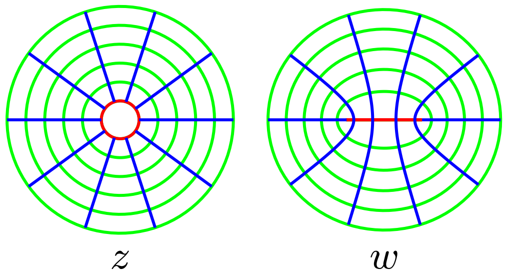


Fascinating complexity and Diversity[3]!

[3] For an excellent exposition of the theory and visualization of fractal sets in connection with complex dynamics see B. Mandelbrot: *The Fractal Geometry of Nature*, W.H. Freeman and Co., 1982 and H-O. Peitgen, P. Richter: *The Beauty of Fractals*, Springer, 1986.

6.5 Angle Preservation of the Joukowsky Transform

Determine the images of the circles $C_r : \varphi \mapsto r e^{i\varphi}$ and of the half lines $H_r : r \mapsto r e^{i\varphi}$ $(0 \le \varphi \le 2\pi,\ r \ge 1)$ under the conformal map[4] $z \mapsto w = z + 1/z$; Note that, for $0 < r \le 1$, C_r as well as H_r are mapped to the same set of curves.



[4]Joukowsky has used this transformation for the design of airfoils.

Solution

Description of the transformation in polar coordinates

$$z = r\mathrm{e}^{\mathrm{i}\varphi} \mapsto w = r\mathrm{e}^{\mathrm{i}\varphi} + \frac{1}{r}\mathrm{e}^{-\mathrm{i}\varphi} = \underbrace{\left(r + \frac{1}{r}\right)\cos\varphi}_{u} + \mathrm{i}\underbrace{\left(r - \frac{1}{r}\right)\sin\varphi}_{v}$$

Images of the circles

- $r = 1$ (red): $w = 2\cos\varphi \quad \rightsquigarrow \quad$ segment $[-1, 1]$ on the real axis
- $r > 1$ (green): $u^2/a^2 + v^2/b^2 = 1$ with $a = r + 1/r$, $b = r - 1/r \quad \rightsquigarrow$ ellipses with lengths a and b of the half axes.

Images of the half lines

addition and subtraction of the equations

$$u = \left(r + \frac{1}{r}\right)\cos\varphi, \quad v = \left(r - \frac{1}{r}\right)\sin\varphi$$

after multiplication of the first (second) equation with $\sin\varphi$ ($\cos\varphi$) $\quad \rightsquigarrow$

$$u\sin\varphi + v\cos\varphi = 2r\cos\varphi\sin\varphi, \quad u\sin\varphi - v\cos\varphi = 2\frac{1}{r}\cos\varphi\sin\varphi$$

multiplication of the two equations and division by $4\cos^2\varphi\sin^2\varphi \quad \rightsquigarrow$

$$\frac{u^2}{4\cos^2\varphi} - \frac{v^2}{4\sin^2\varphi} = 1\,,$$

i.e. the images of the half lines are segments of hyperbolas with $|2\cos\varphi|$ and $|2\sin\varphi|$ lengths of the half axes

6.6 Image of the Polar Grid for a Möbius Transformation

Determine, for the Möbius transformation

$$w = \frac{z - \mathrm{i}}{z + \mathrm{i}},$$

the images of the circles with midpoint 0 and of the lines emanating from the origin.

Resources: Möbius Transformation

Solution

Image of a circle $C: |z| = r$

mapping of the points r, $r\mathrm{i}$, $-r$, $-r\mathrm{i}$ for $r \neq 1$:

$$r \;\mapsto\; \frac{r-\mathrm{i}}{r+\mathrm{i}} = \frac{r^2-1-2r\mathrm{i}}{r^2+1}\,, \qquad r\mathrm{i} \;\mapsto\; \frac{r\mathrm{i}-\mathrm{i}}{r\mathrm{i}+\mathrm{i}} = \frac{r-1}{r+1} = p$$

$$-r \;\mapsto\; \frac{-r-\mathrm{i}}{-r+\mathrm{i}} = \frac{r^2-1+2r\mathrm{i}}{r^2+1}\,, \qquad -r\mathrm{i} \;\mapsto\; \frac{-r\mathrm{i}-\mathrm{i}}{-r\mathrm{i}+\mathrm{i}} = \frac{r+1}{r-1} = q$$

The two points p and q lie on the real axis, while the two other points are complex conjugates of each other. Hence, in view of symmetry, the image of C is a circle with midpoint $(p+q)/2 = (r^2+1)/(r^2-1)$ and radius $|p-q|/2 = 2r/(r^2-1)$.
special case $r = 1$ (unit circle):

$$\mathrm{e}^{\mathrm{i}\varphi} \mapsto w_\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi}-\mathrm{i}}{\mathrm{e}^{\mathrm{i}\varphi}+\mathrm{i}} = \frac{(\mathrm{e}^{\mathrm{i}\varphi}-\mathrm{i})(\mathrm{e}^{-\mathrm{i}\varphi}-\mathrm{i})}{(\mathrm{e}^{\mathrm{i}\varphi}+\mathrm{i})(\mathrm{e}^{-\mathrm{i}\varphi}-\mathrm{i})} = -\mathrm{i}\,\frac{\cos\varphi}{1+\sin\varphi} \in \mathrm{i}\mathbb{R}\,,$$

i.e. C is mapped to the imaginary axis.

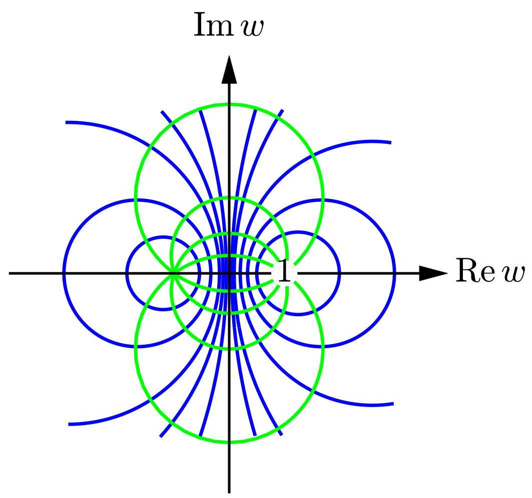
Image of a line $L: z = t\,\mathrm{e}^{\mathrm{i}\varphi}$

L contains the points 0, ∞, and $\mathrm{e}^{\mathrm{i}\varphi}$ with the following images:

$$0 \mapsto -1\,, \quad \infty \mapsto 1\,, \quad \mathrm{e}^{\mathrm{i}\varphi} \mapsto w_\varphi\,.$$

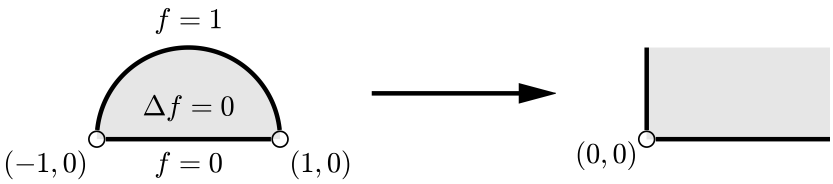
Since circles are mapped to circles or lines, the image of L is a circle passing through $-1, 1$ and w_φ.

Plot of the circles



6.7 Conformal Transformation of a Dirichlet Problem

Solve the Laplace equation $\Delta u = 0$ on the upper half of the unit disc with the depicted boundary values via transformation to the first quadrant.



Resources: Conformal Mapping, Möbius Transformation

Solution

Invariance of the Laplace operator under a conformal map

Composition with a complex differentiable transformation

$$D \ni (x, y) \,\widehat{=}\, z = x + \mathrm{i}y \quad \mapsto \quad w(z) = u(x, y) + \mathrm{i}v(x, y) \in w(D) = Q$$

preserves harmonicity:

$$\Delta g = 0 \text{ on } Q \quad \Longrightarrow \quad \Delta f = 0 \,\text{on}\, D, \quad f(x, y) = g(u(x, y), v(x, y))\,.$$

Hence, the boundary value problem can be solved on the simpler domain $Q:\ u, v > 0$ (quadrant), and then the solution g can be transformed back to the half disc D.

Dirichlet problem on the quadrant in polar coordinates

$Q:\ r > 0,\ 0 < \varphi < \pi/2$ with $u = r\cos\varphi,\ v = r\sin\varphi$

$$\Delta g = \frac{1}{r}\partial_r(r\partial_r g(r, \varphi)) + \frac{1}{r^2}\partial_\varphi^2 g(r, \varphi) = 0, \quad g(r, 0) = 0,\ g(r, \pi/2) = 1$$

apparent solution $g(r, \varphi) = \dfrac{2}{\pi}\varphi = \dfrac{2}{\pi}\arctan\dfrac{v}{u}$

Möbius transformation

$$w(z) = \frac{az + b}{cz + 1}$$

corresponding singular points: $-1 \leftrightarrow 0,\ 1 \leftrightarrow \infty$ (incompatible boundary values at the corners of the domain D or at $0, \infty \in Q$) $\Longrightarrow$

$$w(-1) = 0,\ w(1) = \infty, \quad \text{i.e. } a = b,\ c = -1$$

chosing the image of an additional boundary point, e.g. $w(\mathrm{i}) = \mathrm{i} \Longrightarrow$ $a(\mathrm{i} + 1)/(-\mathrm{i} + 1) \overset{!}{=} \mathrm{i}$, i.e. $a = 1$ and

$$w(z) = \frac{1 + z}{1 - z} = \frac{1 + x + \mathrm{i}y}{1 - x - \mathrm{i}y} = \frac{1 - x^2 - y^2 + 2\mathrm{i}y}{(1 - x)^2 + y^2}$$

Transformation to the half disc

$u(x, y) = \operatorname{Re} w(z) = \dfrac{1 - x^2 - y^2}{(1 - x)^2 + y^2},\ v(x, y) = \operatorname{Im} w(z) = \dfrac{2y}{(1 - x)^2 + y^2}$ $\Longrightarrow$

$$f(x, y) = \frac{2}{\pi}\arctan\frac{v(x, y)}{u(x, y)} = \frac{2}{\pi}\arctan\frac{2y}{1 - x^2 - y^2}$$

correct boundary values on the line segment and the half circle:

$$f(x, 0) = \frac{2}{\pi}\arctan 0 = 0, \qquad \lim_{1 - x^2 - y^2 \to 0^+} f(x, y) = \frac{2}{\pi}\frac{\pi}{2} = 1$$

6.8 Visualization of Complex Iterations with MATLAB®

Visualize the Newton iteration

$$z \leftarrow g(z) = z - \frac{\mathrm{e}^z - 1}{\mathrm{e}^z}$$

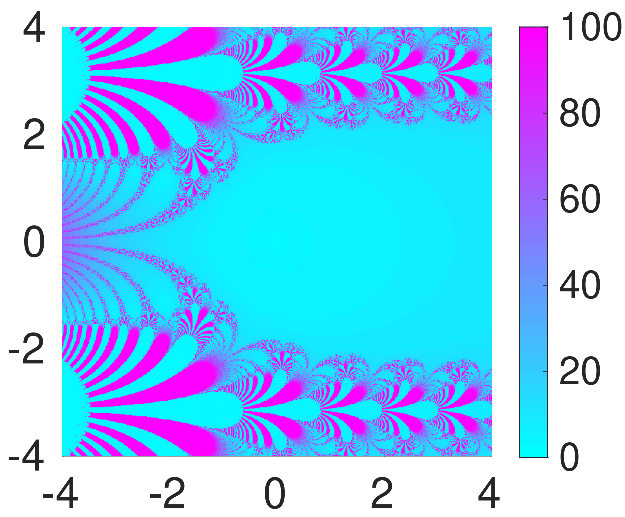
by assigning to each start value $z = x + \mathrm{i}y$ with $-4 \leq x, y \leq 4$ a color index according to the number of iterations required for convergence ($\infty \mathrel{\widehat{=}}$ Divergenz)[5].

Resources: Complex Function

[5] Consider also the function $g(z) = z - (z^3 - 1)/(3z^2)$.

Solution

```
N = 100;   % maximal number of iterations (=N: divergence)
tol = 1.0e-8;   % tolerance (|z-z_old|<tol: convergence)
% start values with distance 0.01 in [-pi,pi]^2
[x,y] = meshgrid([-pi:0.01:pi],[-pi:0.01:pi]); z = x+i*y;
c = 0*z;   % color index
for n=1:N
   z_old = z;
   % simultaneous Newton iteration
   z = z-1+exp(-z);
   % color index +1 for diverging points
   c = c+(abs(z-z_old)>tol);
end
% scaled pixel image of the color distribution
imagesc([-4 4],[-4 4],c), colormap(cool), colorbar
```



rose (color index 100) $\widehat{=}$ no convergence
blue (small color index) $\widehat{=}$ fast convergence

6.9 Complex Potential of Incompressible Flow

The complex potential $f(z) = z + 1/z$ describes the incompressible flow aroud the circle $C : |z| = 1$. Determine the velocity field and plot the equipotential lines and the streamlines.

Resources: Cauchy-Riemann Differential Equations, Harmonic Functions

Solution

Velocity field

$$f(z) = z + \frac{1}{z} = x + \mathrm{i}y + \frac{1}{x+\mathrm{i}y} = x + \mathrm{i}y + \frac{x-\mathrm{i}y}{x^2+y^2} =: u(x,y) + \mathrm{i}v(x,y)$$

$\rightsquigarrow$ (real) potential

$$u(x,y) = \operatorname{Re} f(z) = x + \frac{x}{x^2+y^2}$$

for the velocity field $\vec{V} = \operatorname{grad} u = (u_x, u_y)^{\mathrm{t}}$ with

$$\begin{aligned} u_x(x,y) &= 1 + \frac{x^2+y^2-x(2x)}{(x^2+y^2)^2} = 1 + \frac{y^2-x^2}{(x^2+y^2)^2} \\ u_y(x,y) &= -\frac{2xy}{(x^2+y^2)^2} \end{aligned}$$

Equipotential lines and streamlines

Cauchy-Riemann differential equations $u_x = v_y$, $u_y = -v_x$ $\quad\Longrightarrow$

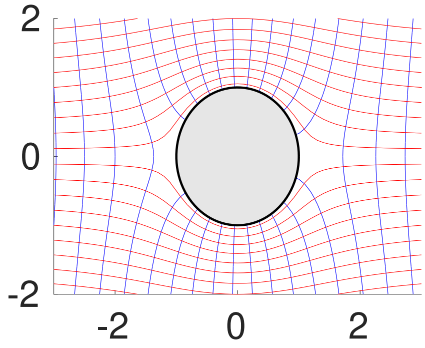
$$\operatorname{grad} v = (v_x, v_y)^{\mathrm{t}} = (-u_y, u_x)^{\mathrm{t}} \perp (u_x, u_y)^{\mathrm{t}} = \operatorname{grad} u$$

$\Longrightarrow$ The level lines of u and v are orthogonal.
$\Longrightarrow$ The streamlines are level curves of

$$v(x,y) = \operatorname{Im} f(z) = y - \frac{y}{x^2+y^2}\,.$$

Visualization with MATLAB®

```
[x,y] = meshgrid([-3:0.1:3], ...
   [-2:0.1:2]);
z = x+i*y; f = z+1./z;
u = real(f); v = imag(f);
hold on
contour(x,y,u,'b');   % blau
contour(x,y,v,'r');   % rot
...
```

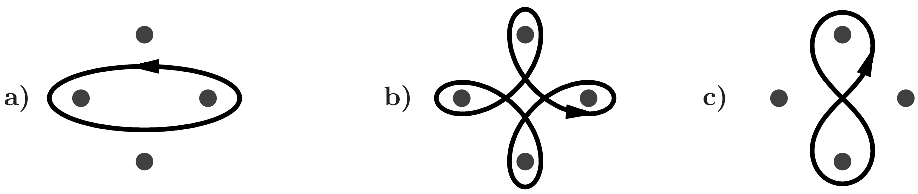


6.10 Complex Line Integrals for Different Paths

Compute

$$\int_C \frac{\mathrm{d}z}{z^4 - 1}$$

for the paths C shown in the figure.



The marked points are the fourth roots of unity.

Resources: Residue Theorem, Residue

Solution

Application of the residue theorem

$$\int_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{z_k} f, \quad f(z) = \frac{1}{z^4-1}$$

with z_k the poles of f enclosed by C in counterclockwise direction
simple poles: roots of unity

$$z_1 = 1, \quad z_2 = \mathrm{i}, \quad z_3 = -1, \quad z_4 = -\mathrm{i}$$

residues: $\operatorname{Res}_{z_k} f = \lim_{z\to z_k}(z-z_k)f(z)$
rule of L'Hôpital and $z_k^4 = 1 \quad \Longrightarrow$

$$\operatorname*{Res}_{z_k} f = \lim_{z\to z_k} \frac{z-z_k}{z^4-1} = \lim_{z\to z_k} \frac{1}{4z^3} = \frac{z_k}{4}$$

Path a)

C encloses z_1 and z_3 in positive direction. Hence,

$$\int\limits_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\left(\operatorname*{Res}_{z_1} f + \operatorname*{Res}_{z_3} f\right) = 2\pi\mathrm{i}\frac{1-1}{4} = 0\,.$$

Path b)

C encloses each pole once in positive direction. Hence,

$$\int\limits_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{k=1}^{4} \operatorname*{Res}_{z_k} f = 2\pi\mathrm{i}\,\frac{z_1+z_2+z_3+z_4}{4} = 0\,.$$

Path c)

C encloses z_2 in positive direction and z_4 in negative direction. Hence,

$$\int\limits_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i}\,\left(\operatorname*{Res}_{z_2} f - \operatorname*{Res}_{z_4} f\right) = 2\pi\mathrm{i}\,\frac{\mathrm{i}-(-\mathrm{i})}{4} = -\pi\,.$$

6.11 The Art of Complex Integration

Compute[6] $\int_0^\infty \sqrt[3]{x}/(x^2+1)\,\mathrm{d}x$.

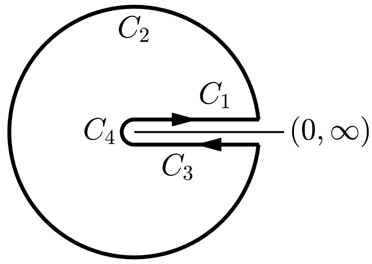
Resources: Residue Theorem, Residue

[6]Solving this problem you are overtrained for any exam which is definitely **not** a disadvantage! Along these lines try $\int_0^\infty \ln x\sqrt{x}/(1+x^2)\,\mathrm{d}x$.

Solution

Application of the residue theorem

The residue theorem is applied for the depicted integration path C consisting of two line segments C_1 und C_3 and two circular segments C_2 und C_4 with radii R and $1/R$ respectively. The cut along the real axis is necessary since on an annulus around 0 no consistent continuous definition of the cube root is possible,



consistent defintion of the cube root on the slit region bounded by C, $\sqrt[3]{r e^{i\varphi}} = \sqrt[3]{r}\, e^{i\varphi/3}$, $r = |z|$ $\quad\Longrightarrow$

$$\sqrt[3]{z} \underset{R\to\infty}{\longrightarrow} \sqrt[3]{|z|},\ z \in C_1, \quad \sqrt[3]{z} \underset{R\to\infty}{\longrightarrow} \sqrt[3]{|z|}\, e^{2\pi i/3},\ z \in C_3$$

simple poles of $f(z) = \dfrac{\sqrt[3]{z}}{z^2+1} = \dfrac{\sqrt[3]{z}}{(z-i)(z+i)}$ at $z = \pm i$ with residues

$$\operatorname*{Res}_{i} f = (z-i)\frac{\sqrt[3]{z}}{z^2+1}\bigg|_{z=i} = \frac{\sqrt[3]{z}}{z+i}\bigg|_{z=i} \underset{i=e^{i\pi/2}}{=} \frac{e^{i\pi/6}}{2i}, \quad \operatorname*{Res}_{-i} f = \cdots = -\frac{e^{i\pi/2}}{2i}$$

residue theorem $\quad\Longrightarrow$

$$\int_C f(z)\, dz = \sum_{k=1}^{4} \int_{C_k} f(z)\, dz = 2\pi i \sum_{z=\pm i} \operatorname{Res}_z f = \pi\left(e^{i\pi/6} - e^{i\pi/2}\right)$$

Limits of the integrals

for $R \to \infty$

- $\int_{C_1} f(z)\, dz \to I = \int_0^\infty \frac{\sqrt[3]{x}}{x^2+1}\, dx$, since $C_1 \to \mathbb{R}_+$
- $\int_{C_3} f(z)\, dz \to \int_\infty^0 \frac{\sqrt[3]{x} e^{2\pi i/3}}{x^2+1}\, dx = -e^{2\pi i/3} \int_0^\infty \frac{\sqrt[3]{x}}{x^2+1} = -e^{2\pi i/3} I$
  in view of the definition of $\sqrt[3]{z}$ and of the orientation of C_3
- $|\int_{C_2} f(z)\, dz| \le 2\pi R \max_{z\in C_2} |f(z)| \le 2\pi R \sqrt[3]{R}/(R^2-1) \to 0$
- $|\int_{C_4} f(z)\, dz| \le 2\pi/R \sqrt[3]{1/R} \max_{|z|=1/R} \frac{1}{|z^2+1|} \to 0$

Computation of the integral

adding the limits of the integrals $\int_{C_k} f(z)\,\mathrm{d}z \quad \rightsquigarrow$

$$I + 0 - \mathrm{e}^{2\pi\mathrm{i}/3} I + 0 = \pi \left(\mathrm{e}^{\mathrm{i}\pi/6} - \mathrm{e}^{\mathrm{i}\pi/2}\right)$$

and, solving for I

$$I = \pi \frac{\mathrm{e}^{\mathrm{i}\pi/6} - \mathrm{e}^{\mathrm{i}\pi/2}}{1 - \mathrm{e}^{2\pi\mathrm{i}/3}} = \pi \frac{\mathrm{e}^{-\mathrm{i}\pi/6} - \mathrm{e}^{\mathrm{i}\pi/6}}{\mathrm{e}^{-\mathrm{i}\pi/3} - \mathrm{e}^{\pi\mathrm{i}/3}} = \pi \frac{\sin(\pi/6)}{\sin(\pi/3)} = \pi \frac{1/2}{\sqrt{3}/2} = \pi/\sqrt{3}$$

6.12 Different Regions of Convergence for Laurent Series

Determine all Laurent series of $f(z) = 1/(z^3 - z^2)$ at $z = -1$.

Resources: Methods of Laurent Expansion

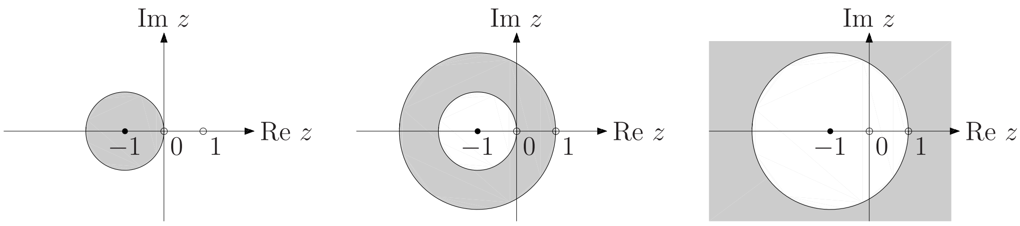
Solution

Regions of convergence

partial fraction decomposition

$$f(z) = \frac{1}{z^3 - z^2} = \frac{1}{z-1} - \frac{1}{z} - \frac{1}{z^2} = \frac{1}{u-2} + \frac{1}{1-u} - \frac{1}{(1-u)^2} \qquad (1)$$

with $u = z + 1 \rightsquigarrow$ 3 regions of convergence, limited by the poles of f



Laurent series

expansion using the formula $\sum_{n=0}^{\infty} q^n = 1/(1-q)$ for geometric series

(i) disc (Taylor series) $D: |u| < 1$:

$$\frac{1}{u-2} = -\frac{1/2}{1-u/2} = -\frac{1}{2}\sum_{n=0}^{\infty} 2^{-n} u^n$$

$$\frac{1}{1-u} = \sum_{n=0}^{\infty} u^n, \quad -\frac{1}{(1-u)^2} = -\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{1-u} = -\sum_{n=0}^{\infty}(n+1)u^n$$

adding the sums according to (1) $\rightsquigarrow$ Taylor coefficient of u^n: $-2^{-n-1} - n$, i.e.

$$f(z) = -\frac{1}{2} - \frac{5}{4}(z+1) - \frac{17}{8}(z+1)^2 + \cdots$$

(ii) annulus $D: 1 < |u| < 2$:
expansion of $1/(u-2)$ as in (i)
geometric series and termwise differentiation $(-\mathrm{d}/\mathrm{d}u)$ $\rightsquigarrow$

$$\frac{1}{1-u} = -\frac{1}{u}\frac{1}{1-1/u} = -\sum_{n=0}^{\infty} u^{-n-1}, \quad -\frac{1}{(1-u)^2} = -\sum_{n=0}^{\infty}(n+1)u^{-n-2}$$

Laurent coefficient of u^n: -2^{-n-1} for $n \geq 0$ and n for $n < 0$, i.e.

$$f(z) = \cdots - \frac{1}{z+1} - \frac{1}{2} - \frac{1}{4}(z+1) + \cdots$$

(iii) complement $D:\ 2<|u|$ of a disc:

$$\frac{1}{u-2}=\frac{1}{u}\,\frac{1}{1-2/u}=\sum_{n=0}^{\infty}2^n u^{-n-1}=\sum_{n=-1}^{-\infty}2^{-n-1}u^n$$

other terms as in (ii) $\quad\rightsquigarrow\quad$ coefficient of u^n: $2^{-n-1}+n$, $n<0$, i.e.

$$f(z)=\frac{1}{(z+1)^3}+\frac{4}{(z+1)^4}+\frac{11}{(z+1)^5}+\cdots$$

6.13 RIEMANN HYPOTHESIS with MATLAB®

The zeta-function is defined by

$$\zeta(z) = \sum_{n=1}^{\infty} n^{-z}, \quad \operatorname{Re} z > 1\,.$$

It admits a unique analytic continuation to $\mathbb{C}\setminus\{1\}$ and can be evaluated with the MATLAB® -command `w = zeta(z)`.

Riemann conjectured that all imaginary zeros z ($z \in \mathbb{C}\setminus\mathbb{R}$) of ζ lie on the line

$$g: \ z = 1/2 + \mathrm{i}\,\mathbb{R}\,.$$

Give several graphic illustrations of this conjecture[7] [8] .

Resources: Complex Function

[7]Relevant for Calculs? NO; but some faszination for mathematics is legitimate.

[8]The Clay Mathematics Institute offers 1000000 $ for a proof of Riemann's conjecture.

Solution

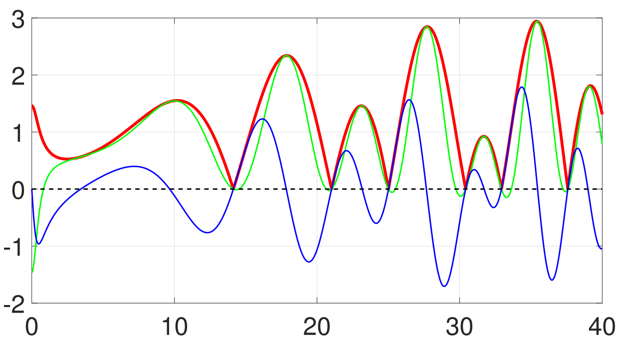
Only mathematically essential parts of the MATLAB® -scripts are listed, without going into details of the graphic specifications.

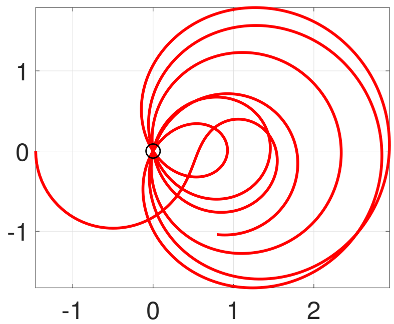
$\zeta(z)$ on the critical line $g:\ x = \mathrm{Re}\, z = 1/2$

```
% values on g, symmetry -> restriction to y = Im z > 0
n = 1000; y_max = 40;
y = y_max*[0:n_pts]/n; z = 1/2 + i*y;
w = zeta(z); u = real(w); v = imag(w);

% graph of |w| (red), u (green), v (blue)
plot(t,abs(w),'-r',t,u,'-g',t,v,'-b')    % left image
% specification of graphic setttings
  ...

% curve t -> (u,v) (rot) and origin (black circle)
plot(u,v,'-r',0,0,'ok')    % right image
% specification of graphic setttings
  ...
```

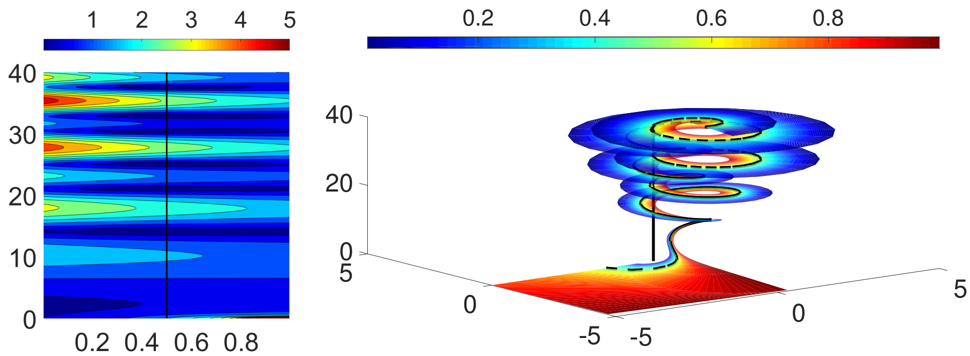




$\zeta(z)$ on the critical strip $D:\ 0 < x = \mathrm{Re}\, z < 1$

```
% grid for D, symmetry -> restriction to y = Im z > 0
y_max = 40;
nx = 50; ny = 200;    % number of points in x- and y-direction
x = [1:nx-1]/nx; y = y_max*[0:ny]/ny;
[X,Y] = meshgrid(x,y); Z = X+i*Y;
W = zeta(Z); U = real(W); V = imag(W);
```

```
% level curves of zeta
d = 1/16; w_max = 2;   % increment and upper bound
contourf(X,Y,abs(W),[[-d/2:d:wmax],inf]); colorbar   % left image
% specification of graphic setttings
  ...
% visualization of the (multiple valued) inverse function of zeta
surf(U,V,Y,X); colorbar   % right image
% specification of graphic setttings
  ...
```



- Left image: dark blue areas mark the location of the zeros on the critical line ($x = \operatorname{Re} z = 0.5$)
- Right image: The helical surface S visualizes $\zeta^{-1} : u + \mathrm{i}v \mapsto x + \mathrm{i}y$. For example, if a yellow ($\widehat{=}$ Wert ≈ 0.6) point on S has horizontal coordinates (u, v) and vertical coordinate y, then $x \approx 0.6$ and $u + \mathrm{i}v = \zeta(x + \mathrm{i}y)$. In particular, the vertical coordinate y parametrizes the critical line g and the colors the real part x of z on the critical strip D. Intersections of the graph with the vertical axis $(0, 0, \mathbb{R})$ correspond to the zeros of ζ.

  Riemann's conjecture is true, if the vertical axis intersects the (infinitely extended) helical surface only in points on the dashed curve with color value 0.5.

Chapter 7

Lexicon

7.1 Complex Numbers

Complex Numbers

representation in rectangular form (cartesian coordinates x and y)

$$z = x + \mathrm{i}y,\ \mathrm{i}^2 = -1, \quad x = \operatorname{Re} z\,(\text{real part}), \quad y = \operatorname{Im} z\,(\text{imaginary part})$$

complex conjugation and absolute value

$$\bar{z} = x - \mathrm{i}y, \quad |z| = \sqrt{x^2 + y^2}$$

MATLAB® :

```
z = x+i*y; x = real(z); y = imag(z);
z_bar = conj(z); z_abs = abs(z);
```

Maple™ :

```
z := x+I*y; x := Re(z); y := Im(z);
z_bar := conjugate(z); z_abs :=abs(z);
```

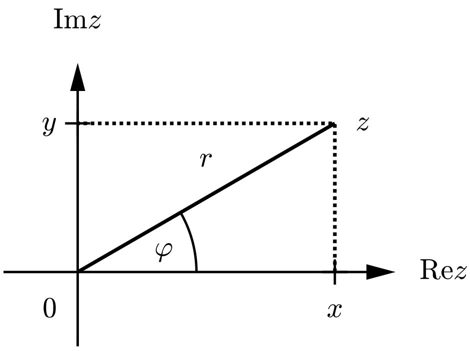
Formula of Euler-Moivre

$$\mathrm{e}^{\mathrm{i}\varphi} = \cos\varphi + \mathrm{i}\sin\varphi, \quad \cos\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi} + \mathrm{e}^{-\mathrm{i}\varphi}}{2}, \quad \sin\varphi = \frac{\mathrm{e}^{\mathrm{i}\varphi} - \mathrm{e}^{-\mathrm{i}\varphi}}{2\mathrm{i}}$$

Polar Form of Complex Numbers

$$z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\varphi}$$

$$\begin{aligned} x &= \operatorname{Re} z = r\cos\varphi \\ y &= \operatorname{Im} z = r\sin\varphi \\ r &= |z| = \sqrt{x^2 + y^2} \\ \varphi &= \arg z = \underbrace{\arctan(y/x)}_{\in[-\pi/2,\pi/2]} + \sigma\pi \end{aligned}$$



$\sigma = 0$ for $x \geq 0$, $\sigma = 1$ for $x < 0 \wedge y \geq 0$ (second quadrant) and $\sigma = -1$ for $x < 0 \wedge y < 0$ (third quadrant)
$\rightsquigarrow \quad \varphi \in (-\pi, \pi]$ (standard interval)
If the alternate standard interval $[0, 2\pi)$ is used, 2π is added for $\varphi \in (-\pi, 0)$.

polar form $re^{i\varphi}$ of special complex numbers z

z	-1	$\pm i$	$\sqrt{3} \pm i$	$1 \pm i$	$1 \pm \sqrt{3}i$
r	1	1	2	$\sqrt{2}$	2
φ	π	$\pm\pi/2$	$\pm\pi/6$	$\pm\pi/4$	$\pm\pi/3$

For a different sign of z ($z \to -z$), π is added to φ.

MATLAB® :

```
z = r*exp(i*phi); phi = angle(z); [phi,r] = cart2pol(x,y);
```

Maple™ :

```
z := r*exp(I*phi); phi := argument(z); polar(z);
```

Complex Arithmetic Operations

$z_k = x_k + iy_k = r_k \exp(i\varphi_k)$

- Addition

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i$$

- Multiplication

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i = r_1 r_2 \exp(i(\varphi_1 + \varphi_2))$$

- Division

$$\frac{z_1}{z_2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}\, i = \frac{r_1}{r_2} \exp(i(\varphi_1 - \varphi_2))$$

Powers of Complex Numbers

- $m \in \mathbb{Z}$: $z = re^{i\varphi} \quad \Longrightarrow \quad z^m = r^m e^{im\varphi}$
- $m = p/q \in \mathbb{Q} \backslash \mathbb{Z}$: $e^{2\pi ik} = 1$ for $k \in \mathbb{N}_0 \quad \Longrightarrow$

$$z^m = r^m e^{i(p\varphi + 2\pi k)/q}, \quad k = 0, \ldots, q-1 \quad \text{multiple valued}$$

infinitely many possible values of z^s for $s \in \mathbb{C} \backslash \mathbb{Q}$

Complex Root

$z = x + \mathrm{i}y = r\,\mathrm{e}^{\mathrm{i}\varphi} \quad \rightsquigarrow$

$$\begin{aligned} \sqrt{z} &= \pm\left(\sqrt{\frac{\sqrt{x^2+y^2}+x}{2}} + \mathrm{i}\sqrt{\frac{\sqrt{x^2+y^2}-x}{2}}\right) \\ &= \pm\sqrt{r}\,\mathrm{e}^{\mathrm{i}\varphi/2} \end{aligned}$$

Circle in the Complex Plane

$|z-a| = s|z-b|$, $s \neq 1$ $\quad$ ($s = 1$ $\quad\rightsquigarrow\quad$ line, bisector of the segment $\overline{ab}$)

$$\text{midpoint}: \; c = \frac{1}{1-s^2}a - \frac{s^2}{1-s^2}b, \quad \text{radius}: \; r = \frac{s}{|1-s^2|}|b-a|$$

$\rightsquigarrow$ $\quad$ parametrization: $t \mapsto c + r\mathrm{e}^{\mathrm{i}t}$

7.2 Differentiation and Conformal Mapping

Domain

connected (non-empty) subset of $\mathbb{R}^n$ or $\mathbb{C}^n$

Complex Function

$\mathbb{C} \supseteq D \ni z \mapsto w = f(z) \in \mathbb{C}$
real form: $f(z) = u(x, y) + \mathrm{i}v(x, y)$, $z = x + \mathrm{i}y$, i.e. $u = \operatorname{Re} f$, $v = \operatorname{Im} f$

Möbius Tranformation

$$z \mapsto w = \frac{az + b}{cz + d},\ ad - bc \neq 0, \quad \underbrace{w \mapsto z = \frac{-dw + b}{cw - a}}_{\text{inverse mapping}}$$

maps circles to circles, interpreting lines as *degenerate circles*
uniquely determined by the images w_k of three points z_k
$\rightsquigarrow$ construction using the invariance of the cross-ratio:

$$\frac{w - w_2}{w - w_3} : \frac{w_1 - w_2}{w_1 - w_3} = \frac{z - z_2}{z - z_3} : \frac{z_1 - z_2}{z_1 - z_3}$$

Complex Exponential Function

$z \mapsto w = \mathrm{e}^z = \mathrm{e}^x(\cos y + \mathrm{i} \sin y)$, $\mathrm{e}^{z+2\pi\mathrm{i}} = \mathrm{e}^z$
maps

- the strip $\operatorname{Im} z \in [s, s + 2\pi)$ to the punctured plane $\mathbb{C}\backslash\{0\}$
- the horizontal line $z = t + \mathrm{i}y$ to the half line $w = s\mathrm{e}^{\mathrm{i}y}$, $s \geq 0$
- the vertical line $z = x + \mathrm{i}t$ to the circle $|w| = \mathrm{e}^x$

Complex Logarithm

$w = \operatorname{Ln}(z) \iff z = \mathrm{e}^w,\ z = x + \mathrm{i}y = r\mathrm{e}^{\mathrm{i}\varphi} \quad \rightsquigarrow$

$$\operatorname{Ln} z = \ln(r) + \mathrm{i}(\varphi + 2\pi k), \quad r = \sqrt{x^2 + y^2},\ \varphi = \underbrace{\arctan(y/x)}_{\in[-\pi/2,\pi/2]} + \sigma\pi \in (-\pi, \pi]$$

with $k \in \mathbb{Z}$ (principal branch: $k = 0$) and $\sigma \in \{-1, 0, 1\}$, depending on the sign of x and y ($\sigma = 0$ for $x \geq 0$, $\sigma = 1$ for $x < 0$ and $y \geq 0$, $\sigma = 1$ for $x < 0$ and $y < 0$)

Complex Derivative and Cauchy-Riemann Differential Equations

$$f'(z) = \lim_{|\Delta z| \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

The existence of f' is a **substantially stronger** condition than the existence of the (real) derivatives of the functions $u(x, y) = \operatorname{Re} f(x + \mathrm{i}y)$, $v(x, y) = \operatorname{Im} f(x + \mathrm{i}y)$ ($f = u + \mathrm{i}v$) since

- u and v satisfy the differential equations

  $$u_x = v_y \quad u_y = -v_x \quad \iff \quad \partial_x f(x + \mathrm{i}y) = -\mathrm{i}\partial_y f(x + \mathrm{i}y)$$

  and u and v are harmonic, i.e. $u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$;

- a complex differentiable function f on a domain D (the domain of analyticity of f) is **infinitely** differentiable on D.

For a function f, expressed in terms of the variables z and $\bar{z} = x - \mathrm{i}y$, complex differentiability is equivalent to $\partial f / \partial \bar{z} = 0$.

Complex Potential

For any harmonic function u ($\Delta u = 0$) on a simply connected domain D exists a conjugate harmonic function v, so that $f(x + \mathrm{i}y) = u(x, y) + \mathrm{i}v(x, y)$ is complex differentiable.

Conformal Mapping

injective complex differentiable function $D \ni z \mapsto w = f(z)$ which is isotropic and angle-preserving:

- For any curve $t \mapsto z(t)$ passing through a point $z_0 = z(t_0)$, the tangent vector $z'(t_0)$ is scaled by the factor $|f'(z_0)|$ and rotated by the angle $\arg f'(z_0)$ when mapped to the curve $t \mapsto w(t) = f(z(t))$.

- The angle between curves, which intersect at z_0, is preserved by the conformal mapping.

Elementary Conformal Mappings

$z \mapsto w = f(z)$

- **scaling** by a factor $s > 0$: $w = sz$
- **rotation** by an angle φ: $w = \mathrm{e}^{\mathrm{i}\varphi} z$
- **disc** $|z| < 1$ $\to$ **half plane** $\operatorname{Im} w > 0$: $w = (1 - \mathrm{i})\dfrac{z + \mathrm{i}}{z + 1}$
- **half plane** $\operatorname{Im} z > 0$ $\to$ **disc** $|w| < 1$: $w = \dfrac{1 + \mathrm{i} - z}{z + \mathrm{i} - 1}$
- **quadrant** $\operatorname{Re} z, \operatorname{Im} z > 0$ $\to$ **half plane** $\operatorname{Im} z > 0$: $w = z^2$
- **strip** $0 < \operatorname{Im} z < \gamma \le 2\pi$ $\to$ **sector** $0 < \arg w < \gamma$: $w = \mathrm{e}^z$
  $\to$ **slit plane** $\mathbb{C} \backslash \mathbb{R}_0^+$ for $\gamma = 2\pi$
- **sector** $0 < \arg z < \alpha$ $\to$ **sector** $0 < \arg w < \beta$: $w = z^{\beta/\alpha}$

Riemann Mapping Theorem

Every simply connected proper subset of the complex plane with sufficiently regular boundary can be mapped conformally onto the unit disc.

7.3 Integration

Complex Integrands

$$\int \underbrace{(u + \mathrm{i}v)}_{f} = \int u + \mathrm{i} \int v, \quad \left| \int f \right| \le \int |f|$$

Complex Line Integral[1]

$$\int\limits_C f \, \mathrm{d}z = \int_a^b f(z(t)) z'(t) \, \mathrm{d}t, \quad C : t \mapsto z(t), \, t \in [a, b]$$

independent of the parametrization for equal orientation, change of sign for opposite orientation ($C \to -C$)

- real notation: $z = x + \mathrm{i}y$, $f = u + \mathrm{i}v \quad \rightsquigarrow$

  $$\int_C f \, \mathrm{d}z = \int_C (u + \mathrm{i}v) \, \mathrm{d}x + \int_C (\mathrm{i}u - v) \, \mathrm{d}y$$

- computation with an antiderivative: $F' = f \quad \Longrightarrow$

  $$\int\limits_C f(z) \, \mathrm{d}z = \big[F\big]_{z_0}^{z_1} = F(z_1) - F(z_0), \quad C : z_0 \to z_1$$

  independent of the path, connecting the two points z_1 and z_2, for complex differentiable functions

  Maple™ :

  ```
  int(f,z0..z1)
  ```

Singularity

of a complex differentiable function at a point a

- weak singularity (removable): $\lim\limits_{z \to a} (z - a) f(z) = 0$
- pole of order n: $|(z - a)^n f(z)| = O(1), \; z \to a, \, n \in \mathbb{N}$ minimal

[1] also known as *Curve Integral*, *Path Integral*, or *Contour Integral*

- essential singularity: $(z-a)^n f(z) \neq O(1) \;\; \forall n \in \mathbb{N}$

Homotopic Curves

$C \overset{D}{\sim} \tilde{C} \quad \Longleftrightarrow \quad \exists$ a continuous deformation in D of C into $\tilde{C}$
simply connected domain D: Every curve $C \subset D$ is homotopic to a point in D.

Cauchy's Theorem

$\int_C f(z)\,\mathrm{d}z = 0$ for a complex differentiable function f on a domain D with boundary C.

Winding Number

$n(C,a) = \frac{1}{2\pi \mathrm{i}} \int_C \frac{\mathrm{d}z}{z-a}$ for a closed path C with $a \notin C$; in particular $n(C,a) = 1$ for a circle with midpoint a and radius r

Cauchy's Integral Formula

$$f^{(n)}(z) = \frac{n!}{2\pi \mathrm{i}} \int\limits_C \frac{f(w)}{(w-z)^{n+1}}\,\mathrm{d}w, \quad z \in D$$

for a complex differentiable function f on a domain D with counterclockwise oriented boundary C; in particular

$$f^{(n)}(z) = \frac{n!}{2\pi} \int_0^{2\pi} f(z + r\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}nt}\,\mathrm{d}t$$

for a circle C with midoint z and radius r

Mean Value Property

$f(z) = \frac{1}{2\pi} \int\limits_0^{2\pi} f(z+r\mathrm{e}^{\mathrm{i}t})\,\mathrm{d}t$ for complex differentiable or harmonic functions f

Maximum Principle

- $\max_{z\in D}|f(z)| \le \max_{z\in C}|f(z)|$ for a complex differentiable function f on a domain D with boundary C
- $\max_{z\in D} u(x,y) \le \max_{(x,y)\in C} u(x,y)$ for harmonic functions u, in particular for the real and imaginary part of a complex differentiable function

Liouville's Theorem

If f is bounded and complex differentiable for all $z \in \mathbb{C}$ then f is constant.

Residue

$$\operatorname*{Res}_{z=a} f(z) = \operatorname*{Res}_{a} f = \frac{1}{2\pi\mathrm{i}} \int\limits_C f(z)\,\mathrm{d}z, \quad D \supset C:\ t \mapsto a + r\mathrm{e}^{\mathrm{i}t},\ r < R$$

for a complex differentiable function f on the annulus $D:\ 0 < |z-a| < R$

- removable singularity: $\operatorname*{Res}_{a} f = 0$
- simple pole: $\operatorname*{Res}_{a} f = \lim\limits_{z\to a} (z-a) f(z)$
- pole of order n: $\operatorname*{Res}_{a} f = \lim\limits_{z\to a} \frac{1}{(n-1)!} (\mathrm{d}/\mathrm{d}z)^{n-1} \left((z-a)^n f(z)\right)$
- essential singularity:
  coefficient c_{-1} of the Laurent series $f(z) = \sum\limits_{n=-\infty}^{\infty} c_n (z-a)^n$

Maple:

```
residue(f(z),z=a)
```

Residue Theorem

$$\int\limits_C f(z)\,\mathrm{d}z = 2\pi\mathrm{i} \sum_k \operatorname*{Res}_{a_k} f$$

for a complex differentiable function f with finitely many singularities a_k on a bounded domain D with counterclockwise oriented boundary C

Trigonometric Integrals

For a rational function, the integral $\int_0^{2\pi} r(\cos t, \sin t)\,\mathrm{d}t$ can be computed with the substitution

$$z = \mathrm{e}^{\mathrm{i}t}, \quad \cos t = \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin t = \frac{1}{2\mathrm{i}}\left(z - \frac{1}{z}\right)$$

which leads to an integral over the unit circle $C : t \mapsto \mathrm{e}^{\mathrm{i}t}$ for which the residue theorem can be applied:

$$\int_C \underbrace{r\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2\mathrm{i}}\left(z-\frac{1}{z}\right)\right)\frac{1}{\mathrm{i}z}}_{f}\,\mathrm{d}z = 2\pi\mathrm{i}\sum_{|a|<1}\operatorname{Res}_a f\,.$$

Integrals of Rational Functions

For a rational function p/q with $\operatorname{Grad} p \leq \operatorname{Grad} q - 2$ and no real poles,

$$\int_{\mathbb{R}} \underbrace{p(x)/q(x)}_{f}\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{\operatorname{Im} a>0}\operatorname{Res}_a f\,.$$

alternatively: $\int \ldots = -2\pi\mathrm{i}\sum_{\operatorname{Im} a<0}\operatorname{Res}_a f$ (sum of residues in the lower half plane)

Integrals Involving Exponential Functions

For a rational function p/q with $\operatorname{Grad} p < \operatorname{Grad} q$ and no real poles,

$$\int_{\mathbb{R}} \underbrace{(p(x)/q(x))}_{f(x)}\,\mathrm{e}^{\mathrm{i}\lambda x}\,\mathrm{d}x = 2\pi\mathrm{i}\sum_{\operatorname{Im} a>0}\operatorname{Res}_{z=a}\left(f(z)\mathrm{e}^{\mathrm{i}\lambda z}\right), \quad \lambda > 0\,.$$

$\lambda < 0 \quad \rightsquigarrow \quad$ negative sum of residues in the lower half plane

7.4 Taylor and Laurent Series

Taylor Series

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!} (z-a)^k, \quad |z-a| < r$$

radius of convergence: $r = (\overline{\lim}_{k\to\infty} |f^{(k)}(a)/k!|^{1/k})^{-1}$ which is equal to the distance of a to the nearest singularity, i.e. to the boundary of the domain of analyticity of f

remainder/error of the Taylor polynomial $p_n(z) = \sum_{k=0}^{n} \ldots$:

$$f(z) - p_n(z) = \frac{1}{2\pi\mathrm{i}} \int\limits_C \frac{f(w)\,\mathrm{d}w}{(w-a)^{n+1}(w-z)} (z-a)^{n+1} = O\left(|z-a|^{n+1}\right), \ z \to a$$

with $C: |w-a| = r' < r$, $r' > |z-a|$ a counterclockwise oriented circle

Maple:

```
taylor(f(z),z=a,n)
```

Methods of Taylor Expansion

- direct computation of the derivatives for the Taylor coefficients
- termwise differentiation or integration of known Taylor series
- comparison of coefficients
- multiplication of Taylor series
- composition of functions

Laurent Series

of a complex differentiable function f in an annulus $D: r_1 < |z-a| < r_2$:

$$f(z) = \sum_{k=-\infty}^{\infty} c_k (z-a)^k, \quad c_k = \frac{1}{2\pi\mathrm{i}} \int\limits_C \frac{f(w)}{(w-a)^{k+1}}\,\mathrm{d}w$$

with $C \subset D$ a counterclockwise oriented circle with midpoint a

Pole of order n: c_{-n+k} is the k-th coefficient of the Taylor series of $g(z) = (z-a)^n f(z)$, i.e. $c_{-n+k} = g^{(k)}(a)/k!$, and $\sum_{k=-n}^{-1} c_k (z-a)^k$ is referred to as **principle part** of the Laurent series.

Maple™ :

```
with(numapprox); laurent(f(z),z=a,n)
```

$\rightsquigarrow$ expansion in a punctured disc

Methods of Laurent Expansion

- application of the analogous techniques for Taylor expansion
- replacing z by $\frac{1}{z-a}$ in known Taylor series
- partial fraction decomposition, using the elementary expansions

$$\begin{aligned} \frac{1}{z-b} &= \frac{1}{a-b} \sum_{n=0}^{\infty} \left(\frac{z-a}{b-a} \right)^n, \quad |z-a| < |b-a| \\ &= \frac{1}{b-a} \sum_{n=1}^{\infty} \left(\frac{b-a}{z-a} \right)^n, \quad |z-a| > |b-a| \end{aligned}$$

  expansion of $1/(z-b)^m$ via differentiation of the Laurent series

- coefficients of a Laurent series in a punctured disc for a pole of order n of f at a:

$$c_{k-n} = \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^k ((z-a)^n f(z)) \Big|_{z=a}$$

7.5 Differential Equations

Regular Point of a Differential Equation

$$pu'' + qu' + ru = 0, \quad f(z) = \sum_{n=0}^{\infty} f_n (z-a)^n,\ f = u, p, q, r, \quad p_0 \neq 0$$

comparing coefficients of z^n in the differential equation $\quad\rightsquigarrow\quad$ recursion

$$\begin{aligned} p_0(n+2)(n+1)\, u_{n+2} \;=\; & -(p_1(n+1)n \,+\, q_0(n+1))\, u_{n+1} \\ & -(p_2 n(n-1) \,+\, q_1 n \,+\, r_0)\, u_n \\ & -(p_3(n-1)n \,+\, q_2(n-1) \,+\, r_1)\, u_{n-1} \\ & -\ldots \end{aligned}$$

($u_{-1} := 0$) which allows to determine $u_2, u_3, \ldots$ successively from the initial conditions $u(a) = u_0$, $u'(a) = u_1$

Maple™ :

```
de := p(z)*diff(u(z),z,z)+q(z)*diff(u(z),z)+r(z)*u(z)=0
c := u(a) = u0, D(u)(a) = u1
dsolve({de,c},u(z),series,order = n)
```

$\rightsquigarrow$ first n terms of the Taylor series of u

Euler's Differential Equation

$$t^2 u''(t) + pu'(t) + qu(t) = 0, \quad t > 0$$

ansatz $u(t) = t^\lambda \quad\rightsquigarrow\quad \lambda(\lambda-1) + p\lambda + q = 0$
general solution

- two real exponents λ_1, λ_2: $u(t) = c_1 t^{\lambda_1} + c_2 t^{\lambda_2}$
- a single real exponent λ: $u(t) = t^\lambda (c_1 + c_2 \ln t)$
- complex conjugate exponents $r \pm s\mathrm{i}$: $u(t) = t^r (c_1 \cos(s \ln t) + c_2 \sin(s \ln t))$

Alternative: substituting $t = \mathrm{e}^\tau$, $u(t) = v(\tau) \quad\rightsquigarrow$

$$v'' + (p-1)v' + qv = 0$$

Singular Point of a Differential Equation

$ru'' + qu' + pu = 0$ with

$$p(z) = \underbrace{p_0}_{\neq 0} + p_1(z-a) + \cdots\,,\quad q(z) = \frac{q_{-1}}{z-a} + \cdots\,,\quad r(z) = \frac{r_{-2}}{(z-a)^2} + \cdots$$

ansatz $u(z) = (z-a)^\lambda(u_0 + u_1(z-a) + \cdots)$ and comparison of the coefficients of $(z-a)^{\lambda+n-2}$ in the differential equation $\rightsquigarrow$

$$\begin{aligned}\underbrace{(p_0(n+\lambda)(n+\lambda-1) + q_{-1}(\lambda+n) + r_{-2})}_{\varphi(\lambda+n)}\, u_n = \\ &- (p_1(n+\lambda-1)(n+\lambda-2) + q_0(\lambda+n-1) + r_{-1})\, u_{n-1} \\ &- (p_2(n+\lambda-2)(n+\lambda-3) + q_1(\lambda+n-2) + r_0)\, u_{n-2} \\ &- \cdots\end{aligned}$$

for $n = 0, 1, \ldots$ and with $u_k = 0$ for $k < 0$
For a solution λ of the characteristic equation $(n = 0)$

$$\varphi(\lambda) = p_0\lambda(\lambda-1) + q_{-1}\lambda + r_{-2} = 0\,,$$

a nonzero coefficient u_0 can be chosen arbitrarily. Then, if $\varphi(\lambda+n) \neq 0\ \forall\ n > 0$, the recursion successively determines $u_1, u_2, \ldots$. Otherwise, i.e. if the zeros of φ differ by an integer, a solution corresponding to the exponent with smaller real part must be constructed with a different technique, e.g. with variation of constants.

Maple™ :

```
de := p(z)*diff(p(z),z,z)+q(z)*diff(u(z),z)+r(z)*u(z) = 0
dsolve(de,series,order = n)
```

$\rightsquigarrow$ first n terms of the general solution

Bessel Differential Equation

$$z^2u''(z) + zu'(z) + (z^2 - \lambda^2)u(z) = 0$$

$\rightsquigarrow$ Bessel functions $u(z) = J_{\pm\lambda}(z) = \left(\frac{z}{2}\right)^{\pm\lambda} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(\pm\lambda+n+1)} \left(\frac{z}{2}\right)^{2n}$

For $\pm\lambda \in \mathbb{N}$ the functions $J_\pm$ are not linearly independent. In this case, a second, linearly independent solution of the differential equation is a so-called Bessel function of the second kind.

Special Bessel functions:

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}, \quad J_{1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{\sin z}{\sqrt{z}},\ J_{-1/2}(z) = \sqrt{\frac{2}{\pi}} \frac{\cos z}{\sqrt{z}}$$

Integral representation: $J_n(x) = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \cos(nt - x \sin t)\, \mathrm{d}t,\ n \in \mathbb{N}_0$

Hypergeometric Differential Equation

$$z(1-z)u''(z) + (c - (a+b+1)z)u'(z) - abu(z) = 0$$

regular solution (hypergeometric function) for $-c \notin \mathbb{N}_0$

$$u(z) = F(a, b, c, z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n$$

with $(t)_0 = 1$, $(t)_1 = t$, $(t)_2 = t(t+1)$, ...